Scaling, structure functions and all that…

S. C. Chapman

Notes for C0907

- SCALING: Some generic concepts: universality, turbulence, fractals and multifractals, stochastic models
- RESCALING PDFS AND STRUCTURE FUNCTIONS
- FINITE LENGTH TIMESERIES, UNCERTAINTIES, EXTREMES-'real data' examples
- 'BURST' MEASURES- waiting times, avalanche distributions

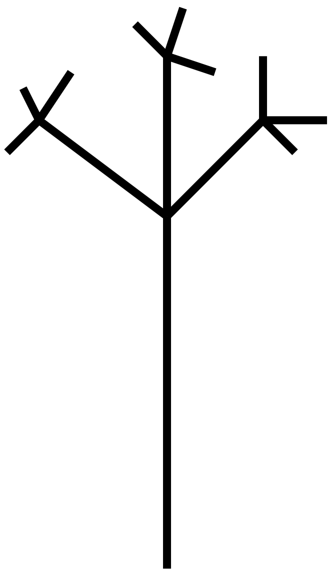
Scaling

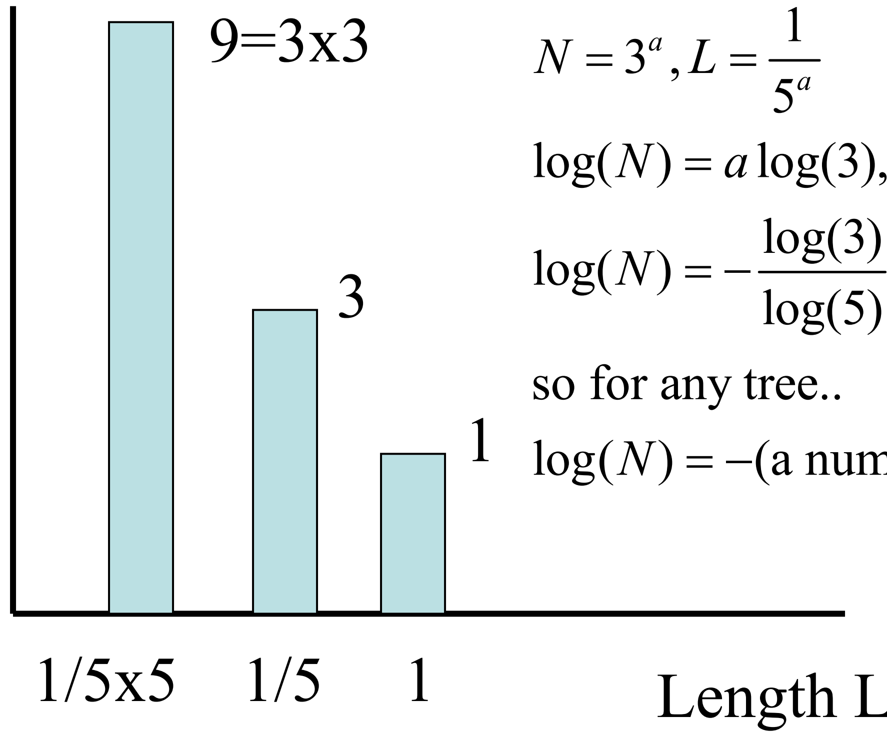
Some ideas and examples

Scaling and universality-Branches on a self-similar tree

Each branch grows 3 new branches, 1/5 as long as itself..

Number N of branches of length L





$$N = 3^a, L = \frac{1}{5^a}$$

$$\log(N) = a\log(3), \log(L) = -a\log(5)$$

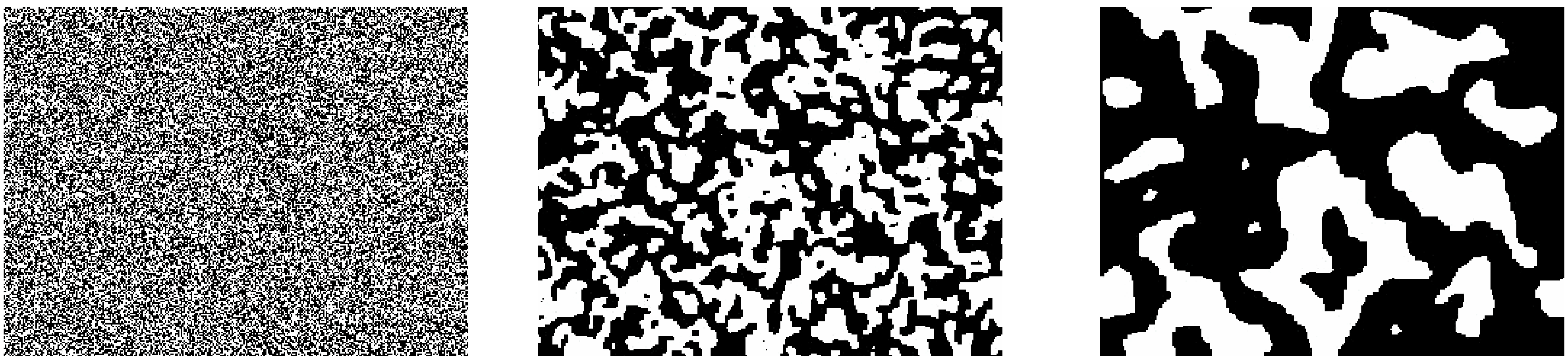
$$\log(N) = -\frac{\log(3)}{\log(5)}\log(L)$$

so for any tree..

$$\log(N) = -(\text{a number})\log(L)$$

Segregation/coarsening- a selfsimilar dynamics

Rules: each square changes to be like the majority of its neighbours
Coarsening, segregation, selfsimilarity

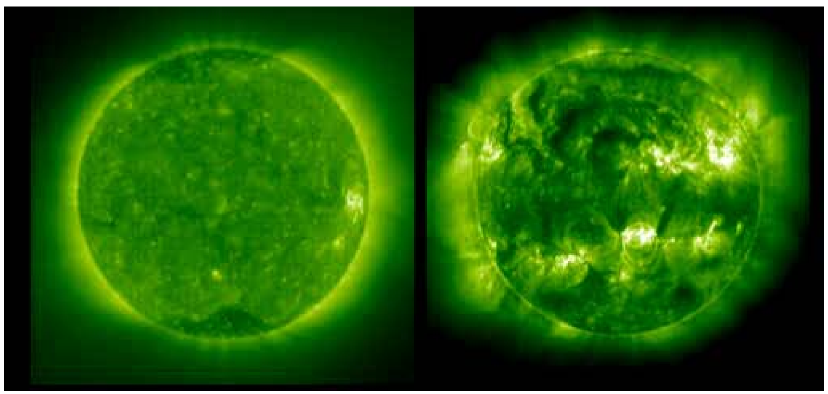


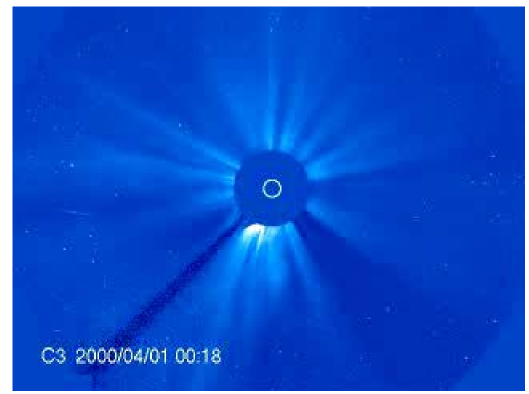
Courtesy P. Sethna

Solar corona over the solar cycle

SOHO-EIT image of the corona at solar minimum and solar maximum - Magnetic field structure

