CO906 viva questions 2011

March 1, 2011

1 ODEs and time stepping algorithms

1. Write down the integral form of the solution of the ode

$$\frac{dv}{dt} = G(v)$$

and use it to derive

- The forward Euler method.
- The backward Euler method.
- The implicit trapezoidal method.

2. A 2nd order Runge-Kutta scheme for the ode

$$\frac{dv}{dt} = G(v)$$

is

$$\begin{aligned} g_1 &= G(v_i) \\ g_2 &= G(v_i + h\, g_1) \\ v_{i+1} &= v_i + \frac{1}{2}\, h\, [g_1 + g_2] \end{aligned}$$

Show that the stepwise error is $O(h^3)$.

3. Use Taylor's Theorem to derive the following finite difference formulae

$$\begin{aligned} \frac{dv}{dt}(t_i) &\approx \frac{v_{i+1} - v_{i-1}}{2\,h} \\ \frac{d^2v}{dt^2}(t_i) &\approx \frac{v_{i+1} - 2\,v_i + v_{i-1}}{h^2} \end{aligned}$$

4. Describe a problem for which it would be efficient to use an adaptive timestepping algorithm and one for which it would not. Discuss how you would implement adaptive timestepping and the advantages and disadvantages of doing so.

5. The implicit trapezoidal method for the ODE

$$\frac{dv}{dt} = G(v)$$

is

$$v_{i+1} = v_i + \frac{1}{2}\, h \,[G(v_i) + G(v_{i+1})].$$

Discuss the distinction between implicit and explicit algorithms. Show how the above algorithm can be made explicit using the predictor–corrector concept.

6. Consider the linear boundary value problem

$$\begin{aligned} \frac{d^2v}{dx^2} + v - x^2 &= 0 \\ v[0] &= a \\ v[1] &= b. \end{aligned}$$

Show that the finite difference approximation to this equation is equivalent to a set of linear equations and describe how you would solve them numerically. Consider the related nonlinear boundary value problem

$$\begin{aligned} \frac{d^2v}{dx^2} + v^2 - x^2 &= 0 \\ v[0] &= a \\ v[1] &= b. \end{aligned}$$

Describe how you would solve this problem using the shooting method.

2 Parabolic PDEs, FTCS method, Crank-Nicholson method, Stability

1. Discuss the differences between elliptic, parabolic and hyperbolic PDEs. Discuss also the meaning and differences between periodic, Dirichlet and Neumann boundary conditions for a PDE.

2. Derive the FTCS scheme for the diffusion equation:

$$v_{i,j+1} = v_{i,j} + \delta\left[v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\right],$$

where $\delta = \frac{Dh}{(\Delta x)^2}$. Write separately the equations at $i = 0$ and $i = N-1$ for the case of periodic boundary conditions. Can you convert the entire system of equations into matrix form?

3. Derive the Crank-Nicholson scheme for the diffusion equation:

$$-\frac{\delta}{2}v_{i+1,j+1} + (1+\delta)v_{i,j+1} - \frac{\delta}{2}v_{i-1,j+1} = \frac{\delta}{2}v_{i+1,j} + (1-\delta)v_{i,j} + \frac{\delta}{2}v_{i-1,j},$$

where $\delta = \frac{Dh}{(\Delta x)^2}$. Write separately the equations at $i = 0$ and $i = N-1$ for the case of periodic boundary conditions. Can you convert the entire system of equations into matrix form?

4. Consider the FTCS scheme for the diffusion equation on the interval $[x_L, x_R]$:

$$v_{i,j+1} = v_{i,j} + \delta\left[v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\right],$$

where $\delta = \frac{Dh}{(\Delta x)^2}$. Write down the Dirichlet and Neumann boundary conditions. Describe how to implement them within the FTCS scheme.

5. What does it mean for a finite difference scheme for a PDE to be *stable*? Demonstrate using a Neumann stability analysis that the FTCS scheme is conditionally stable. How does this affect the choice of timestep?

6. Show using Neumann stability analysis that the Crank-Nicholson scheme is unconditionally stable. How does this affect the choice of timestep?

