Stochastic models of complex systems

Problem sheet 1

Sheet counts $25/100$ homework marks, [x] indicates weight of the question.
* Questions do not enter the mark.

1.1 Generators and eigenvalues [10]

(a) Consider the **Fibonacci numbers** $(F_n : n \in \mathbb{N}_0)$ defined by the recursion

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2) \quad \text{with} \quad F_0 = 0,\ F_1 = 1\ .$$

Write $\binom{F_{n+1}}{F_n} = M\binom{F_n}{F_{n-1}}$ as a discrete-time dynamical system with $M \in \mathbb{R}^{2\times 2}$.
Compute the eigenvalues of M and show that

$$F_n = \frac{\eta^n - (1-\eta)^n}{\sqrt{5}} \quad \text{where} \quad \eta = \frac{1+\sqrt{5}}{2} \quad \text{is the Golden ratio .}$$

(b) Consider the continuous-time Markov chain X with generator $G = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix}$.

i. Draw a diagram for X (i.e. connect the three states by their jump rates), and give the transition matrix P^Y of the corresponding jump chain Y.

ii. Convince yourself that $P'(0) = G$, $P''(0) = G^2$ etc., and that for some matrix B,

$$P(t) = \exp(tG) = B^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\lambda_2 t} & 0 \\ 0 & 0 & e^{-\lambda_3 t} \end{pmatrix} B\ .$$

Compute λ_2 and λ_3. Use this to compute $p_{11}(t)$, i.e. determine the coefficients in

$$p_{11}(t) = a + b\, e^{-\lambda_2 t} + c\, e^{-\lambda_3 t}\ .$$

iii. What is the stationary distribution $\boldsymbol{\pi}^*$ of X?

(c)* Derive a recursion relation for the generating function $G(s) = \sum_n F_n s^n$ of the Fibonacci numbers and solve it. Sketch $G(s)$. For which $s \geq 0$ is it well defined?

1.2 Branching processes [10]

Let $Z = (Z_n : n \in \mathbb{N})$ be a branching process, defined recursively by

$$Z_0 = 1\ , \quad Z_{n+1} = X_1^n + \ldots + X_{Z_n}^n \quad \text{for all } n \geq 0\ ,$$

where the $X_i^n \in \mathbb{N}$ are iidrv's denoting the offspring of individuum i in generation n.

(a) Is Z a Markov process? What is its state space S? Is the process irreducible? What are the communicating classes of S and the stationary distributions?

(b) Consider a geometric offspring distribution $X_i^n \sim Geo(p)$, i.e.

$$p_k = \mathbb{P}(X_i^n = k) = p\,(1-p)^k\ , \quad p \in (0,1)\ .$$

Compute the prob. generating function $G(s) = \sum_k p_k s^k$ as well as $\mathbb{E}(X_i^n)$ and $Var(X_i^n)$. Sketch $G(s)$ for (at least) three (wisely chosen) values of p and compute the probability of extinction as a function of p.

(c) Consider a Poisson offspring distribution $X_i^n \sim Poi(\lambda)$, i.e.

$$p_k = \mathbb{P}(X_i^n = k) = \frac{\lambda^k}{k!}\,e^{-\lambda}\ , \quad \lambda > 0\ .$$

Repeat the same analysis as in (b). (The equation for the probability of extinction cannot be solved in this case, find an approximate solution.)

(d) For Poisson offspring show that $\sum_{i=1}^k X_i^n \sim Poi(k\lambda)$.
Use this to compute the elements p_{ij} of the transition matrix P of Z.

(e)* For geometric offspring with $p = 1/2$, show that $G_n(s) = \frac{n-(n-1)s}{n+1-ns}$ and compute $\mathbb{P}(Z_n = 0)$. If T is the (random) time of extinction, what is its distribution and its expected value?

1.3 Random walk [5]

(a) Consider a simple symmetric random walk on $\{1,\ldots,L\}$ with
- periodic boundary conditions, i.e. $p_{L,L-1} = p_{L,1} = p_{1,L} = p_{1,2} = 1/2$,
- closed boundary conditions, i.e. $p_{L,L-1} = p_{L,L} = p_{1,1} = p_{1,2} = 1/2$,
- reflecting boundary conditions, i.e. $p_{L,L-1} = p_{1,2} = 1$,
- absorbing boundary conditions, i.e. $p_{L,L} = p_{1,1} = 1$.
(All transition probabilities which are not specified above are 0.)
In each case, sketch the transition matrix $P = (p_{ij})_{ij}$ of the process, decide whether the process is irreducible, and give all stationary distributions $\boldsymbol{\pi}^*$.
(Hint: Use detailed balance.)

(b)* Consider an undirected connected graph (G, E) without loops and double edges. A simple random walk on (G, E) has transition probabilities $p_{i,j} = e_{i,j}/c_i$, where c_i is the number of outgoing edges in vertex i, and $e_{i,j} \in \{0, 1\}$ denotes the presence of an edge (i, j).
Find a formula for the stationary distribution $\boldsymbol{\pi}^*$.
Does your formula also hold on a directed, strongly connected graph?