SOHO- LASCO image of the outer corona near solar maximum





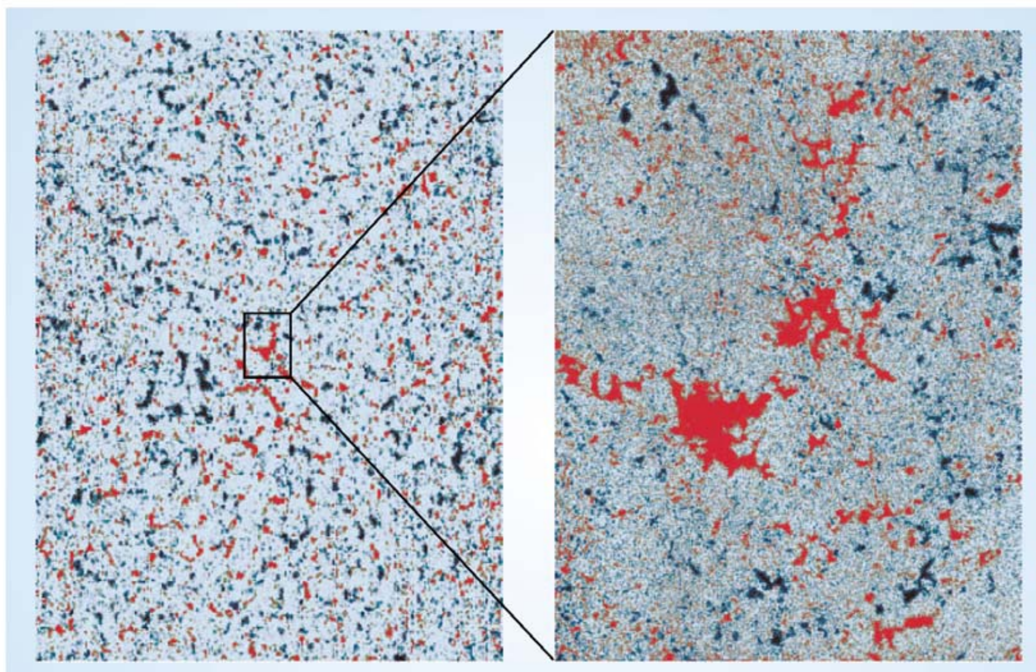
The solar wind is accelerated at the corona- open question….

'Fractal –like' patches of magnetic polarity on the quiet sun

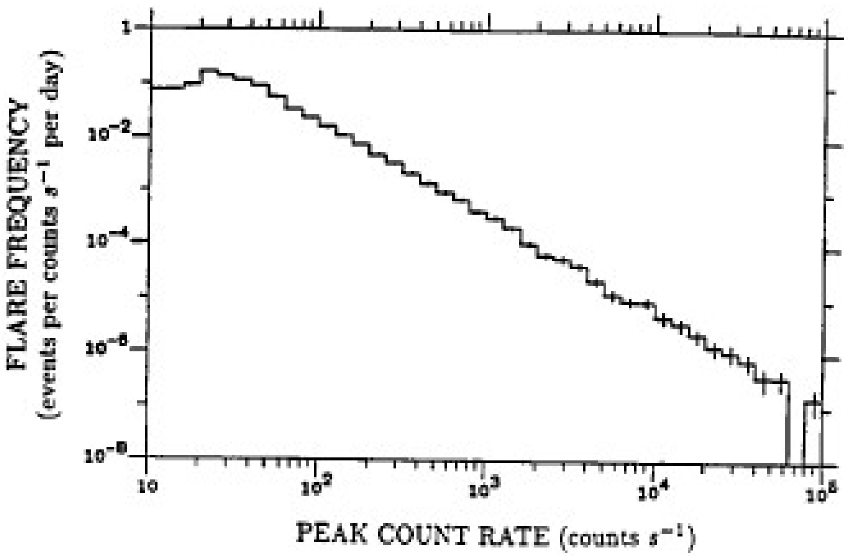
Patches of opposing polarity –

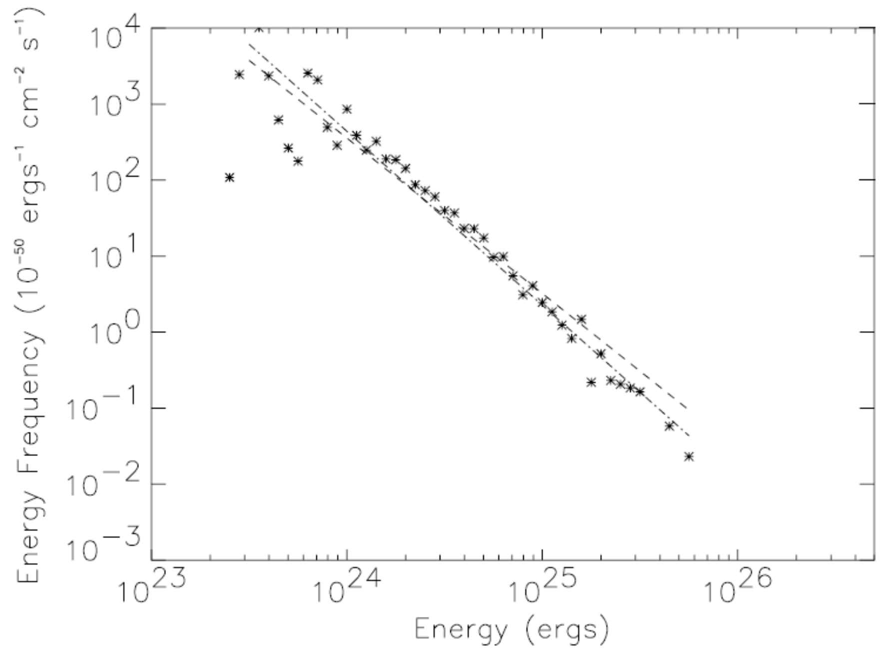
Zeeman effect photosphere, quiet sun,

(Stenflo, Nature 2004, See eg Janssen et al A&A 2003, Bueno et al Nature 2004+..) - ***spatial***



Power law statistics of flares





Peak flare count rate *Lu&Hamilton ApJ 1991*

TRACE nanoflare events *Parnell&Judd ApJ 2000*

-temporal

Solar wind at 1AU power spectra- suggests inertial range of (anisotropic MHD) turbulence. Multifractal scaling in velocity and magnetic field components.. AND something else in B magnitude..

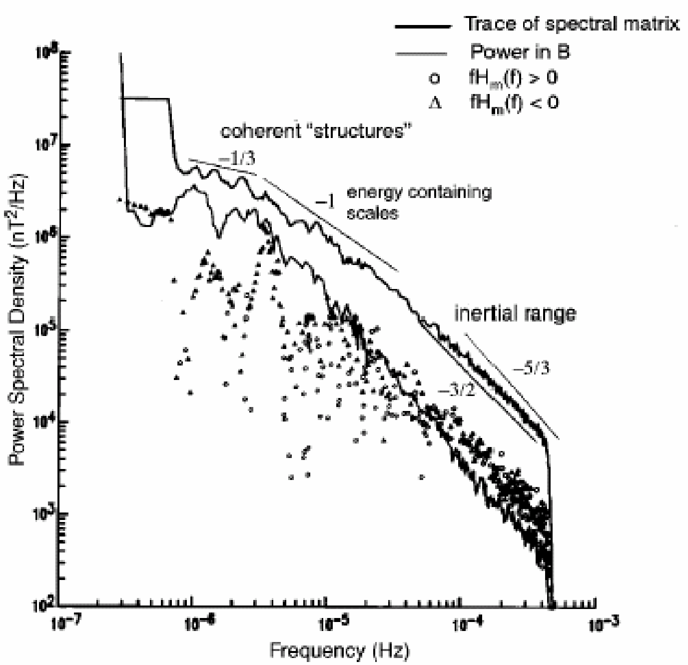


FIG. 1. A power spectrum of the solar wind magnetic field from a time series spanning more than a year. The upper curve is the trace of the power spectral matrix of the three components of B, the lower solid curve is the power in |B|, and the circles and triangles are the positive and negative values of the [reduced (Ref. 91)] magnetic helicity (Refs. 39 and 92) respectively.

