Stochastic Models of Complex Systems

Problem sheet 2

2.1 For each of the following models decide whether $(X_n : n \in \mathbb{N})$ is a Markov chain. If yes provide the state space and the transition probabilities, decide whether it is irreducible and give all stationary distributions.

(a) **Ehrenfest urn model of diffusion**
A total of M balls is distributed over two urns. In each time step, one ball is chosen uniformly at random and is transferred from its urn to the other urn. Let X_n be the number of balls in the left urn after n time steps.

(b) **Wright-Fisher model**
Consider M balls of two types in one urn. In each step, M balls are drawn with replacement from the urn, which form the new 'generation'. Let X_n be the number of balls of type 1 in generation n.

[6]

2.2 Consider the **contact process** $(\eta_t : t \geq 0)$ on the complete graph $\Lambda = \{1, \ldots, L\}$ (all sites connected) with state space $S = \{0,1\}^L$ and transition rates

$$c(\eta, \eta^x) = \eta(x) + \lambda\big(1 - \eta(x)\big) \sum_{y \neq x} \eta(y) ,$$

where $\eta, \eta^x \in S$ are connected states such that $\eta^x(y) = \begin{cases} 1 - \eta(x) & , y = x \\ \eta(y) & , y \neq x \end{cases}$,
(η with site x flipped).

(a) Let $N_t = \sum_{x \in \Lambda_L} \eta_t(x) \in \{0, \ldots, L\}$ be the number of infected sites at time t. Show that $(N_t : t \geq 0)$ is a Markov chain with state space $\{0, \ldots, L\}$ by computing the transition rates $c(n, m)$ for $n, m \in \{0, \ldots, L\}$.
Write down the master equation for the process.

(b) Is the process $(N_t : t \geq 0)$ irreducible, does it have absorbing states?
What are the stationary distributions?

(c) Assume that $\mathbb{E}\big(N_t^k\big) = \mathbb{E}(N_t)^k$ for all $k \geq 1$. This is called a **mean-field assumption**, meaning basically that we replace the random variable N_t by its expected value.
Use this assumption to derive the **mean-field rate equation** for $\rho(t) := \mathbb{E}(N_t)/L$,

$$\frac{d}{dt}\,\rho(t) = f\big(\rho(t)\big) = -\rho(t) + L\lambda\big(1 - \rho(t)\big)\rho(t) .$$

(d) Analyze this equation by finding the stable and unstable stationary points via $f(\rho^*) = 0$.
What is the prediction for the stationary density ρ^* depending on λ?

[10]

2.3 The totally asymmetric simple exclusion process **(TASEP)** with open boundaries is an exclusion process on the one-dimensional lattice $\Lambda = \{1, \ldots, L\}$ with transition rates

$$10 \xrightarrow{1} 01 \quad \text{in the bulk, and} \quad |0 \xrightarrow{\rho_l} |1\,, \quad 1| \xrightarrow{1-\rho_r} 0| \quad \text{at the boundaries}\,.$$

So particles jump one site to the right with rate 1 if possible and are injected and ejected at the boundary, where the system is coupled to reservoirs with densities $\rho_l, \rho_r \in [0,1]$. The state space is $S = \{0,1\}^L$ and we denote a particle configuration by $\eta = (\eta_x : x \in \Lambda)$.

(a) Draw the initial occupation numbers η_x independently with $\eta_x \sim Be(\rho_l)$ for $x < L/2$ and $\eta_x \sim Be(\rho_r)$ for $x \geq L/2$. Then simulate the process using random sequential update, and record the configuration η in regular time intervals Δt up to time T. Visualize the time evolution (e.g. by using 'image' in MATLAB) for the following situations (three cases each)

$$\begin{aligned} &\rho_l = 1,\ 0.8,\ 0.6 \qquad \text{and} \quad \rho_r = 0 \qquad\qquad \text{(traffic light)} \\ &\rho_l = 0.2, \quad \text{and} \qquad \rho_r = 0.6,\ 0.8,\ 1 \qquad \text{(end of traffic jam)}\,. \end{aligned}$$

Suggested parameter values are $L = 200$, $T = 400$, $\Delta t = 2$.
Interpret your findings in a few sentences.

(b) Initialize the system with $\eta_x = 0$ for all $x \in \Lambda$ and measure the total density of particles $\rho(t) = \frac{1}{L}\sum_{x\in\Lambda} \eta_x(t)$ as a function of time for the parameter values

$$(\rho_l, \rho_r) = (0.2, 0.2),\ (0.8, 0.2) \quad \text{and} \quad (0.8, 0.8)\,.$$

Plot $\rho(t)$ for $t \leq T$ large enough to predict the limiting behaviour $\lim_{t\to\infty} \rho(t)$.
Interpret your findings in a few sentences.

(c) Study the effect of a narrow road or a hill, by changing the jump rate in the bulk for $x \geq L/2$ from 1 to 0.8. Use $\rho_l = \rho_r = \rho$ and initialize $\eta_x \sim Be(\rho)$ independently for all $x \in \Lambda$. Simulate the process for $\rho = 0.2,\ 0.4,\ 0.6,\ 0.8$ and visualize the profiles as in (a), for e.g. $L = 200$, $T = 400$ and $\Delta t = 2$. Interpret your findings in a few sentences.

[11]

2.4 Adapt your programme from Q2.3 to simulate a generalized TASEP, using now periodic boundary conditions on the lattice $\Lambda = \{1, \ldots L\}$. The jump rates should depend on the neighbourhood configuration in the following way:

$$0100 \xrightarrow{1} 0010\,, \quad 1101 \xrightarrow{\alpha} 1011\,, \quad 0101 \xrightarrow{\beta} 0011\,, \quad 1100 \xrightarrow{\gamma} 1010\,.$$

For this model, the average stationary current is a function of the number N of particles, or the density $\rho = N/L$. It is defined by $j(\rho) = \mathbb{E}\big(c(\eta, \eta^{x,x+1})\big)$, where $c(\eta, \eta^{x,x+1})$ is the jump rate of a particle from x to $x+1$ as given above.

(a) Making use of the ergodic theorem, measure the **fundamental diagram**, i.e. $j(\rho)$ as a function of the density. For fixed lattice size L (e.g. 500) vary the number of cars N to get j for $\rho = 0, 0.1, \ldots, 0.9, 1$. Do this for $\alpha = \beta = \gamma = 1$ (usual TASEP) and at least two other choices of rates. Explain what your choices correspond to in terms of driver behaviour if you interpret this as a traffic model.

(b) Calculate the stationary current using a mean-field approximation for the parameters you chose in (a), and compare this to your measurement results in a plot.
Discuss how the different shapes of the fundamental diagram are related to your choice of rates.

[9]