Stochastic Models of Complex Systems

Hand-out 2

Poisson process, random sequential update, exponentials

Let $X \sim Poi(\lambda)$ be a **Poisson** random variable with intensity $\lambda \geq 0$, i.e.

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!}\, e^{-\lambda} \quad \text{for all } k \in \mathbb{N}_0 \, .$$

We have $\mathbb{E}(X) = \lambda$, $\mathrm{Var}(X) = \lambda$ and the probability generating function of X is

$$G_X(s) = \mathbb{E}(s^X) = \sum_{k=0}^{\infty} s^k \frac{\lambda^k}{k!}\, e^{-\lambda} = e^{\lambda(s-1)} \, .$$

Therefore, if $X_i \sim Poi(\lambda_i)$, $i = 1, \ldots, n$ are independent Poisson, then the sum is also Poisson,

$$S = \sum_{i=1}^{n} X_i \sim Poi(\lambda_1 + \ldots + \lambda_n) \, .$$

For $\alpha \in [0, 1]$, an α-**thinning** $\alpha \circ X$ of an integer random variable $X \in \mathbb{N}_0$ is defined as

$$\alpha \circ X = \sum_{k=1}^{X} Z_k \quad \text{with} \quad Z_k \sim Be(\alpha) \in \{0, 1\} \quad \text{iid Bernoulli} \, .$$

For Poisson variables we have

$$X \sim Poi(\lambda),\ \alpha \in [0,1] \quad \Rightarrow \quad \alpha \circ X \sim Poi(\alpha\lambda) \, .$$

This follows directly from computing the generating function

$$G_{\alpha\circ X}(s) = \mathbb{E}\big(e^{\sum_{k=1}^{X} Z_k}\big) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, e^{-\lambda} \mathbb{E}(s^{Z_k})^n = G_X(G_Z(s)) = e^{\lambda\alpha(s-1)}$$

where we have used $G_Z(s) = 1 - \alpha + \alpha s = 1 + \alpha(s-1)$.

A **Poisson process** $N = (N_t : t \geq 0) \sim PP(\lambda)$ with **rate** $\lambda > 0$ is a Markov chain with independent stationary increments, and $N_t \sim Poi(\lambda t)$ for all $t \geq 0$. We know from lectures that the holding times of the chain are independent $Exp(\lambda)$ variables with mean $1/\lambda$. The above properties for Poisson random variables imply the following for processes:

- **Adding Poisson processes.**
  Let $N^i \sim PP(\lambda_i)$ be independent Poisson processes, and define their sum $M = (M_t : t \geq 0)$ via $M_t := N_t^1 + \ldots + N_t^n$ for all $t \geq 0$. Then $M \sim PP(\lambda_1 + \ldots + \lambda_n)$ is a Poisson process.

- **Thinning.**
  An α-thinning $\alpha \circ N$ of a Poisson process $N \sim PP(\lambda)$ is defined via $(\alpha \circ N)_t = \alpha \circ N_t$ for all $t \geq 0$, i.e. independently keep jumps with probability α. Then $\alpha \circ N \sim PP(\alpha\lambda)$ is again a Poisson process.

Random sequential update.

The properties of Poisson processes can be used to set up an efficient sampling algorithm for stochastic particle systems, often called random sequential update (an adaption of the 'Gillespie algorithm'). Here we focus on a system with state space $S = \{0,1\}^\Lambda$ with lattice Λ and flip dynamics, for example the contact process (see picture).

To resolve the full dynamics on site $x \in \Lambda$, the sampling rate should be $r_x = \max_{\eta \in S} c(\eta, \eta^x)$ determined by the fastest process. From the graphical construction the independent PPs on each site add up, and the next possible event in the whole system happens at rate $R = \sum_{x\in\Lambda} r_x$. By the thinning property, the probability that it happens on site x is given by $p_x = r_x/R$. This leads to the following algorithm to construct a sample path for the particle system:
Pick η_0 from the initial distribution and set $t = 0$. Then repeat iteratively:

(1) update the time counter by $t+ = Exp(R)$,
(2) pick a site x with probability p_x,
(3) update (flip) site x with probability $c(\eta, \eta^x)/r_x$.

For example, for the contact process on $\Lambda = \{1, \ldots, L\}$ with periodic boundary conditions and rates

$$c(\eta, \eta^x) = \eta(x) + \lambda\big(1 - \eta(x)\big)\big(\eta(x-1) + \eta(x+1)\big)$$

we have $r_x = r = \max\{1, 2\lambda\}$, and thus $p_x = 1/L$ choosing sites uniformly and $R = rL$.
For particle hopping like in exclusion processes an analogous construction works with the extra step of choosing a target site between (2) and (3).

Simplified time counter.
Since $R = O(L)$ is of order of the system size, the increments $\tau_i \sim Exp(R)$ of the time counter are of order $1/L$. By the **scaling property** $\alpha Exp(\beta) \sim Exp(\beta/\alpha)$ of exponential rv's (check!), we have

$$\tau_i \sim Exp(R) \sim \frac{1}{R}\tilde{\tau}_i \quad \text{with normalized} \quad \tilde{\tau}_i \sim Exp(1)\ .$$

To simulate up to a time $T = O(1)$ we therefore need of order $RT = O(L)$ sampling increments τ_i. The time counter of the simulation is then

$$t = \sum_{i=1}^{RT} \tau_i = \frac{1}{R}\sum_{i=1}^{RT} \tilde{\tau}_i = T + O(L^{-1/2}) \to T \quad \text{as } L \to \infty$$

by the law of large numbers. So if we just replace the increments τ_i by their mean $1/R$, i.e. use

(1)' update the time counter by $t+ = 1/R$

instead of the computationally more expensive (1), the error in t is of order $L^{-1/2}$ by the central limit theorem. This is often negligible for large L unless one is interested in very precise time statistics.

Further related properties of exponentials.
Let $\tau_1, \tau_2, \ldots$ be a sequence of independent $Exp(\lambda_i)$ rv's. Then

- $\min\{\tau_1, \ldots, \tau_n\} \sim Exp(\lambda_1 + \ldots + \lambda_n)$ (related to the sum of Poisson processes),
- If $\lambda_i = \lambda$ are identical, and $N \sim Geo(p)$ is an independent geometric rv with mean $1/p$, then

$$\sum_{i=1}^{N} \tau_i \sim Exp(p\lambda) \quad \text{(related to the marginal waiting time on a site } x)\ .$$

This can be proved by direct computation $\big(\mathbb{P}(\min \tau_i > t) = \prod_i \mathbb{P}(\tau_i > t)\big)$ and using generating/characteristic functions, respectively (try it!).