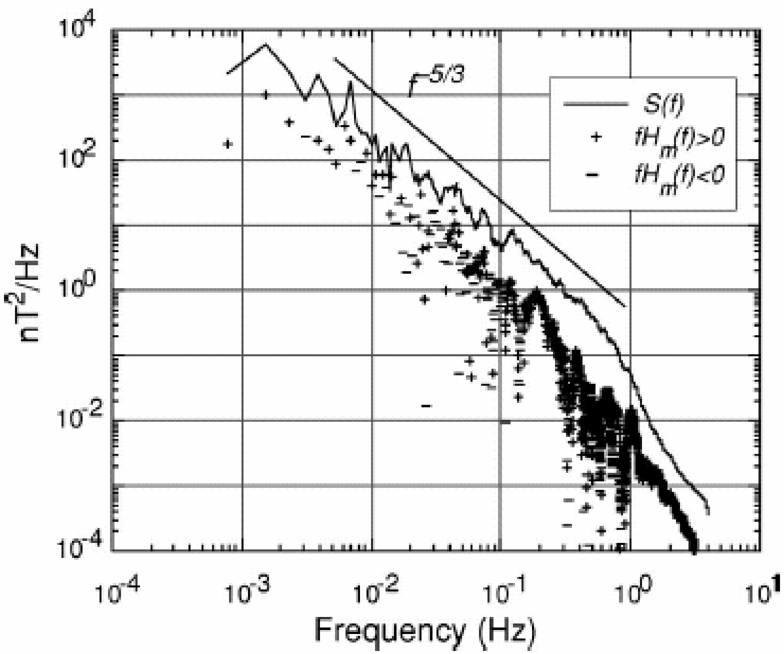
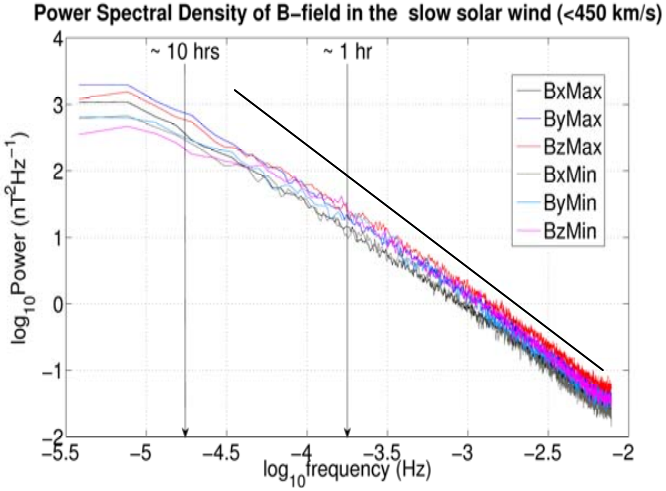
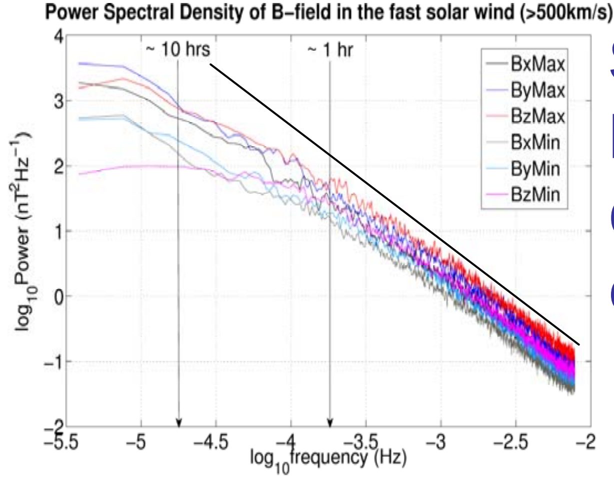


FIG. 2. A power spectrum of Mariner 10 data from 0.5 AU showing the dissipation range of magnetic fluctuations. Also shown are positive and negative values of $fH_m(f)$.

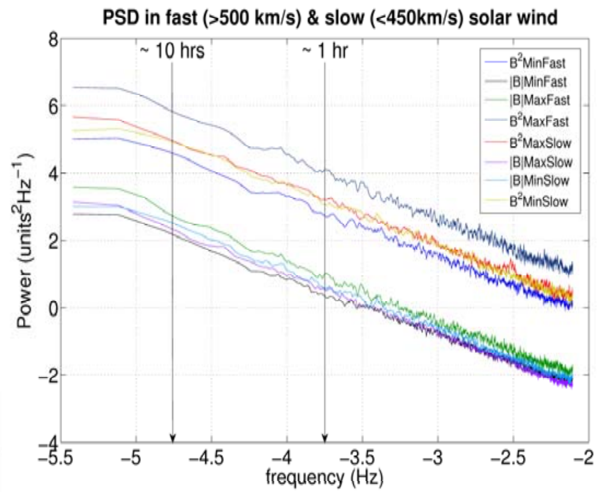
Goldstein and Roberts, POP 1999, See also Tu and Marsch, SSR, 1995

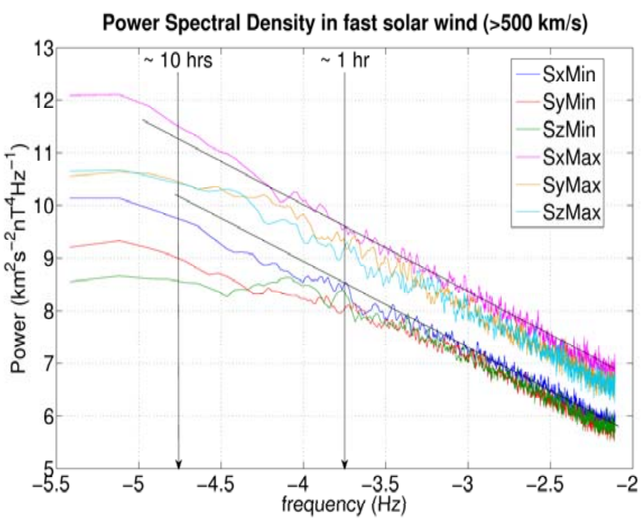
Scaling in Poynting flux S_x- dominated by coronal signature?

