CO906 worksheet 3

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1 Individual work

1.1 Stability analysis of the advection equation

Consider the 1-d advection equation

$$\frac{\partial v}{\partial t} + c\,\frac{\partial v}{\partial x} = 0 \tag{1}$$

1. Using the same reasoning which we used previously for the diffusion equation, derive the Crank-Nicholson scheme for integrating Eq. (1):

$$\beta v_{i+1,j+1} + v_{i,j+1} - \beta v_{i-1,j+1} = -\beta v_{i+1,j} + v_{i,j} + \beta v_{i-1,j} \qquad i = 1 \ldots N-1 \tag{2}$$

   where $\beta = \frac{c\,h}{4\,\Delta x}$.

2. Write down the supplementary equations for $i = 0$ and $i = N$ for the case of periodic boundary conditions. Is this finite difference scheme implicit or explicit? Why?

3. Write a short pseudo-code indicating how you would implement this algorithm to solve Eq. (1).

4. Perform a Neumann stability analysis to determine its stability properties.

2 Group work

2.1 Numerical solution of the linear advection equation

Consider the scalar conservation law:

$$\frac{\partial v}{\partial t} = -\frac{\partial F(v)}{\partial x}. \tag{3}$$

with the initial condition of your choice and periodic boundary conditions.

1. For Eq. (3) write down
   - The FTCS scheme.
   - The Lax scheme.
   - The Lax-Wendroff scheme.

2. Consider the linear advection equation for which $F(v) = -c\,v$. Write codes which solve Eq. (3) with this flux function using the FTCS, Lax and Lax-Wendroff methods. For each, produce a plot illustrating the evolution of the solution in time.

3. For a fixed wave speed, c, and grid spacing, Δx, characterise the stability of each by varying h and quantitatively comparing your numerical solutions to the exact solution. Do the results agree with what you expect from lectures? What can you say empirically about the stability of the Lax-Wendroff method?

2.2 A continuous model of traffic flow

Suppose $\rho(x,t)$ is the density of cars on a single lane road. The number of cars in the stretch of road $[x_1, x_2]$ is

$$n(t) = \int_{x_1}^{x_2} \rho(x,t)\,dx. \tag{4}$$

Since cars are neither created nor destroyed, the rate of change of $n(t)$ is equal to the difference between the number of cars flowing into $[x_1, x_2]$ at x_1 and the number following out at x_2. If all cars moved at the same speed, v, we would simply have $v\rho(x_1,t)$ cars per unit time flowing in at x_1 and $v\rho(x_2,t)$ flowing out at x_2. However cars do not move with uniform speed - we adjust our driving speed according to the density of traffic. Thus, v should be a function of ρ and we write

$$\begin{aligned} \frac{dn}{dt} &= v(\rho(x_1,t))\,\rho(x_1,t) - v(\rho(x_2,t))\,\rho(x_2,t) && (5)\\ &= -\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(v(\rho)\,\rho\right). && (6)\end{aligned}$$

From this we conclude

$$\begin{aligned} &\int_{x_1}^{x_2} \left(\frac{d\rho}{dt} + \frac{\partial}{\partial x}\left(v(\rho)\,\rho\right)\right) dx = 0 \\ \Rightarrow\ &\frac{d\rho}{dt} + \frac{\partial}{\partial x}\left(v(\rho)\,\rho\right) = 0 && (7)\end{aligned}$$

One plausible model for the dependence of the traffic speed on the density is

$$v(\rho) = c\left(1 - \frac{\rho}{\rho_0}\right). \tag{8}$$

1. What are the meanings of c and ρ_0? Explain why Eq. (8) is a plausible model. Can you think of any others?
2. Suppose at $t = 0$, the traffic density is

   $$\rho(x,0) = \frac{\rho_0}{2}\left(1 - \tanh\left(\frac{x}{L}\right)\right). \tag{9}$$

   With this initial condition, how does the traffic look like at the initial time? Suppose we want to use this initial condition to model a long line of traffic which moves away from a junction after a traffic light turns green at $t = 0$. Suggest a meaning of L in this case?
3. By choosing appropriate rescalings, nondimensionalise Eq. (7) and show that the non-dimensional version can be written as

   $$\frac{d\rho}{dt} + (1 - 2\,\rho)\frac{\partial \rho}{\partial x} = 0. \tag{10}$$

   What is the corresponding initial condition?
4. Modify your code from Q 2.1 to solve this problem. You will have to think a little about what boundary conditions to implement since periodic boundary conditions are not appropriate for this problem. Produce a plot illustrating how the traffic density evolves in time from the given initial configuration. Explain your results.
5. Pick reasonable values, including units, for the parameters L, c, ρ_0 appearing in the original dimensional problem. Based on your numerical solution, how long will it take the traffic at x=-100 m to be travelling at 1/4 the maximum speed? Is this reasonable?