Stochastic Models of Complex Systems

Problem sheet 3

3.1 The **voter model** $(\eta_t : t \geq 0)$ on a general lattice Λ with state space $S = \{0,1\}^L$ is given by jump rates

$$c(\eta, \eta^x) = \sum_{y\in\Lambda} p_{x,y}\Big(\eta(x)\big(1-\eta(y)\big) + \eta(y)\big(1-\eta(x)\big)\Big)$$

with $p_{x,y} \geq 0$. In the following we consider the one-dimensional case $\Lambda_L = \{1,\ldots,L\}$ with periodic boundary conditions, where only nearest neighbours influence each other with rate 1.

(a) Write down the generator $\mathcal{L}f$ acting on test functions $f : S \to \mathbb{R}$ for this process. Discuss whether the process is ergodic, and give all stationary distributions.

(b) Let $N(\eta) = \sum_{x\in\Lambda} \eta_x$ be the number of individuals with opinion 1 and denote by $\rho(t) = \frac{1}{L}\mathbb{E}_{\pi(t)}\big(N(t)\big)$ their average fraction at time t. Use that

$$\frac{d}{dt}\rho(t) = \frac{1}{L}\,\mathbb{E}_{\pi(t)}(\mathcal{L}N)$$

to show that $\rho(t) = \rho(0)$ for all $t \geq 0$.
Therefore, for an initial condition with $\rho(0) = 1/2$, what is the limit of $\pi(t)$ as $t \to \infty$?

(c) Let $f_n(\eta) = \delta_{n,N(\eta)}$ which is 1 if $N(\eta) = n$ and 0 otherwise. Use the mean-field assumption

$$\mathbb{E}_{\pi(t)}\Big(\eta(x)\big(1-\eta(y)\big)\delta_{n,N(\eta)}\Big) = \frac{n}{L}\Big(1-\frac{n}{L}\Big)p_n(t)\ ,$$

and the same method as in (b) to derive the Master equation for $p_n(t) = \mathbb{P}\big(N(\eta_t) = n\big)$

$$\frac{d}{dt}p_n(t) = 2(n-1)\Big(1-\frac{n-1}{L}\Big)p_{n-1}(t) + 2(n+1)\Big(1-\frac{n+1}{L}\Big)p_{n+1}(t) - 4n\Big(1-\frac{n}{L}\Big)p_n(t)$$

for $n = 1,\ldots,L-1$. How does the equation look for $n = 0, L$?

(d) In the limit of large system size, consider the (random) fraction $X_t = N(\eta_t)/L \in [0,1]$ of individuals with opinion 1. For the process $(X_t : t \geq 0)$ the density is then given by $f(t,x) = \lim_{L\to\infty} Lp_n(t)$. Under a proper time rescaling $s = t/L^\alpha$ (give the value of α) derive the Fokker-Planck equation

$$\frac{\partial}{\partial s}f(s,x) = 2\,\frac{\partial^2}{\partial x^2}\Big(x(1-x)f(s,x)\Big)$$

from the Master equation in the limit $L \to \infty$.
Hint: Write $g(t, x\pm\frac{1}{L}) := \frac{n\pm1}{L}(1-\frac{n\pm1}{L})Lp_{n\pm1}(t)$ in the Master equation and do a Taylor expansion around $x = n/L$ up to second order.

[10]

(e)* $(X_s : s \geq 0)$ as given in (d) is also called a **Wright-Fisher diffusion**.
Show from the Fokker-Planck equation that $\mathbb{E}(X_s) = \mathbb{E}(X_0)$ for all $s > 0$ and discuss the limit of X_s as $s \to \infty$, similarly to (b).

3.2 Let $X = (X_n : n \in \mathbb{N})$ be a simple random walk on $\mathbb{Z}$ with transition probabilities

$$p_{i,i+1} = 1/2 + \epsilon\ , \quad p_{i,i-1} = 1/2 - \epsilon \quad \text{for all } i \in \mathbb{Z}\ .$$

Rescale time $t = \Delta t\, n$ and derive the Fokker-Planck equation for an appropriate scaling of space and ϵ, analogous to the derivation of Section 2.1. What is the right scaling of the asymmetry $\epsilon(\Delta t)$ to get a limit with non-zero drift and diffusion?

[4]

3.3 (a) Let $\xi \sim N(0, 1)$ be a Gaussian random variable with mean 0 and variance 1. Then consider the continuous time stochastic process $X_t = \sqrt{t}\xi$. Show that $X_t \sim N(0, t)$. Is X a Brownian motion? (Justify your answer.)

(b) Let B and $\tilde{B}$ be a two independent standard Brownian motions in $\mathbb{R}$ and $\rho \in [-1, 1]$ a constant. Then consider the process $X_t = \rho B_t + \sqrt{1 - \rho^2}\tilde{B}_t$. Show that X is again a standard Brownian motion. (Hint: use covariances)

[4]

(c)* Let B be a Brownian motion. What is the distribution of $B_s + B_t$ for $0 \leq s \leq t$?

(d)* Scaling property: Let B be a standard Brownian motion in $\mathbb{R}^d$. Show that for $\lambda > 0$, $B_\lambda = \big(\lambda^{-1/2} B_{\lambda t} : t \geq 0\big)$ is a standard Brownian motion in $\mathbb{R}^d$.

3.4 Consider the contact process $(\eta_t : t \geq 0)$ as defined in Q2.2, but now on the one-dimensional lattice $\Lambda_L = \{1, \ldots, L\}$ with connections only between nearest neighbours and periodic boundary conditions.
The critical value λ_c is defined such that the infection on the infinite lattice $\Lambda = \mathbb{Z}$ started from the fully infected lattice dies out for $\lambda < \lambda_c$, and survives for $\lambda > \lambda_c$. It is known numerically up to several digits, depends on the dimension, and lies in the interval $[1, 2]$ in our case.

(a) Simulate the process with initial condition $\eta(x) = 1$ for all $x \in \Lambda$ and several values of $\lambda \in [1, 2]$. Plot the number of infected individuals $N_t = \sum_{x\in\Lambda_L} \eta_t(x)$ as a function of time averaging over 100 realizations in a double-logarithmic plot.
What is the expected behaviour of N_t depending on λ for times up to order L?
For a given system size L, find the window of interest choosing $\lambda = 1, 1.2, \ldots, 1.8, 2$ and then use increments 0.01 for λ to find an estimate of the critical value $\lambda_c(L) \in [1, 2]$.
Repeat this for different lattice sizes, e.g. $L = 64, 128, 256, 512$, and plot your estimates of $\lambda_c(L)$ against $1/L$. Extrapolate to $1/L \to 0$ to get an estimate of $\lambda_c = \lambda_c(\infty)$.
This approach is called **finite size scaling**, in order to correct for **finite size effects** which influence the critical value. [12]

(b) Simulate the process for $L = 128$ with initial condition $\eta(x) = 1$ for all $x \in \Lambda$ and several (at least 3) values of λ around $\lambda_c(L)$. After an equilibration time $\tau_{equ} = L$, sample from the distribution of the number of infections $N_t = \sum_{x\in\Lambda_L} \eta_t(x)$, i.e. over a time interval of length $\tau_{meas} = L$ count the fraction of time N_t spent in n for each $n \in \{0, \ldots, L\}$. Average this measurement over 100 realizations and plot your estimate of the distribution for all values of λ in a single plot (it might be a good idea to use a log-scale on the y axis). Explain the form of the observed curves. [6]

(c)* Repeat the analysis of (a) on the fully connected graph Λ_L, and compare your estimate of λ_c with the mean-field prediction from Q2.2.