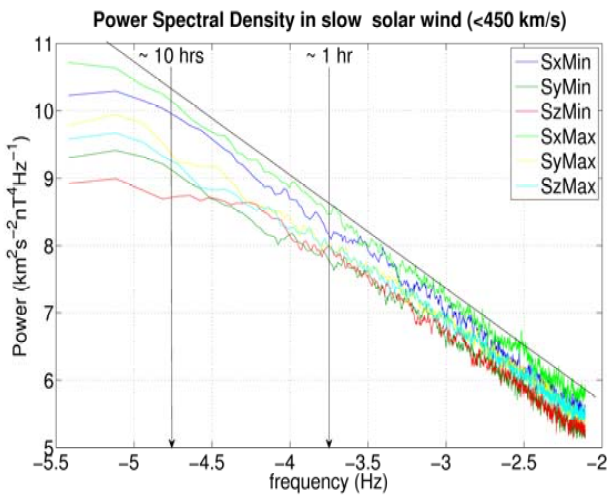




Shown: *log-log* plots of PSD of 3 day intervals averaged over 1 year
ACE solar max (2000); solar min (2007)
Plotted: |**B**|, B^2 and normalized **S**=-[**B**(**v.B**)-**v**B^2]
Fast $v>500kms^{-1}$ and slow $v<450kms^{-1}$



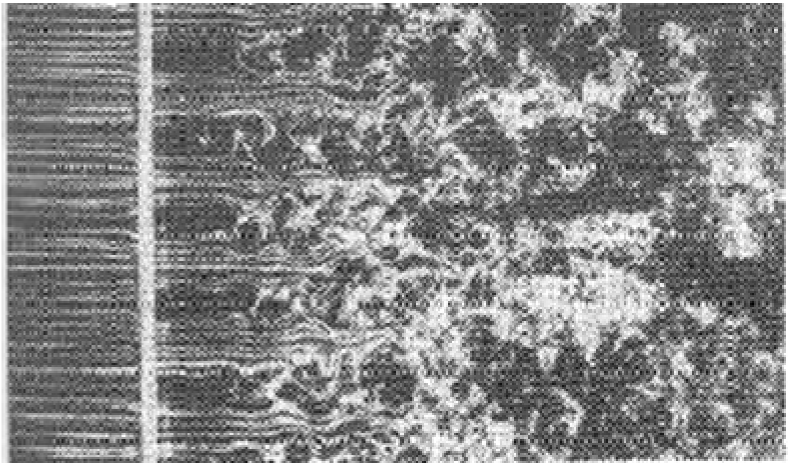




Turbulence

a la Komogorov, intermittency

Turbulence



Dynamics are complex
Statistics are simple
Assume:
Isotropic
Stationary
Homogeneous

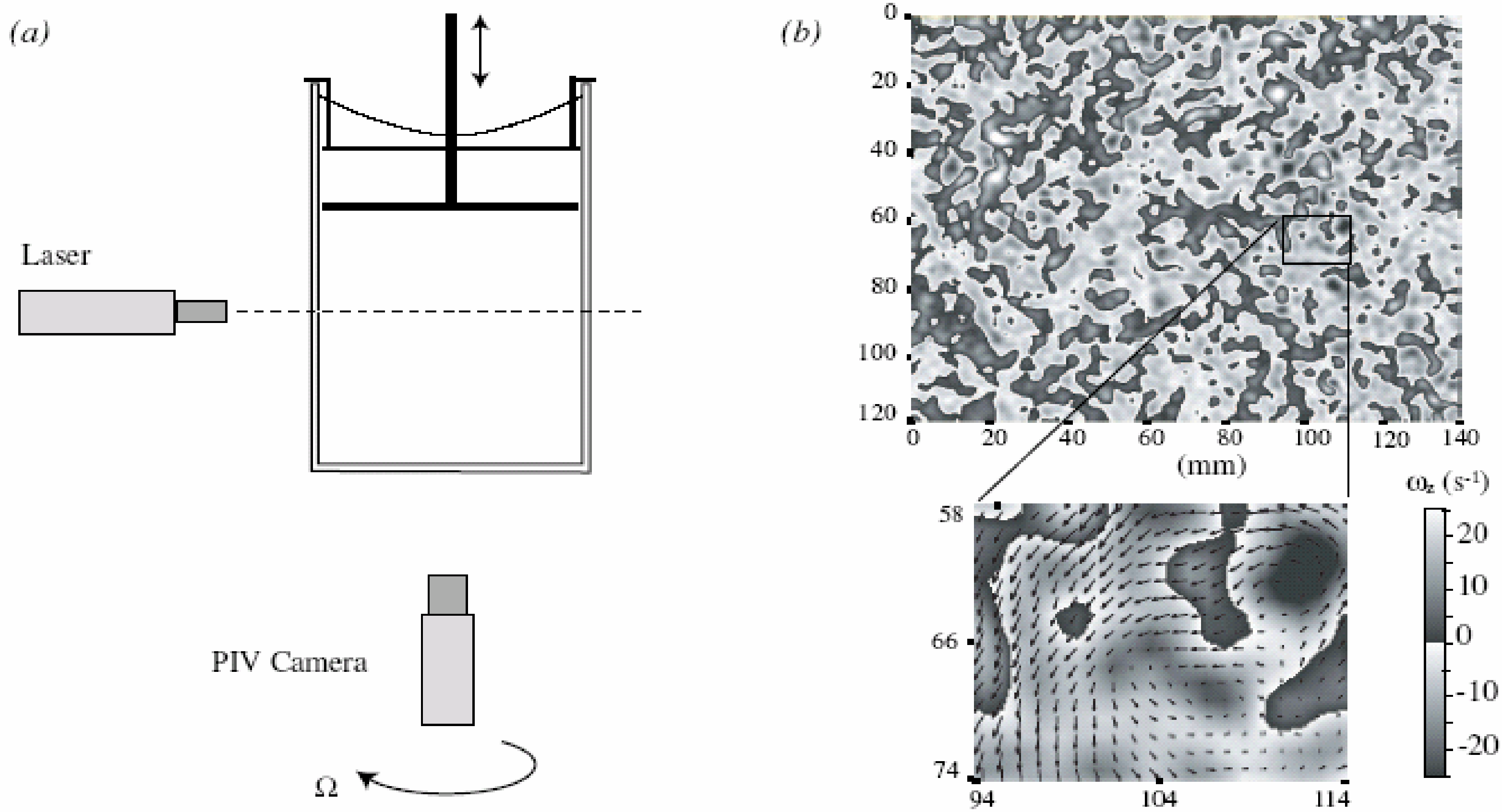
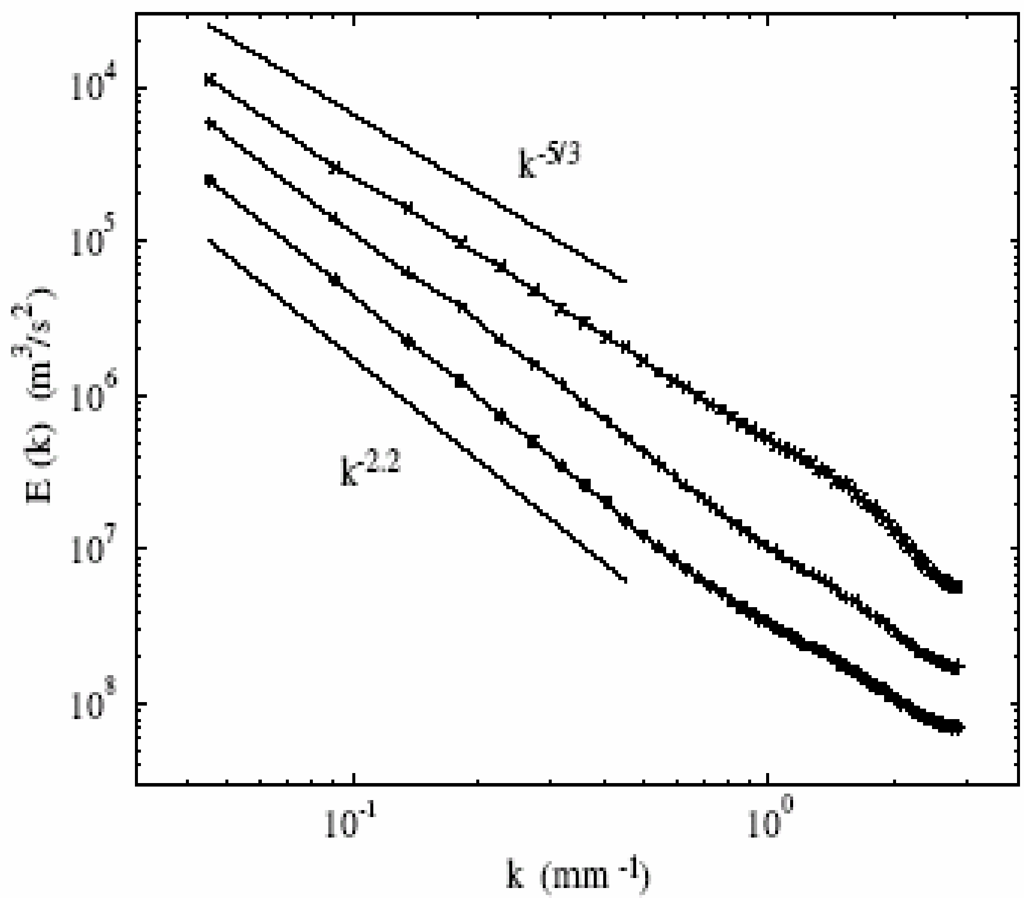


