CO906 worksheet 1

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Due: 25-01-10

1 Individual work

1.1 Taylor's Theorem

Questions

(a) Write down Taylor's Theorem with the Lagrange form of the error.

(b) Write down the Taylor expansion of the following functions up to and including terms of order t^3:

- $\sin(t)$
- $\sin\left(\frac{\pi}{2}(1+\lambda t)\right)$ where $\lambda \in \mathbb{R}$
- $\sin\left(\frac{\pi}{2}\,e^{\lambda t}\right)$ where $\lambda \in \mathbb{R}$

(c) For the exponential function $v(t) = e^{\lambda t}$, find explicitly the value of ξ in the remainder term of the first order Taylor expansion.

1.2 Lotka–Volterra Equations

The Lotka–Volterra equations are a simple predator–prey model in theoretical ecology:

$$\frac{dv_1}{dt} = v_1\left(\alpha - \beta\,v_2\right) \tag{1}$$
$$\frac{dv_2}{dt} = -v_2\left(\gamma - \delta\,v_1\right)$$

where $v_1(t)$ represents the population density of prey and $v_2(t)$ representing the population density of predators. α, β, γ δ are parameters. The initial populations are $(v_1(0), v_2(0)) = (V_1, V_2)$.

Questions

(a) Find the fixed points.

(b) Show explicitly that the quantity

$$Q = v_2^{\alpha}\,e^{-\beta\,v_2}\,v_1^{\gamma}\,e^{-\delta\,v_1}$$

is conserved.

1.3 Error Analysis

Consider the general first order ODE

$$\frac{dv}{dt} = g(v), \tag{2}$$

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Questions

(a) Use Taylor's Theorem to approximate the following quantities as power series in h up to and including terms or order h^3:

$$g(v(t_0) + h\, g(v(t_0))) \tag{3}$$

$$g(v(t_0 + h)) \tag{4}$$

where t_0 is a constant.

(b) Consider the following timestepping algorithm for Eq. (2):

$$
\begin{aligned}
g_1 &= g(v_i) \\
g_2 &= g\!\left(v_i + \frac{2}{3}\, h\, g_1\right) \\
v_{i+1} &= v_i + h\left[\frac{1}{4}\, g_1 + \frac{3}{4}\, g_2\right].
\end{aligned}
$$

Show explicitly that the stepwise error in this algorithm is $O(h^3)$. The expressions obtained for (3) and (4) should come in useful.

1.4   Nondimensionalisation

Here is the one-dimensional Burgers' equation for a field, $v(x,t)$, which is a function of space , x, and time, t:

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = \nu\,\frac{\partial^2 v}{\partial x^2}. \tag{5}$$

Questions

(a) If $v(x,t)$ has units of velocity, what are the units of ν?

(b) Show how to choose units such that the nondimensional equation is

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = \frac{1}{\rho}\,\frac{\partial^2 v}{\partial x^2} \tag{6}$$

with $\rho = \frac{V L}{\nu}$ where V is a characteristic velocity and L is a characteristic length. What are the dimensions of the parameter, ρ?

1.5   A simple model of traffic flow

Suppose $\rho(x,t)$ is the density of cars on a single lane road. The number of cars in the stretch of road $[x_1, x_2]$ is

$$n(t) = \int_{x_1}^{x_2} \rho(x,t)\,dx. \tag{7}$$

Since cars are neither created nor destroyed, the rate of change of $n(t)$ is equal to the difference between the number of cars flowing into $[x_1, x_2]$ at x_1 and the number following out at x_2. If all cars moved at the same speed, v, we would simply have $v\rho(x_1, t)$ cars per unit time flowing in at x_1 and $v\rho(x_2, t)$ flowing out at x_2. However cars do not move with uniform speed - we adjust our driving speed according to the density of traffic. Thus, v should be a function of ρ and we write

$$
\begin{aligned}
\frac{dn}{dt} &= v(\rho(x_1,t))\,\rho(x_1,t) - v(\rho(x_2,t))\,\rho(x_2,t) \tag{8} \\
&= -\int_{x_1}^{x_2} \frac{\partial}{\partial x}\,(v(\rho)\,\rho). \tag{9}
\end{aligned}
$$

From this we conclude

$$\int_{x_1}^{x_2} \left(\frac{d\rho}{dt} + \frac{\partial}{\partial x} \left(v(\rho) \, \rho \right) \right) \, dx = 0$$

$$\Rightarrow \quad \frac{d\rho}{dt} + \frac{\partial}{\partial x} \left(v(\rho) \, \rho \right) = 0 \tag{10}$$

The function

$$v(\rho) = c \left(1 - \frac{\rho}{\rho_0} \right) \tag{11}$$

is a plausible model for the dependence of the speed on the density.

