CO906 viva questions 2010

February 11, 2010

1 Chapter 1: ODEs and time stepping algorithms

1. Write down an example of a second order non-autonomous ODE and show how to reduce it to a system of first order ODEs.
2. Write down the integral form of the solution of the ode
$$\frac{dv}{dt} = G(v)$$
and use it to derive
   - The forward Euler method.
   - The backward Euler method.
   - The implicit trapezoidal method.
3. A 2nd order Runge-Kutta scheme for the ode
$$\frac{dv}{dt} = G(v)$$
is
$$\begin{aligned} g_1 &= G(v_i) \\ g_2 &= G(v_i + h\, g_1) \\ v_{i+1} &= v_i + \frac{1}{2}\, h\, [g_1 + g_2] \end{aligned}$$
Show that the stepwise error is $O(h^3)$.
4. Use Taylor's Theorem to derive the following finite difference formulae
$$\begin{aligned} \frac{dv}{dt}(t_i) &\approx \frac{v_{i+1} - v_{i-1}}{2\, h} \\ \frac{d^2v}{dt^2}(t_i) &\approx \frac{v_{i+1} - 2\, v_i + v_{i-1}}{h^2} \end{aligned}$$

5. Describe a problem for which it would be efficient to use an adaptive timestepping algorithm and one for which it would not. Discuss how you would implement adaptive timestepping and the advantages and disadvantages of doing so.

6. The implicit trapezoidal method for the ODE
$$\frac{dv}{dt} = G(v)$$
is
$$v_{i+1} = v_i + \frac{1}{2}\,h\,[G(v_i) + G(v_{i+1})].$$
Discuss the distinction between implicit and explicit algorithms. Show how the above algorithm can be made explicit using the predictor–corrector concept.

2 Chapter 2: Intro to PDEs, Method of Characteristics, Travelling Waves and Similarity Solutions

1. Discuss the differences between elliptic, parabolic and hyperbolic PDEs. Discuss also the meaning and differences between periodic, Dirichlet and Neumann boundary conditions for a PDE.

2. Write down a 1st order PDE of your choice. Write down the corresponding characteristic equations and describe in pictures how you can obtain the numerical solution of the PDE from these equations.

3. Consider the linear advection equation,
$$\frac{\partial v}{\partial t} + c\,\frac{\partial v}{\partial x} = 0$$
with the initial condition, $v(x,0) = \mathrm{e}^{-x^2}$. Using the Method of Characteristics, obtain the solution.

4. What is a travelling wave? Find a travelling wave solution of the linear wave equation:
$$\frac{\partial^2 v}{\partial t^2} = c^2\,\frac{\partial^2 v}{\partial x^2}.$$
Note: I do not expect you to know how to fit the solution to the initial data.

5. Consider the diffusion equation

$$\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2}.$$

What is a similarity solution? Using the ansatz $v(x,t) = t^a F(\xi)$ where $\xi = x t^b$ deduce that $b = -1/2$ if this ansatz is to solve the diffusion equation. Hence derive the similarity equation for the diffusion equation:

$$D \frac{d^2 F}{d\xi^2} + \frac{1}{2} \xi \frac{dF}{d\xi} - a\, F = 0$$

You do not need to know how to solve it.

6. We considered in lectures and in classes the following model of traffic flow:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left[c\, (1 - \frac{\rho}{\rho_0})\, \rho \right] = 0$$

Explain the meaning of this model and how you studied it numerically. Describe some of the phenomena which you found for different initial conditions. Does the model reasonably describe real traffic flow?

3 Chapter 3: Parabolic PDEs, FTCS method, Crank-Nicholson method, Stability

1. Derive the FTCS scheme for the diffusion equation:

$$v_{i,j+1} = v_{i,j} + \delta \left[v_{i+1,j} - 2 v_{i,j} + v_{i-1,j} \right],$$

where $\delta = \frac{Dh}{(\Delta x)^2}$. Write separately the equations at $i = 0$ and $i = N-1$ for the case of periodic boundary conditions. Can you convert the entire system of equations into matrix form?