Figure 1. (*a*) Schematic of the experimental set-up. The water tank, the oscillating grid and the camera are in the rotating frame, while the pulsed laser remains in the laboratory frame. (*b*) Example of vertical vorticity field measured by PIV.

Power spectrum, transition from 3D-2D flow



Intermittent turbulence-topology

Consider simple finite sized scaling system, scale lengths l_j

$\lambda = \left(l_{j-1}/l_j\right)^3, l = 1 \ldots N$ with N levels

from a smallest size $l_1 = \eta$ to the system size $l_N = L$, m_j patches on lengthscale l_j

Non space filling, intermittent patches: $< m_j^q > l_j^{\gamma(q)} = < m_{j-1}^q > l_{j-1}^{\gamma(q)} = < m_N^q > L^{\gamma(q)}$

Fractal support: $\dfrac{\varepsilon_{\mathrm{j}}^*}{l_j^\alpha} = \dfrac{\varepsilon_{\mathrm{j-1}}^*}{l_{j-1}^\alpha} = \dfrac{\varepsilon_{\mathrm{N}}^*}{L^\alpha}$ where $\varepsilon_{\mathrm{j}}^*$ is 'active quantity' per patch, lengthscale l_j

Conservation:

$\varepsilon_j = m_j \varepsilon_{\mathrm{j}}^*$

$< \varepsilon_j > = \varepsilon_0$ which fixes $\gamma(1) = \alpha$ or $\mu(1) = 0$

when these combine to give:

$$< \varepsilon_j^q > = (\varepsilon_N^*)^q < m_N^q > \left(\frac{l_j}{L}\right)^{[\alpha q - \gamma(q)]} = \varepsilon_0^q \left(\frac{l_j}{L}\right)^{-\mu(q)}$$

Intermittency-
as a deviation from a space filling cascade (Kolmogorov turbulence)

velocity difference across an eddy $d_r v = v(l+r) - v(l)$

eddy time $T(r)$ and energy transfer rate $\varepsilon_\mathrm{r} \propto \dfrac{d_r v^2}{T}$

have T as the eddy turnover time $T \propto {r}/{d_r v}$ so that $\varepsilon_\mathrm{r} \propto \dfrac{d_r v^3}{r}$

If the flow is non- intermittent $\langle \varepsilon_r{}^p \rangle = \bar{\varepsilon}^p, r$ independent for any p

$\Rightarrow \ \langle d_r v^p \rangle \propto r^{p/3} \bar{\varepsilon}^{p/3} \sim r^{\zeta(p)}$ -$\zeta(p) = \alpha p$ linear with p – *selfsimilar*(*fractal*) scaling

intermittency correction- r dependence $\langle \varepsilon_r{}^p \rangle \propto \bar{\varepsilon}^p \left(r/L \right)^{\tau(p)}$

$\Rightarrow \ \langle d_r v^p \rangle \propto r^{p/3} \bar{\varepsilon}^{p/3} \left(L/r \right)^{\tau\left(p/3\right)} \sim r^{\zeta(p)}$- $\zeta(p)$ quadratic in p

$\langle \varepsilon_r \rangle = \bar{\varepsilon}$ independent of r (steady state) so $\tau(1) = 0$,

$\Rightarrow$ $\zeta(p)$ must monotonically increase (and $\zeta(p) > 1$ for some p)

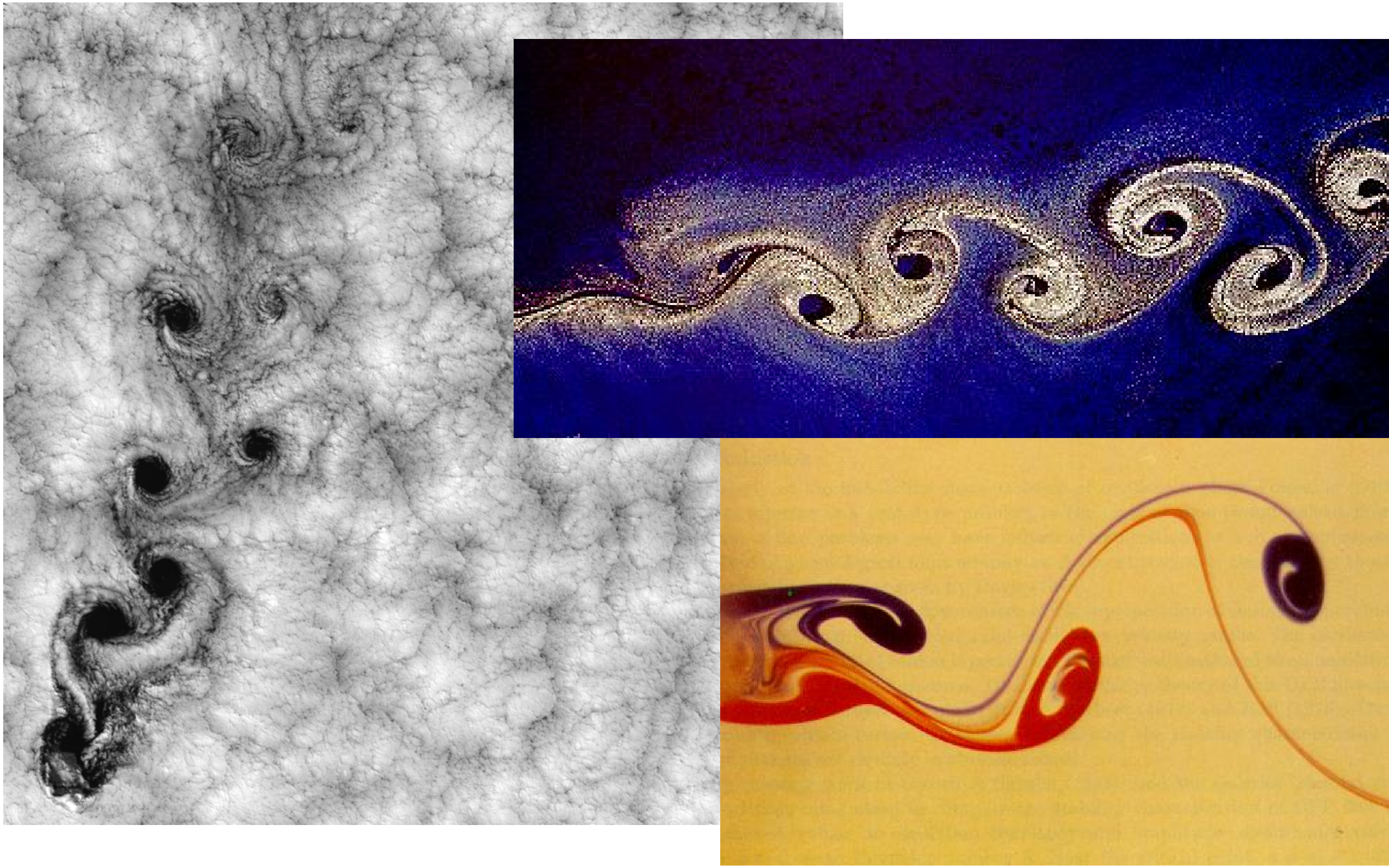
in situ single point observations take $r \equiv t$: measure $\zeta(p)$ from $< d_t v^p > \sim t^{\zeta(p)}$

