CO906 worksheet 2

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1 Individual work

1.1 Self-similar solutions of the diffusion equation on an unbounded domain

Consider the diffusion equation

$$\frac{\partial v}{\partial t} = D\,\frac{\partial^2 v}{\partial x^2} \tag{1}$$

Referring to the self-similar solutions worked out in Chap 2 of the notes, shift the time variable by an appropriate constant in order to write down the explicit solution of (1) for each of the following initial conditions:

Questions

(a)

$$v(x,0) = A\,\mathrm{e}^{-\frac{x^2}{\sigma^2}}$$

(b)

$$v(x,0) = \frac{v_L + v_R}{2} - \left(\frac{v_L - v_R}{2}\right)\mathrm{erf}\left(\frac{x}{\sigma}\right)$$

Note that if the solution which you write down does not reproduce the appropriate initial condition when you set $t = 0$ then it is wrong!

1.2 Diffusion on a finite interval with Dirichlet boundary conditions

Consider Eq. (1) on the interval $[x_L, x_R]$ with the initial condition

$$v(x,0) = \left(\frac{v_R - v_L}{\mathrm{e}^{-\frac{x_R - x_L}{\sigma}} - 1}\right)\left(\mathrm{e}^{-\frac{x - x_L}{\sigma}} - 1\right) + v_L, \tag{2}$$

and the Dirichlet boundary conditions

$$\begin{aligned} v(x_L,t) &= v_L \\ v(x_R,t) &= v_R. \end{aligned} \tag{3}$$

(a) Show that the initial condition (2) satisfies the boundary conditions (3).

(b) Find a stationary (ie time–independent) solution of (1) which satisfies the boundary condtions (3).

1.3 Diffusion on a finite interval with Neumann boundary conditions

Consider Eq. (1) on the interval $[x_L, x_R]$ with the Neumann boundary conditions

$$\begin{aligned} \frac{\partial v}{\partial x}(x_L, t) &= 0 \\ \frac{\partial v}{\partial x}(x_R, t) &= 0. \end{aligned} \tag{4}$$

Consider

$$M(t) = \int_{x_L}^{x_R} v(x,t)\,dx. \tag{5}$$

$M(t)$ corresponds to the total amount of the diffusing material in the interval $[x_L, x_R]$.

(a) Show that

$$\frac{dM}{dt} = -D\left(\frac{\partial v}{\partial x}(x_L, t) - \frac{\partial v}{\partial x}(x_R, t)\right). \tag{6}$$

(b) Show that $M(t)$ is conserved when Eq. (1) is solved with the Neumann boundary conditions, Eqs.(4).

(c) What is the stationary state in this case?

1.4 A nonlinear conservation law

Recall the traffic model which we studied in Chap.2:

$$\frac{\partial v}{\partial t} + (1-2v)\,\frac{\partial v}{\partial x} = 0 \tag{7}$$

For this equation write down

(a) The FTCS scheme.

(b) The Lax scheme.

(c) The Lax-Wendroff scheme.

2 Group work (week 3)

2.1 Solving the diffusion equations with the FTCS scheme

We shall study Eq.(1) with the initial condition:

$$v(x,0) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{8}$$

(a) Write the FTCS equations for this equation discretised on N spatial points, $x_0, x_1, \ldots x_{N-1}$ with $x_0 = -L$ and $x_{N-1} = L$, with periodic boundary conditions.

(b) Write a code which implements the FTCS scheme to solve the diffusion equation and plot on the same axes, some successive snapshots of the solution in order to demonstrate the time evolution. Create three plots for values of δ equal to 0.45, 0.50 and 0.55. For which values of δ is the numerical algorithm stable?

(c) Take $\mu = 0$ and $\sigma \ll L$. Choose the timestep so that $\delta = 0.45$. The solution at early times should be close to the self-similar solution you wrote down in question 1.1. with appropriate choices of the constants. Compare the numerical and analytical solutions qualitatively by plotting them together for several different times. Explain why the numerical solution deviates from the self-similar solution at later times.

(d) Verify empirically that the total mass is conserved by measuring

$$M(t_j) = \int_{-L}^{L} v(x,t_j)\,dx \approx \sum_{i=0}^{N-1} v_{i,j}\,\Delta x$$

and plotting it as a function of time.

(e) Modify your code to implement the Dirichlet and Neumann boundary conditions

$$v(-L,t) = v(L,t) = 0 \tag{9}$$

$$\frac{\partial v}{\partial x}(-L,t) = \frac{\partial v}{\partial x}(L,t) = 0. \tag{10}$$

Compare the early and late time behaviours for each of these boundary conditions to the periodic case. Can you give a physical interpretation of the different boundary conditions in terms of what happens to material diffusing to the boundaries?

2.2 Crank–Nicholson Method

In this question we will explore the improved stability properties of the Crank-Nicholson method and attempt to demonstrate empirically that the self-similar solutions of the diffusion equation worked out in the notes are "attracting" for appropriate classes of initial and boundary conditions.

(a) Download the sample code from the class website. It demonstrates how to use GSL to solve the 5×5 linear system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \\ b \end{pmatrix} \tag{11}$$

Modify the sample code (or write your own code) to solve the corresponding $N \times N$ problem. Show how this linear system arised from a discretisation of the boundary value problem

$$\frac{d^2 v}{dx^2} = 0 \qquad (12)$$

on the interval $[x_L, x_R]$ with the boundary conditions $v(x_L) = a$, $v(x_R) = b$?

(b) Modify your code from Q.2.1 to use the Crank–Nicholson method to solve Eq.(1) with Dirichlet boundary conditions and Gaussian initial data.

(c) Demonstrate empirically that the Crank-Nicholson method remains stable even when $\delta > \frac{1}{2}$.

(d) Consider the initial data

$$v(x,0) = \begin{cases} 0 & x < -\sigma \\ \frac{3}{4\sigma^3}(\sigma - x)(\sigma + x) & -\sigma \leq x \leq \sigma \\ 0 & x > \sigma. \end{cases} \qquad (13)$$

Demonstrate that for large intermediate times [1] the solution approaches the gaussian self similar solution. That is to say, as t increases, the solutions at different times should collapse onto a single curve if you rescale of v and x appropriately. You should explain how you arrive at the scaling you use.

[1] By intermediate times we mean times large compared to the characteristic time (which you should be able to construct from the parameters of the equation and initial data) but smaller than the time it takes for the solution starts to feel the boundaries. You might need to increase the size of the spatial domain to ensure that a range of intermediate times exist.

3 Group work (week 4)

3.1 Hyperbolic schemes for scalar conservation laws

Consider the 1 dimensional scalar conservation law

$$\frac{\partial v}{\partial t} = -\frac{\partial F(v)}{\partial x}. \tag{14}$$

(a) For the case of the linear advection equation where $F(v) = -c\,v$, write codes to solve Eq. (14) with periodic boundary conditions using the FTCS, Lax and Lax-Wendroff schemes. Plot some snapshots illustrating the time evolution of the numerical solution in each case.

(b) For a fixed wave speed, c, and grid spacing, Δx, characterise the stability of each method by varying h. Do the results agree with what you expect from lectures? What can you say empirically about the stability of the Lax-Wendroff method?

(c) Modify your code to solve your favourite nonlinear conservation law (like the traffic model or inviscid Burgers' equation) and describe qualitatively how the numerical solutions you obtain compare with those obtained using the method of charteristics.