Questions

(a) What are the meanings of c and ρ_0?

(b) Suppose at $t = 0$, the traffic density is

$$\rho(x, 0) = \frac{\rho_0}{2} \left(1 - \tanh \left(\frac{x}{L} \right) \right). \tag{12}$$

   What kind of traffic configuration does this initial condition represent? What is the meaning of L?

(c) By choosing appropriate units, show that the non-dimensional version of Eq. (10) is

$$\frac{d\rho}{dt} + (1 - 2\rho) \frac{\partial \rho}{\partial x} = 0. \tag{13}$$

   What is the corresponding initial condition?

(d) Write down the characteristic equations for Eq. (13).

2   Group work (week 1)

2.1   Getting started with the CSC computing environment

This task is just to make sure that everyone can run the example codes and plot the results.

- Download and unpack the tarball setup.tgz from the class website.

- Compile the code and link to the gsl library using the Makefile provided.

- Run the code on the CoW by appropriately modifying the PBS script provided.

- The code produces a file containing plots of the functions $\mathrm{erf}(t)$, $t\,\mathrm{erf}(t)$ and $t^2\,\mathrm{erf}(t)$.

The sample code, while performing a trivial task, illustrates a large number of concepts and techniques which will be useful generally:

- Linking functions (in this case, erf) from an external library (in this case, GSL).

- How to split a large code into several files.

- Using make to automate complicated compilation and linking tasks.

- How to submit jobs to the CoW using PBS.

- Reading command line arguments into a code

- Dynamic memory allocation using calloc();

- How to read parameters from an external file.

- How to get a program to time itself.

- Producing output at fixed time intervals even as the time increment varies.

Questions

(a) Plot the functions $\mathrm{erf}(t)$, $t\,\mathrm{erf}(t)$ and $t^2\,\mathrm{erf}(t)$ generated by the code using the graphics application of your choice and a printout of the graph included in your report.

(b) The file `f_x_t.dat` included with the tarball contains a 51x51 grid of values, $(t_i, x_j, f(x_j, t_i)$ obtained from the following function of two variables:

$$f(x,t) = \frac{\cos(x-t)^2}{1 + t\cos(x-t)^2}$$

over the range $-2.5 < x < 2.5$ and $0 < t < 5$. From this data, produce a plot of $f(x,t)$ vs t for $x = -2, -1, 0, 1, 2$ and a plot of $f(x,t)$ vs x for $t = 0, 1, 2, 3, 4$. Both plots should also contain the correponding exact expressions for comparison. See fig. 1.

2.2   Numerical error analysis

Download and unpack the tarball euler.tgz from the class website. The code uses the simple Euler method to solve the equation used as an example in the notes:

$$\frac{d^2v}{dt^2} + 2\,t\frac{dv}{dt} - \alpha v = 0, \tag{14}$$

for the particular case of $\alpha = 0$, $v(0) = 0$, $\frac{dv}{dt}(0) = \frac{2}{\sqrt{\pi}}$, Eq. (14) has a simple exact solution:

$$v(t) = \mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-s^2}\,ds, \tag{15}$$
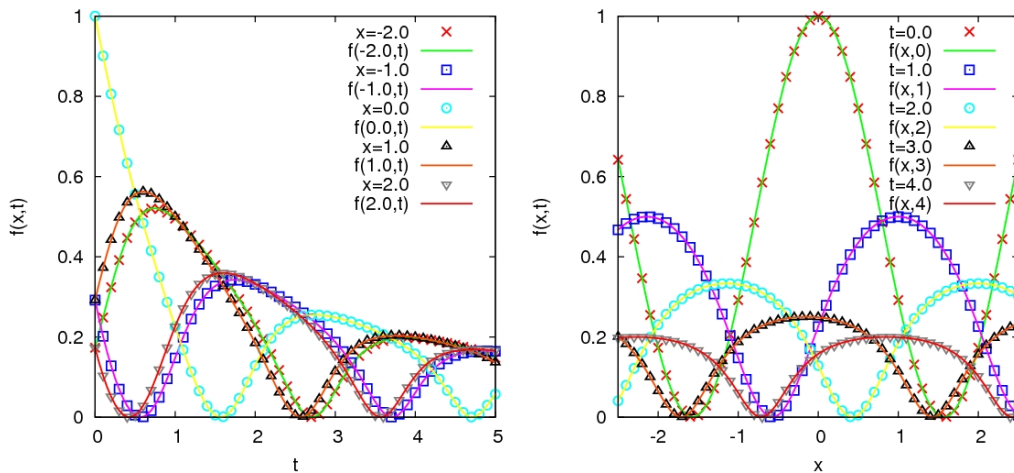
Figure 1: Plotting sections through grid data for functions of two variables is something we will need to do a lot.