$p = 6$ needed to measure $\tau(2)$! predicted from phenomenology

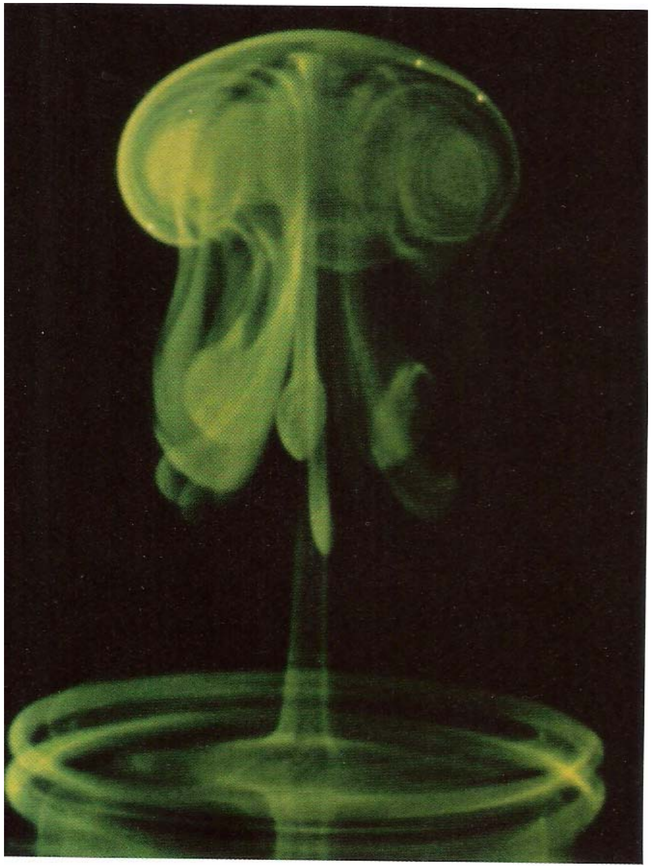
Scaling and similarity

Buckingham Π theorem (‘dimensional analysis’) of systems that show scaling

Similarity in action…



Similarity in action…



Peck and Sigurdson, A Gallery of Fluid Motion, CUP(2003)

Universality- 1 d.o.f.

Pendulum

$$F = mg, F_t = mg\sin\theta, a_t = l\frac{d^2\theta}{dt^2}$$

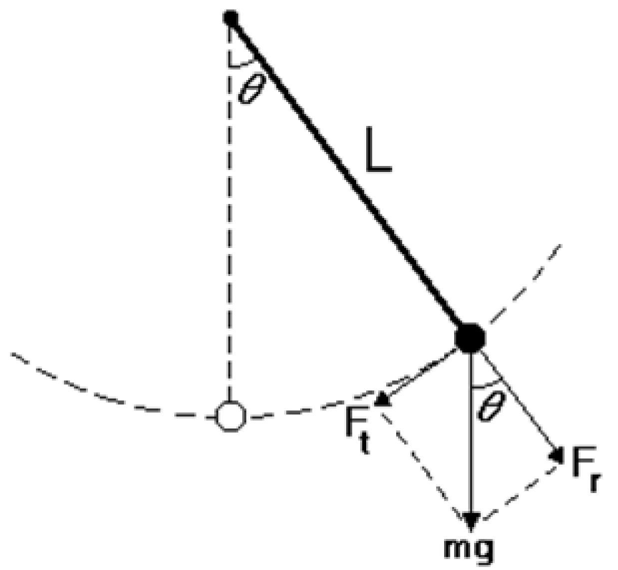
$$F_t = ma_t; \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\omega^2\frac{\partial V}{\partial\theta}$$

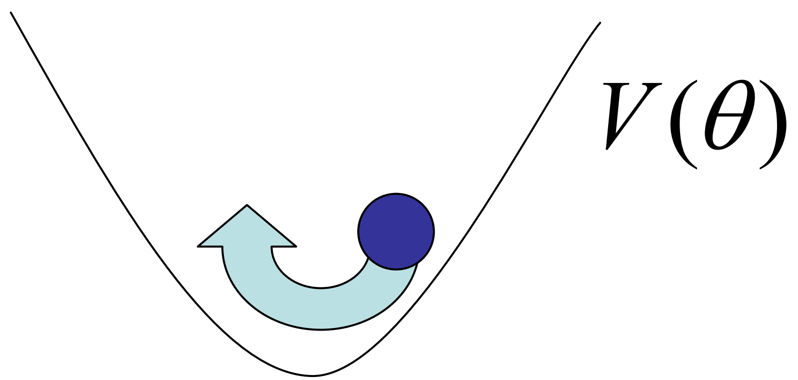
$$V(\theta) = 1 - \cos(\theta) \sim \frac{\theta^2}{2} + \ldots$$

same behaviour at

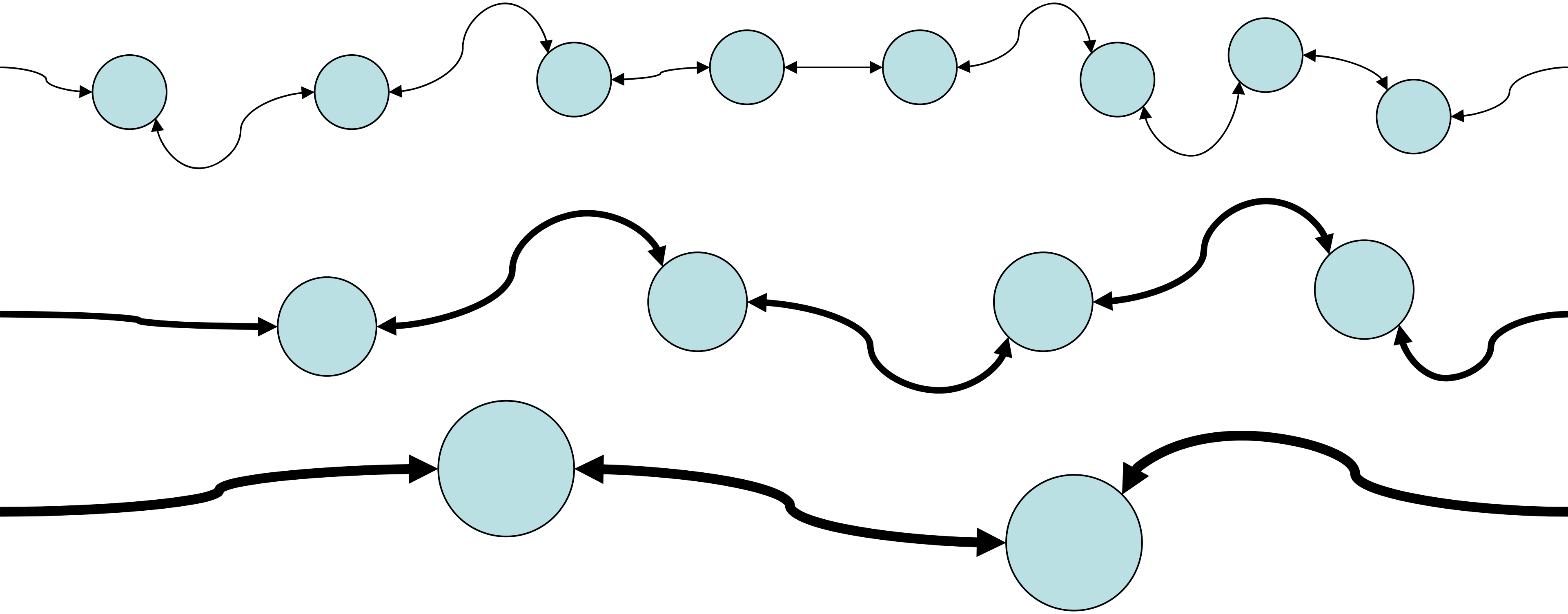
any local minimum in $V(\theta)$

(insensetive to details)





Universality- many d.o.f.



