

Stochastic Models of Complex Systems

Hand-out 3

Characteristic function, Gaussians, LLN, CLT

Let X be a real-valued random variable with PDF f_X . The **characteristic function** $\phi_X(t)$ is defined as the Fourier transform of the PDF, i.e.

$$\phi_X(t) = \mathbb{E}(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f_X(x) dx \quad \text{for all } t \in \mathbb{R}.$$

As the name suggests, ϕ_X uniquely determines (characterizes) the distribution of X and the usual inversion formula for Fourier transforms holds,

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_X(t) dt \quad \text{for all } x \in \mathbb{R}.$$

The characteristic function plays a similar role as the probability generating function for discrete random variables. Moments can be recovered via

$$\frac{\partial^k}{\partial t^k} \phi_X(t) = (i)^k \mathbb{E}(X^k e^{itX}) \Rightarrow \mathbb{E}(X^k) = (i)^{-k} \frac{\partial^k}{\partial t^k} \phi_X(t) \Big|_{t=0}. \quad (1)$$

Also, if we add independent random variables X and Y , their characteristic functions multiply,

$$\phi_{X+Y}(t) = \mathbb{E}(e^{it(X+Y)}) = \phi_X(t) \phi_Y(t). \quad (2)$$

Furthermore, for a sequence X_1, X_2, \dots of real-valued random variables we have

$$X_n \rightarrow X \quad \text{in distribution, i.e. } f_{X_n}(x) \rightarrow f_X(x) \forall x \in \mathbb{R} \Leftrightarrow \phi_{X_n}(t) \rightarrow \phi_X(t) \forall t \in \mathbb{R}. \quad (3)$$

A real-valued random variable $X \sim N(\mu, \sigma^2)$ has **normal** or **Gaussian** distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \geq 0$ if its PDF is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Properties.

- The characteristic function of $X \sim N(\mu, \sigma^2)$ is given by

$$\phi_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} + itx\right) dx = \exp\left(i\mu t - \frac{1}{2}\sigma^2 t^2\right).$$

To see this (try it!), you have to complete the squares in the exponent to get

$$-\frac{1}{2\sigma^2} (x - (it\sigma^2 + \mu))^2 - \frac{1}{2}t^2\sigma^2 + it\mu,$$

and then use that the integral over x after re-centering is still normalized.

- This implies that linear combinations of independent Gaussians X_1, X_2 are Gaussian, i.e.

$$X_i \sim N(\mu_i, \sigma_i^2), a, b \in \mathbb{R} \Rightarrow aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2).$$

This holds even for correlated X_i , where the variance is $a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\text{Cov}(X_1, X_2)$ with covariance $\text{Cov}(X_1, X_2) = \mathbb{E}((X_1 - \mu_1)(X_2 - \mu_2))$ and the mean remains unchanged.

Let X_1, X_2, \dots be a sequence of iidrv's with mean μ and variance σ^2 and set $S_n = X_1 + \dots + X_n$. The following two important limit theorems are a direct consequence of the above.

Weak law of large numbers (LLN)

$$S_n/n \rightarrow \mu \quad \text{in distribution as } n \rightarrow \infty .$$

There exists also a strong form of the LLN with almost sure convergence which is harder to prove.

Central limit theorem (CLT)

$$\frac{S_n - \mu n}{\sigma\sqrt{n}} \rightarrow N(0, 1) \quad \text{in distribution as } n \rightarrow \infty .$$

The LLN and CLT imply that for $n \rightarrow \infty$, $S_n \simeq \mu n + \sigma\sqrt{n}\xi$ with $\xi \sim N(0, 1)$.

Proof. With $\phi(t) = \mathbb{E}(e^{itX_i})$ we have from (2)

$$\phi_n(t) := \mathbb{E}(e^{itS_n/n}) = (\phi(t/n))^n .$$

(1) implies the following Taylor expansion of ϕ around 0:

$$\phi(t/n) = 1 + i\mu \frac{t}{n} - \frac{\sigma^2}{2} \frac{t^2}{n^2} + o(t^2/n^2) ,$$

of which we only have to use the first order to see that

$$\phi_n(t) = \left(1 + i\mu \frac{t}{n} + o(t/n)\right)^n \rightarrow e^{it\mu} \quad \text{as } n \rightarrow \infty .$$

By (3) and uniqueness of characteristic functions this implies the LLN.

To show the CLT, set $Y_i = \frac{X_i - \mu}{\sigma}$ and write $\tilde{S}_n = \sum_{i=1}^n Y_i = \frac{S_n - \mu n}{\sigma}$.

Then, since $\mathbb{E}(Y_i) = 0$, the corresponding Taylor expansion (now to second order) leads to

$$\phi_n(t) := \mathbb{E}(e^{it\tilde{S}_n/\sqrt{n}}) = \left(1 - \frac{t^2}{2n} + o(t^2/n)\right)^n \rightarrow e^{-t^2/2} \quad \text{as } n \rightarrow \infty ,$$

which implies the CLT. □

Multivariate case.

All of the above can be generalized to multivariate random variables in \mathbb{R}^d . $\mathbf{X} = (X_1, \dots, X_d)$ is a **multivariate Gaussian** if it has PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d \det \Sigma}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})^T\right) \quad \text{with } \mathbf{x} = (x_1, \dots, x_d) ,$$

where $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d)$ is the vector of means $\mu_i = \mathbb{E}(X_i)$ and $\Sigma = (\sigma_{ij} : i, j = 1, \dots, d)$ is the **covariance matrix** with entries

$$\sigma_{ij} = \text{Cov}(X_i, X_j) = \mathbb{E}((X_i - \mu_i)(X_j - \mu_j)) .$$

Σ is symmetric and invertible (unless in degenerate cases with vanishing variance). The characteristic function of \mathbf{X} is given by

$$\phi_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}(e^{it\mathbf{X}^T}) = \exp\left(it\boldsymbol{\mu}^T - \frac{1}{2}\mathbf{t}\Sigma\mathbf{t}^T\right) , \quad \mathbf{t} \in \mathbb{R}^d .$$

(In the above notation \mathbf{x} is a row and \mathbf{x}^T a column vector, and scalar products are written as $\mathbf{x}\mathbf{x}^T$.)