CO906 worksheet 2

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1 Group work

1.1 Solving the diffusion equation using the FTCS scheme

The objective of this exercise is to implement the FTCS scheme to solve the one dimensional diffusion equation:

$$\frac{\partial v}{\partial t} = D\,\frac{\partial^2 v}{\partial x^2} \tag{1}$$

with different boundary conditions and to study the stability properties of the algorithm.

Questions

(a) Show that

$$v(x,t) = \frac{1}{\sqrt{4\pi D(t-t_0)}}\,\mathrm{e}^{-\frac{(x-\mu)^2}{4D(t-t_0)}} \tag{2}$$

is a solution of Eq. (1). How would you interpret this solution? What role is played by each of the parameters μ, t_0 and D?

(b) Write the FTCS equations for this equation discretised on N spatial points, $x_0, x_1, \ldots x_{N-1}$ with $x_0 = -L$ and $x_{N-1} = L$, with periodic boundary conditions.

(c) Write a code which implements the FTCS scheme to solve the diffusion equation with the initial condition given by Eq. (2). Choose your parameters such that the boundaries of the domain are initially "far away" from the important part of the solution. Quantify what you mean by "far away". Choose the timestep, h, such that

$$\delta = \frac{D\,h}{(\Delta x)^2} = 0.45. \tag{3}$$

Plot, on the same axes, five snapshots of the solution taken at times chosen appropriately to give a good illustration of the time evolution.

(d) The solution at early times should be close to the exact solution, Eq. (2). Compare the numerical and analytical solutions qualitatively by plotting them together for several different times. Explain why the numerical solution deviates from Eq. (2) at later times.

(e) Show empirically that the FTCS algorithm is only conditionally stable and produce numerical evidence to support the stability criterion derived in the lectures.

(f) Verify empirically that the total mass is conserved by measuring

$$M(t_j) = \int_{-L}^{L} v(x,t_j)\,dx \approx \sum_{i=0}^{N-1} v_{i,j}\,\Delta x$$

and plotting it as a function of time.

(g) Modify your code to implement the Dirichlet and Neumann boundary conditions

$$v(-L,t) = v(L,t) = 0 \tag{4}$$
$$\frac{\partial v}{\partial x}(-L,t) = \frac{\partial v}{\partial x}(L,t) = 0. \tag{5}$$

Compare the early and late time behaviours for each of these boundary conditions to the periodic case with the same initial condition. Can you give a physical interpretation of the different boundary conditions in terms of what happens to material diffusing to the boundaries?

1.2 The Black-Scholes formula

In this exerise we will apply what you have learned about the diffusion equation to solve the Black-Scholes equation and obtain the Black-Scholes formula numerically using the Crank-Nicholson method.

Consider a trader operating in an envoronment where the bank interest rate is r and price fluctuations are described by a geometric brownian motion with volatility σ_0. Suppose at some time $T-t$, s/he is offered the option to buy an asset currently valued at S at some fixed time T in the future for a fixed price, K. Clearly this option is worth something since the trader stands to gain if the price at time T exceeds K but stands to lose nothing since s/he is not obliged to buy if the price at time T is lower than K. But how much is this option worth? The answer is provided by the Black-Scholes equation:

$$\frac{\partial C}{\partial t} = -\frac{\sigma_0^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} + rC \tag{6}$$

where $C(S,t)$ is the value of the option at time $t < T$. This equation must be supplemented with the "final" condition:

$$C(S,T) = \max\{S-K, 0\} \tag{7}$$

and the boundary conditions:

$$C(0,t) = 0 \tag{8}$$
$$C(S,t) \to S \quad \text{as } S \to \infty. \tag{9}$$

Questions

(a) Make an attempt to briefly motivate the boundary and final conditions mentioned above.

(b) Show that the following change of variables:

$$x = \log\frac{S}{K} \tag{10}$$
$$\tau = T - t \tag{11}$$

converts Eq. (6) into a diffusion equation with advection and source terms:

$$\frac{\partial C}{\partial \tau} = \frac{\sigma_0^2}{2}\frac{\partial^2 C}{\partial x^2} + \left(r - \frac{\sigma_0^2}{2}\right)\frac{\partial C}{\partial x} - rC \tag{12}$$

in which Eq. (7) is converted into an *initial* condition.