Keep coarsegraining-

rescaled system ‘looks the same’ (selfsimilar), insensitive to details

Similarity and universality

- Different systems, same physical model
- The same function (suitably normalized) can describe them
- This function is universal (the details do not matter)
- The values of the normalizing parameters are not universal
- How can we find the physical model (solution)?
- Particularly useful in nonlinear systems which are ‘hard’ to solve – i.e. turbulence!
- ‘Classical’ inertial range turbulence- self similarity, intermittency…
- Leads to *order/control parameters*

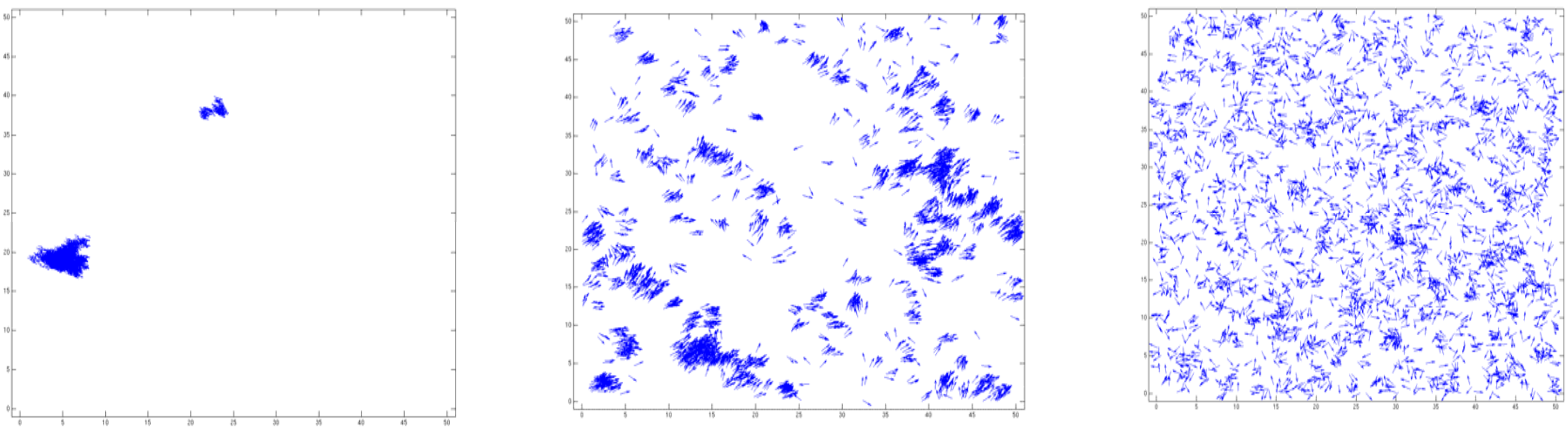
Competition between order and disorder

Rules: random fluctuation plus 'following the neighbours'

$$\mathbf{x}_{n+1}^{k} = \mathbf{x}_{n}^{k} + \mathbf{v}_{n}^{k} dt, \quad \left|\mathbf{v}_{n}^{k}\right| \text{ constant}$$

$$\theta_{n+1}^{k} = \left\langle \theta_{n}^{k} \right\rangle_{k \cap R} + \delta\theta, \ \delta\theta = \left[-\eta, \eta\right] \text{ iid random variable}$$

order parameter: total speed $\frac{1}{N}\left|\sum_{i=1}^{N} \mathbf{v}_i\right|$



Vicsek bird model

Low noise- flocking

High noise- random walks

Phase transition- cf linear models for ferromagnets (EW, linear Ising)
(birds=short range interacting spins + motion)

Buckingham π theorem

System described by $F(Q_1 ... Q_p)$ where $Q_{1..p}$ are the relevant macroscopic variables

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.)

there are $M = P - R$ distinct dimensionless groups.

Then $F(\pi_{1..M}) = C$ is the general solution for this universality class.

To proceed further we need to make some intelligent guesses for $F(\pi_{1..M})$

See e.g. *Barenblatt, Scaling, self - similarity and intermediate asymptotics, CUP, [1996]*

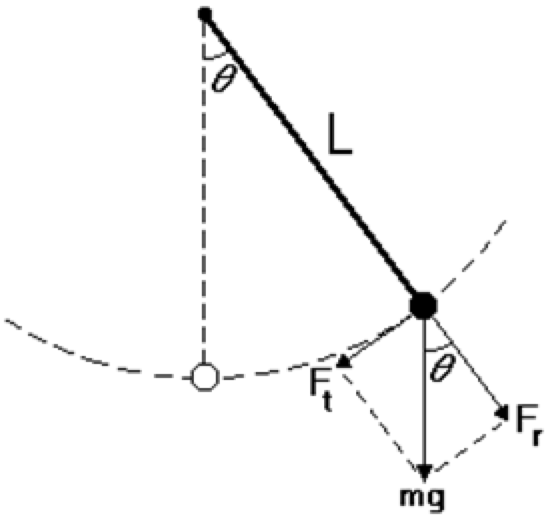
also *Longair, Theoretical concepts in physics, Chap 8, CUP [2003]*

Example: simple (nonlinear) pendulum

System described by $F(Q_1 ... Q_p)$ where Q_k is a macroscopic variable

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups



Step 1: write down the relevant macroscopic variables:

variable	dimension	description
θ_0	$-$	angle of release
m	$[M]$	mass of bob
τ	$[T]$	period of pendulum
g	$[L][T]^{-2}$	gravitational acceleration
l	$[L]$	length of pendulum

Step 2: form dimensionless groups: $P = 5, R = 3$ so $M = 2$

$\pi_1 = \theta_0, \pi_2 = \dfrac{\tau^2 l}{g}$ and no dimensionless group can contain m

then solution is $F(\theta_0, \tau^2 l/g) = C$

Step 3: make some simplifying assumption: $f(\pi_1) = \pi_2$ then the period: $\tau = f(\theta_0)\sqrt{l/g}$

NB $f(\theta_0)$ is universal ie same for all pendula-

we can find it knowing some other property eg conservation of energy..