Questions

(a) Write Eq. (14) as a 3-dimensional first order system:

$$\frac{d\mathbf{v}}{dt} = \mathbf{G}(\mathbf{v}). \tag{16}$$

(b) Write down the exact solution, $\mathbf{v}_e(t)$, of this first order system corresponding to the exact solution, Eq. (15), when $\alpha = 0$.

(c) From numerical explorations, or otherwise, describe how the solutions changes when $\alpha \neq 0$.

(d) Returning to the case $\alpha = 0$, the global error in the numerical solution, $\widetilde{\mathbf{v}}(t)$, over the interval $[0, T]$ is

$$E(T) = \int_0^T |\widetilde{\mathbf{v}}(\tau) - \mathbf{v}_e(\tau)| \ d\tau \tag{17}$$

which we can approximate by

$$E(T) = \sum_i |\widetilde{\mathbf{v}}(t_i) - \mathbf{v}_e(t_i)| \ h. \tag{18}$$

Show *empirically* that $E(T)$ is proportional to h as $h \to 0$,.

2.3   Code validation

Download the 4 precompiled copies of the Lotka–Volterra code from the class website and the parameter file, parameter.txt. The code is run by adding the the following command to an appropriate PBS script and submitting it to the CoW:

```
LotkaVolterra.v1  parameters.txt outputFile.dat
```

Each code *claims* to solve Eqs. (1) using the improved Euler algorithm. Each gives plausible results as can be seen in Fig. 2. However only one of the codes is correct. Your task is to figure out which one.
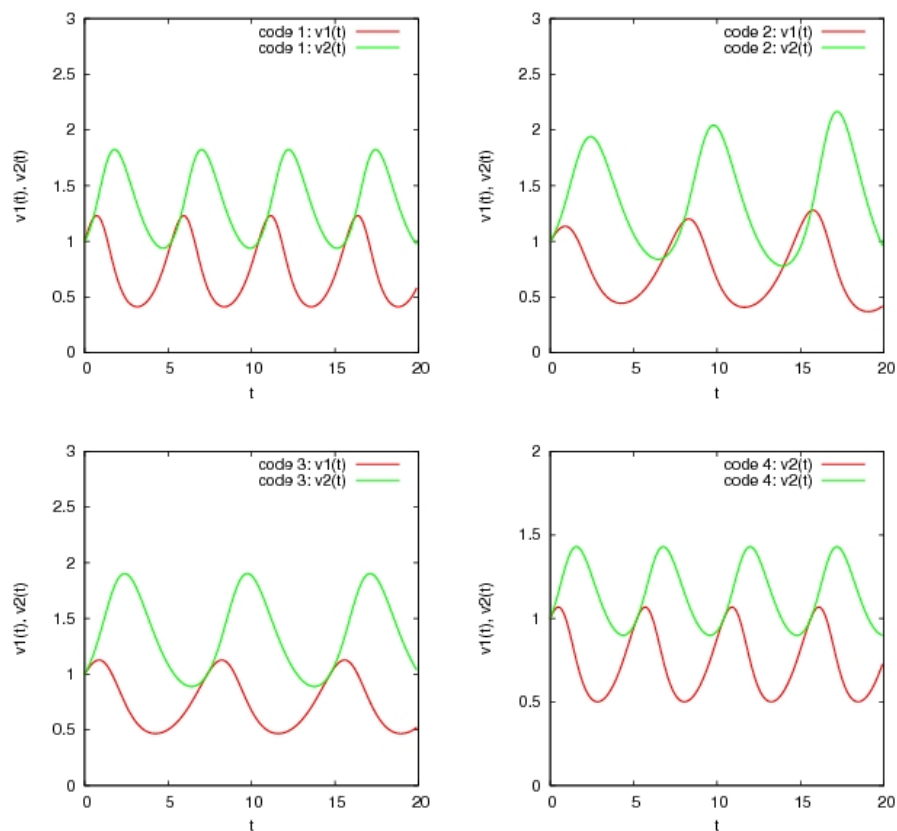
Figure 2: "Solutions" of Eqs. (1) obtained using 4 versions of the code.