2. Derive the Crank-Nicholson scheme for the diffusion equation:

$$-\frac{\delta}{2} v_{i+1,j+1} + (1+\delta) v_{i,j+1} - \frac{\delta}{2} v_{i-1,j+1} = \frac{\delta}{2} v_{i+1,j} + (1-\delta) v_{i,j} + \frac{\delta}{2} v_{i-1,j},$$

where $\delta = \frac{Dh}{(\Delta x)^2}$. Write separately the equations at $i = 0$ and $i = N-1$ for the case of periodic boundary conditions. Can you convert the entire system of equations into matrix form?

3. Consider the FTCS scheme for the diffusion equation on the interval $[x_L, x_R]$:

$$v_{i,j+1} = v_{i,j} + \delta \left[v_{i+1,j} - 2v_{i,j} + v_{i-1,j} \right],$$

where $\delta = \frac{Dh}{(\Delta x)^2}$. Write down the Dirichlet and Neumann boundary conditions. Describe how to implement them within the FTCS scheme.

4. What does it mean for a finite difference scheme for a PDE to be *stable*? Demonstrate using a Neumann stability analysis that the FTCS scheme is conditionally stable. How does this affect the choice of timestep?

5. Show using Neumann stability analysis that the Crank-Nicholson scheme is unconditionally stable. How does this affect the choice of timestep?

6. Discuss the relative advantages and disadvantages of the FTCS scheme compared to the Crank-Nicholson scheme when applied to the diffusion equation taking into account ease of implementation, computational cost and stability.

4 Chapter 4: Hyperbolic PDEs, Conservation Laws, Neumann Stability Analysis, Lax Scheme, Lax-Wendroff Scheme

1. Show that the linear wave equation,

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}.$$

can be written as a system of conservation laws

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\partial}{\partial x} \left[A\, \mathbf{v} \right]$$

where $\mathbf{v}$ is a two-vector and A is a matrix.

2. Derive the FTCS scheme for the linear advection equation:

$$v_{i,j+1} = v_{i,j} - \frac{1}{2} \gamma \left(v_{i+1,j} - v_{i-1,j} \right),$$

where $\gamma = \frac{ch}{\Delta x}$. Show that it is unconditionally unstable.

3. Consider the linear advection equation. Starting from the FTCS scheme, write down the Lax scheme:

$$v_{i,j+1} = \frac{1}{2}(v_{i+1,j} + v_{i-1,j}) - \frac{1}{2}\,\gamma\,(v_{i+1,j} - v_{i-1,j})$$

where $\gamma = \frac{ch}{\Delta x}$. Show that it is conditionally stable if the CFL criterion, $\gamma < 1$ is satisfied.

4. Discuss, in general terms why hyperbolic equations are trickier to solve using finite difference methods than parabolic ones. Consider the scalar conservation law:

$$\frac{\partial v}{\partial t} = -\frac{\partial F(v)}{\partial x}.$$

The Lax-Wendroff scheme is

$$\begin{aligned} v_{i,j+1} \quad = \quad & v_{i,j} - \frac{h}{2\Delta x}\left[F_{i+1,j} - F_{i-1,j}\right] + \frac{1}{2}\frac{h^2}{\Delta x}\left[F'_{i+\frac{1}{2},j}\left(\frac{F_{i+1,j} - F_{i,j}}{\Delta x}\right)\right. \\ & \left. -F'_{i-\frac{1}{2},j}\left(\frac{F_{i,j} - F_{i-1,j}}{\Delta x}\right)\right]. \end{aligned}$$

Discuss the origin of this method (I do not expect a full derivation) and why it is an improvement on the FTCS and Lax schemes.

5. Write down the Lax scheme for a scalar conservation law. Adapt it to a system of two conservation laws such as that arising from the linear wave equation:

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\partial}{\partial x}\left[\mathbf{F}(\mathbf{v})\right]$$

with

$$\begin{aligned} \mathbf{v} &= \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} \\ \mathbf{F} &= \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}\begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} \end{aligned}$$

6. Explain the concept of numerical diffusion with reference to the examples of the FTCS and Lax schemes applied to the advection equation. Is it good or bad for numerical solution of hyperbolic problems?