(c) Introduce a new function $v(x,\tau)$ defined by

$$C(x,\tau) = \mathrm{e}^{ax+b\tau}\, v(x,\tau). \tag{13}$$

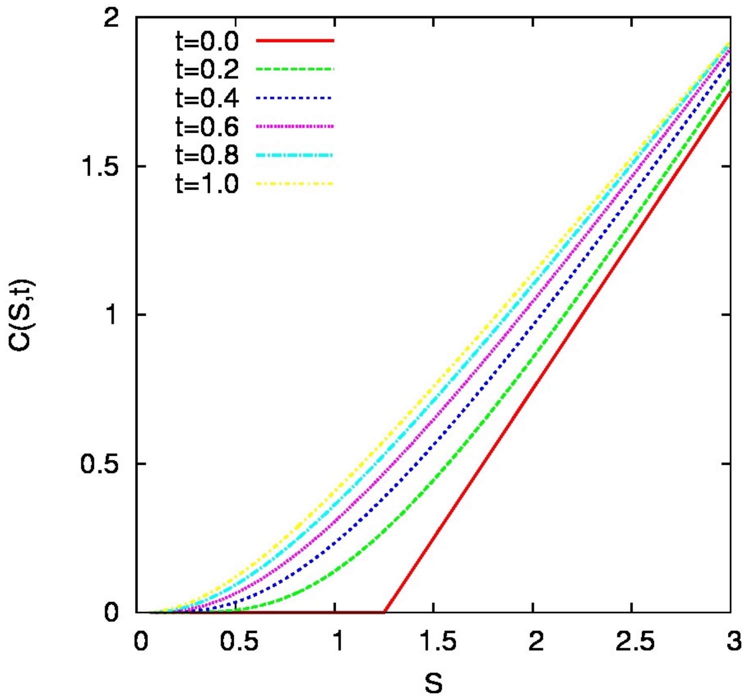

Figure 1: Numerical solution of the Black-Scholes equation.

Show that it is possible to find values of a and b such that $v(x,\tau)$ satisfies the standard diffusion equation

$$\frac{\partial v}{\partial t} = \frac{1}{2}\sigma_0^2 \frac{\partial^2 v}{\partial x^2}, \tag{14}$$

with the initial condition

$$v(x,0) = V_0(x) = K \max\left\{ \mathrm{e}^{(1-a)x} - \mathrm{e}^{-ax}, 0 \right\}. \tag{15}$$

(d) Modify your code from Q.1.1 to use the Crank–Nicholson method to solve Eq.(1) on the spatial interval $[L_1, L_2]$with Dirichlet boundary conditions and Gaussian initial data, Eq. (2).

(e) Demonstrate empirically that the Crank-Nicholson method remains stable even when $\delta > \frac{1}{2}$.

(f) Choose $L_1 = -3$, $L_2 = 2$, $r = 0.1$ $\sigma_0 = 1.2$, $K = 1.25$ and $T = 1.0$. Use your code to solve the option pricing problem, Eq. (14), with the initial data given by Eq. (15) and the Dirichlet boundary conditions:

$$v(L_1, t) = V_0(L_1) \tag{16}$$
$$v(L_2, t) = V_0(L_2). \tag{17}$$

(this is only an approximation of the true boundary conditions so you should expect to get erroneous behaviour when the solution starts to feel the boundaries). When the solution is converted back into the original variables you should get something like Fig. 1.

(g) Does the solution make sense? Explore what happens as the volality is varied.

1.3 Relaxation methods for elliptic equations

Consider Poisson's equation:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \rho(x, y) \tag{18}$$

on the square $(x, y) \in [-1, 1] \times [-1, 1]$. Take the function ρ to be

$$\rho(x, y) = -9 \sin(3x) - 4(x^2 + y^2) \sin(2xy). \tag{19}$$

The solution is

$$\sin(3x) + \sin(2xy) \tag{20}$$

Questions

(a) Verify the solution, Eq. (20).

(b) Use the solution to construct the appropriate Dirichlet boundary conditions and implement the Jacobi method for Eq. (18) on a 100×100 grid using a relaxation step of $0.2\,(\Delta x)^2$.

(c) Plot the residual (error) as a function of the number of iterations. Characterise its rate of decrease. Does it converge to zero? Explain.