Questions

(a) Devise and run a set of tests which determine which code has been written correctly. You should include in your report an explanation of the tests you performed on the codes, (supported by graphs), in order to reach your conclusions.

(b) For the version of the code which you believe is the correct one, estimate how the global numerical error varies with the step size.

(c) Modify the correct version code (or write your own) to use the RK4 algorithm instead of the improved Euler and show empirically that the estimated global error is now $O(h^4)$.

3 Group work (week 2)

3.1 Method of Characteristics for the Traffic Model

Refer back to the traffic model introduced in Q1.5.

Questions

(a) Choose plausible values for ρ_0, c and L (there is no right answer). Don't forget to include the appropriate units.

(b) Write a code to solve Eq. (13) with the non–dimensionalised initial data you obtained in Q. 1.5 by choosing 200 equally spaced points in the spatial interval $[-10, 10]$ and solving the characteristic equations for each of these points over a specified time interval, $[0, T]$. You may use any time-stepping algorithm of you wish except the forward Euler method. Include in your report plots of the global solution, the characteristic curves in the $x - t$ plane and some snapshots of the traffic density at successive times.

(c) Using your numerical solution of the non-dimensional problem, Eq. (13), what is the density of cars (in cars/m)

- at $x = 100$ m after 60 s?
- at $x = 0$ m after 100 s?
- at $x = -100$ m after 60 s?

(d) Investigate what happens if you start from the "reflected" initial condition

$$\rho(x, 0) = \frac{\rho_0}{2}\left(1 + \tanh\left(\frac{x}{L}\right)\right). \tag{19}$$

Give a heuristic interpretation of your results. What do the results say about the model?

3.2 Travelling Waves in Burgers Equation

Recall Burgers' Equation, Eq. (6), from Q 1.4. Its *inviscid* counterpart is the *first order* pde:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0. \tag{20}$$

Let us consider the following initial condition.

$$v(x, 0) = \frac{v_R + v_L\, e^{\alpha x}}{1 + e^{\alpha x}}. \tag{21}$$

with $v_L > v_R > 0$. In this question, we explore the extent to which it is correct to think of Eq. (20) as the $\nu \to 0$ limit of Eq. (6)

Questions

(a) Write down the nondimensional versions of Eq. (6), Eq. (20) and the initial condition Eq. (21). You should use the initial condition, Eq. (21), to choose specific length and velocity scales.

(b) For a given value of ν, v_L and v_R, use what we learned in class about travelling waves to choose the value of α for which it is easy to write down the solution of Eq. (6) with the initial condition Eq. (21).
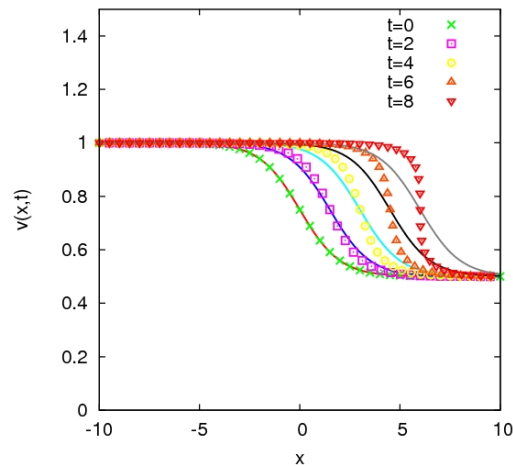
Figure 3: Comparison of the solutions of Eq. (6 (solid lines) and Eq. (20) (points) with the initial condition Eq. (21) at a series of successive times.

(c) Write down the characteristic equations for Eq. (20) and adapt your code from Q. 3.1 to solve them. Plot some graphs comparing the solution to Eq. (20) obtained via characteristics to the corresponding travelling wave solution of Eq. (6). You should observe something like Fig. 3.

(d) Show either analytically or numerically that the characteristic curves associated with Eq. (20) intersect after a finite time for the initial data given by Eq. (21) indicating that the solution ceases to exist after this time.

(e) What happens to the travelling wave solution of Eq. (6) as $\nu \to 0$? Does it exist for all time if ν is very small but finite?

(f) What is your assessment of the statement that one may think of Eq. (20) as the $\nu \to 0$ limit of Eq. (6)?