# Stochastic models of complex systems

## Problem sheet 1

**1.1 Generators and eigenvalues**

The analysis of linear dynamical systems shares a lot of the structure with Markov chains.

(a) Consider the **harmonic oscillator** $\phi : \mathbb{R} \to \mathbb{R}$ given by the equation

$$\frac{d^2}{dt^2}\,\phi(t) = \ddot{\phi}(t) = -k\phi \quad \text{with} \quad k > 0\,.$$

Using the vector valued notation $\boldsymbol{x}(t) = \binom{\phi(t)}{\dot{\phi}(t)}$ write the system in the form

$$\frac{d}{dt}\,\boldsymbol{x}(t) = M\boldsymbol{x}(t) \quad \text{with} \quad M \in \mathbb{R}^{2\times 2}\,.$$

Compute the eigenvalues $\lambda_1$ and $\lambda_2$ of $M$ and find a solution of the form

$$\phi(t) = a\,e^{\lambda_1 t} + b\,e^{\lambda_2 t}\,,$$

fixing $a, b \in \mathbb{R}$ by the initial conditions $\phi(0) = 1$, $\dot{\phi}(0) = 0$.

(b) Consider the **Fibonacci numbers** $(F_n : n \in \mathbb{N}_0)$ defined by the recursion

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2) \quad \text{with} \quad F_0 = 0,\ F_1 = 1\,.$$

Write $\binom{F_{n+1}}{F_n} = M\binom{F_n}{F_{n-1}}$ as a discrete-time dynamical system with $M \in \mathbb{R}^{2\times 2}$.
Compute the eigenvalues of $M$ and show that

$$F_n = \frac{\eta^n - (1-\eta)^n}{\sqrt{5}} \quad \text{where} \quad \eta = \frac{1+\sqrt{5}}{2} \quad \text{is the Golden ratio}\,. \qquad [6]$$

(c)* Derive a recursion relation for the generating function $G(s) = \sum_n F_n s^n$ of the Fibonacci numbers and solve it. Sketch $G(s)$. For which $s \geq 0$ is it well defined?

**1.2 Branching processes**

Let $Z = (Z_n : n \in \mathbb{N})$ be a branching process, defined recursively by

$$Z_0 = 1\,, \quad Z_{n+1} = X_1^n + \ldots + X_{Z_n}^n \quad \text{for all } n \geq 0\,,$$

where the $X_i^n \in \mathbb{N}$ are iidrv's denoting the offspring of individuum $i$ in generation $n$.

(a) Consider a geometric offspring distribution $X_i^n \sim Geo(p)$, i.e.

$$p_k = \mathbb{P}(X_i^n = k) = p\,(1-p)^k\,, \quad p \in (0,1)\,.$$

Compute the prob. generating function $G(s) = \sum_k p_k s^k$ as well as $\mathbb{E}(X_i^n)$ and $Var(X_i^n)$. Sketch $G(s)$ for (at least) three (wisely chosen) values of $p$ and compute the probability of extinction as a function of $p$.

(b) Consider a Poisson offspring distribution $X_i^n \sim Poi(\lambda)$, i.e.

$$p_k = \mathbb{P}(X_i^n = k) = \frac{\lambda^k}{k!}\,e^{-\lambda}\,, \quad \lambda > 0\,.$$

Repeat the same analysis as in (a). [8]

(c)* For geometric offspring with $p = 1/2$, show that $G_n(s) = \frac{n-(n-1)s}{n+1-ns}$ and compute $\mathbb{P}(Z_n = 0)$. If $T$ is the (random) time of extinction, what is its distribution and its expected value?

## 1.3 Toom's model (A probabilistic cellular automaton)

Consider a fixed population of $L$ individuals on a one-dimensional lattice $\Lambda = \{1, \ldots, L\}$ with periodic boundary conditions. At time $t = 0$ each site $i$ is occupied by a type $X_0(i) \in \{1, \ldots, N\}$, where $N$ is the total number of types. Time is counted in discrete generations $t = 0, 1, \ldots$, and the lattice at odd times is shifted by $1/2$. So in generation $t+1$ each individual has two parents, from one of which it inherits the type. The type $X_{t+1}(i + 1/2)$ is determined by $X_t(i)$ and $X_t(i + 1)$ according to the following rules, for $x \neq y \in \{1, \ldots, N\}$:

$$xx \to x \ , \quad yy \to y \ , \quad xy, yx \to \begin{cases} x \ , & \text{with prob. } \hat{p}_{xy} \\ y \ , & \text{with prob. } \hat{p}_{yx} = 1 - \hat{p}_{xy} \end{cases} .$$

For now we focus on $\hat{p}_{xy} = 1/2$ for all $x, y$, i.e. all types have the same 'fitness'.

(a) Simulate the dynamics of this process (e.g. using MATLAB) up to generation $T$ in a $T \times 2L$ matrix (with time growing upwards). Initialize the matrix with 0s, then fill the even sites in the bottom row with initial condition $X_0(i) = i$ (i.e. a different type on each site). Then occupy the odd sites in the next row according to the dynamics of the process until all rows are filled.
Visualise the matrix using e.g. the MATLAB function 'image'.
(Make sure that the empty sites show up in white or a very light colour, you might have to replace 0 by another (negative) value.)
You may use the suggested parameter values $L = 100$, $T = 500$ or any other that make sense (it is a good idea to vary them to get a feeling for the model). Address the following points, supported by appropriate visualisations:
- Explain the emerging patterns in a couple of sentences.
- What are the stationary distributions for the process $X_t$, i.e. what will happen when you run the simulation long enough?
- How long will it roughly take to reach stationarity (depending on $L$)? Test your answer using three values for $L$, e.g. 10, 50 and 100.

(b) Let $N_t$ be the number of individuals of a given species at generation $t$. Show that $(N_t : t \in \mathbb{N})$ is actually a random walk, give the state space and the transition probabilities.
Is the walk irreducible? What are the stationary distributions?

Now consider only a few types, but with different fitnesses.

(c) Consider two types $1, 2$ with $\hat{p}_{12} = q > 1/2$, $\hat{p}_{21} = 1 - q$ (i.e. 1 is fitter than 2), with initial condition $X_0(i) = 2$ for all $i$ except for a single site with type 1.
Produce a few visualisations (e.g. with $L = 100$, $T = 100$) for different values of $q$ including one where type 1 survives.
What are the stationary distributions for this asymmetric model?

(d)* Can you compute the survival probability of type 1?

(e) Consider three species with a 'cyclic' interaction

$$\hat{p}_{12} = \hat{p}_{23} = \hat{p}_{31} = q > 1/2 \quad \text{and} \quad \hat{p}_{21} = \hat{p}_{32} = \hat{p}_{13} = 1 - q < 1/2 \ .$$

Initialize the system with random types $X_0(i) \sim U\big(\{1, 2, 3\}\big)$ for all $i$. Produce a couple of visualisations (e.g. with $L = 100$, $T = 500$) for $q = 0.6$, 0.8 and 1, and explain the patterns you see.

[14]