3 Relaxation methods for elliptic PDE's

1. Consider the Poisson equation on the square $(x, y) \in [-L, L] \times [-L, L]$:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \rho(x, y).$$

Give an example of some boundary conditions appropriate to this problem. Derive a finite difference approximation of this equation and show that it is equivalent to solving a set of linear equations. What difficulties are associated with solving this set of equations when the grid spacing is small?

2. Derive the Jacobi relaxation method for solving the Laplace equation:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

with Dirichlet boundary conditions. How would you initialise the iteration? Describe what you would expect to see as the iteration proceeds.

3. Consider the Jacobi method for the solution of the Poisson equation:

$$v_{ij}^{(k+1)} = v_{ij}^{(k)} + \Delta t \left[\frac{v_{i+1j}^{(k)} + v_{i-1j}^{(k)} - 4v_{ij}^{(k)} + v_{ij+1}^{(k)} + v_{ij-1}^{(k)}}{(\Delta x)^2} - \rho_{ij} \right]$$

How would you select the value for the relaxation parameter, Δt? Describe how you would expect the residual and the global error to behave as a function of the number of iterations and explain why.

4. Use what you know about finite difference algorithms for the diffusion equation to explain why one cannot use arbitrarily large "steps" of the fictitious time when you are doing Jacobi iteration for an elliptic equation. Describe some methods for improving the rate of convergence of the Jacobi method.

4 Hyperbolic PDEs, Conservation Laws, Neumann Stability Analysis, Lax Scheme, Lax-Wendroff Scheme

1. Show that the linear wave equation,

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}.$$

can be written as a system of conservation laws

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\partial}{\partial x} \left[A\, \mathbf{v} \right]$$

where $\mathbf{v}$ is a two-vector and A is a matrix.

2. Derive the FTCS scheme for the linear advection equation:

$$v_{i,j+1} = v_{i,j} - \frac{1}{2}\gamma \left(v_{i+1,j} - v_{i-1,j} \right),$$

where $\gamma = \frac{ch}{\Delta x}$. Show that it is unconditionally unstable.

3. Consider the linear advection equation. Starting from the FTCS scheme, write down the Lax scheme:

$$v_{i,j+1} = \frac{1}{2}(v_{i+1,j} + v_{i-1,j}) - \frac{1}{2}\,\gamma\,(v_{i+1,j} - v_{i-1,j})$$

where $\gamma = \frac{ch}{\Delta x}$. Show that it is conditionally stable if the CFL criterion, $\gamma < 1$ is satisfied. What is the interpretation of the CFL criterion in terms of wave propagation?

4. Discuss, in general terms why hyperbolic equations are trickier to solve using finite difference methods than parabolic ones. Consider the scalar conservation law:

$$\frac{\partial v}{\partial t} = -\frac{\partial F(v)}{\partial x}.$$

The Lax-Wendroff scheme is

$$\begin{aligned} v_{i,j+1} &= v_{i,j} - \frac{h}{2\Delta x}\left[F_{i+1,j} - F_{i-1,j}\right] + \frac{1}{2}\frac{h^2}{\Delta x}\left[F'_{i+\frac{1}{2},j}\left(\frac{F_{i+1,j} - F_{i,j}}{\Delta x}\right)\right. \\ &\qquad \left. -F'_{i-\frac{1}{2},j}\left(\frac{F_{i,j} - F_{i-1,j}}{\Delta x}\right)\right]. \end{aligned}$$

Discuss the origin of this method (I do not expect a full derivation) and why it is an improvement on the FTCS and Lax schemes.

5. Write down the Lax scheme for a scalar conservation law. Adapt it to a system of two conservation laws such as that arising from the linear wave equation:

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\partial}{\partial x}\left[\mathbf{F}(\mathbf{v})\right]$$

with

$$\begin{aligned} \mathbf{v} &= \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} \\ \mathbf{F} &= \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}\begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} \end{aligned}$$

6. Explain the concept of numerical diffusion with reference to the examples of the FTCS and Lax schemes applied to the advection equation. Is it good or bad for numerical solution of hyperbolic problems?