Example: fluid turbulence, the Kolmogorov '5/3 power spectrum'

System described by $F(Q_1 ... Q_p)$ where Q_k is a macroscopic variable

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups

Step 1: write down the relevant variables (incompressible so energy/mass):

variable	dimension	description
$E(k)$	$[L]^3[T]^{-2}$	energy/unit wave no.
ε_0	$[L]^2[T]^{-3}$	rate of energy input
k	$[L]^{-1}$	wavenumber

Step 2: form dimensionless groups: $P = 3, R = 2,$ so $M = 1$

$$\pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}$$

Step 3: make some simplifying assumption:

$F(\pi_1) = \pi_1 = C$ where C is a non universal constant, then: $E(k) \sim \varepsilon_0^{2/3} k^{-5/3}$

Buchingham π theorem (similarity analysis)

universal scaling, anomalous scaling

System described by $F(Q_1 \ldots Q_p)$ where Q_k is a relevant macroscopic variable

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups

Turbulence:

variable	dimension	description
$E(k)$	$[L]^3[T]^{-2}$	energy/unit wave no.
ε_0	$[L]^2[T]^{-3}$	rate of energy input
k	$[L]^{-1}$	wavenumber

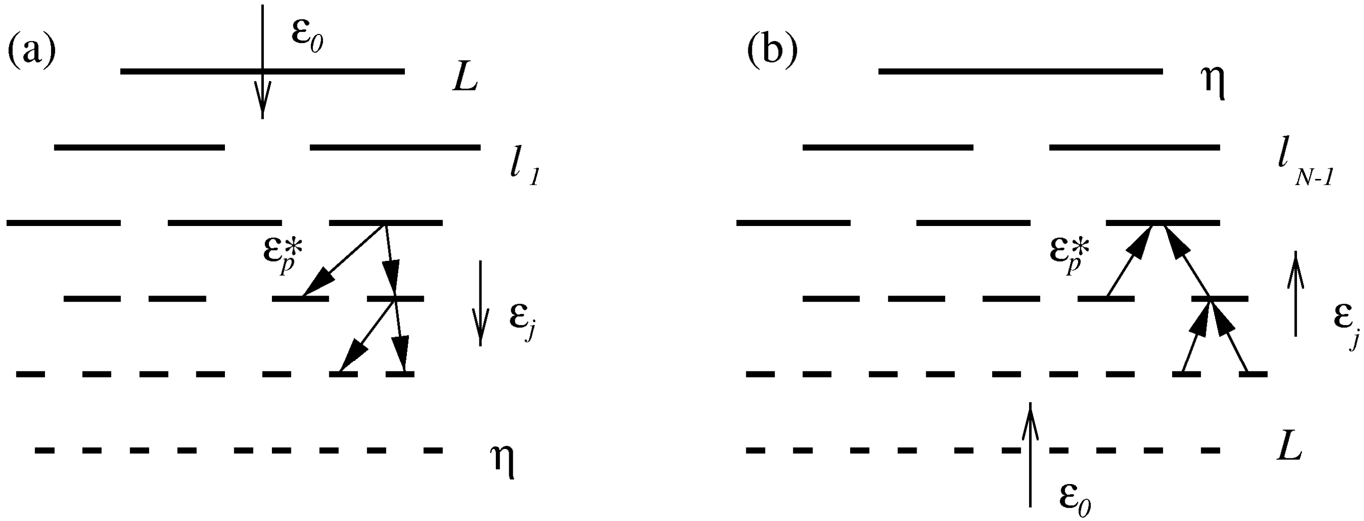
$$M = 1, \pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}, E(k) \sim \varepsilon_0^{2/3} k^{-5/3}$$

introduce another characteristic speed....

variable	dimension	*description*
$E(k)$	$[L]^3[T]^{-2}$	energy/unit wave no.
ε_0	$[L]^2[T]^{-3}$	rate of energy input
k	$[L]^{-1}$	wavenumber
v	$[L][T]^{-1}$	characteristic speed

$$M = 2, \pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}, \pi_2 = \frac{v^2}{Ek} \text{ let } \pi_1 \sim \pi_2^{\alpha}, E(k) \sim k^{-(5+\alpha)/(3+\alpha)}$$

Turbulence and ‘degrees of freedom’



- System is driven on one lengthscale (L) and dissipates on another (η) –forward cascade
- Inverse cascade- same thing, just the other way around
- System has many degrees of freedom i.e. structures on many lengthscales (eddies here)
- System is scaling- structures, processes can be rescaled to ‘look the same on all scales’
- These structures transmit some dynamical quantity from one lengthscale to another that is, over all the d.o.f.
- There is conservation of flux of the dynamical quantity- here energy transfer rate
- Steady state (not equilibrium) means energy injection rate balances energy dissipation rate on the average

Homogeneous Isotropic Turbulence and Reynolds Number

Step 1: write down the relevant variables:

variable	dimension	description
L_0	$[L]$	driving scale
η	$[L]$	dissipation scale
U	$[L][T]^{-1}$	bulk (driving) flow speed
ν	$[L]^2[T]^{-1}$	viscosity

Step 2: form dimensionless groups: $P = 4, R = 2,$ so $M = 2$

$\pi_1 = \frac{UL_0}{\nu} = R_E, \pi_2 = \frac{L_0}{\eta}$ and importantly $\frac{L_0}{\eta} = f(N),$ where N is no. of d.o.f

Step 3: d.o.f from scaling ie $f(N) \sim N^{\alpha}$ here $\frac{L_0}{\eta} \sim N^3$, or $N^{3\beta}$ or $\frac{L_0}{\eta} \sim \lambda^{N/3}$ or ...

Step 4: assume steady state and conservation of the dynamical quantity, here energy...

transfer rate $\varepsilon_r \sim \frac{u_r^3}{r}$, injection rate $\varepsilon_{inj} \sim \frac{U^3}{L_0}$, dissipation rate $\varepsilon_{diss} \sim \frac{\nu^3}{\eta^4}$ - gives $\varepsilon_{inj} \sim \varepsilon_r \sim \varepsilon_{diss}$

this relates π_1 to π_2 giving: $R_E = \frac{UL_0}{\nu} \sim \left(\frac{L_0}{\eta}\right)^{4/3} \sim N^{\alpha}, \alpha > 0$ thus N grows with R_E

Generalize the idea of a Reynolds Number

... a control parameter for the onset of 'disorder'

(turbulence, burstiness)

The above is true for other systems with:

$P = 4, R = 2\ (L, T)$, so $M = 2$

$\pi_1 = R_E$ the Reynolds Number

$\pi_2 = f(N)$ where N is the number of degrees of freedom

flux of some dynamical quantity is conserved- steady state

scaling so $f(N) \sim N^{\alpha}$

gives $\pi_1 = f(\pi_2)$ or $R_E = f(N)$

Avalanching systems and scaling behaviour

Avalanche models: add grains slowly, redistribute only if local gradients exceeds a critical value

Suggested as a model for bursty transport and energy release in plasmas- solar corona, magnetotail, edge turbulence in tokamaks (L-H), accretion disks

Avalanching systems

- Threshold for avalanching
- Avalanches are much faster than feeding rate
- Avalanches on all sizes, no characteristic size
- Feeding rate=outflow rate on average only
- System moves through many metastable states- rather than toward an equilibrium

Avalanche model (Self Organized Criticality and all that...)

Step 1:

variable	dimension	description
L_0	$[L]$	system size
δl	$[L]$	grid size
h	$[S][T]^{-1}$	average driving rate per node
ε	$[S][T]^{-1}$	system average dissipation/loss

Step 2: form dimensionless groups: $P = 4, R = 2,$ so $M = 2$

$$\pi_1 = \frac{h}{\varepsilon} = R_A, \pi_2 = \frac{L_0}{\delta l} = f(N)$$ where N is no. of d.o.f.

Step 3: d.o.f from scaling ie $f(N) \sim N^{\alpha}, N \sim \left(\frac{L_0}{\delta l}\right)^{\alpha}$ with Euclidean dimension $D \geq \alpha > 0$

Step 4: assume steady state and conservation of the dynamical quantity, here sand...S

conservation of flux of sand gives $h \times (\text{no of nodes}) \sim \varepsilon$

so $h\left(\frac{L_0}{\delta l}\right)^{D} \sim \varepsilon$ this relates π_1 to π_2 giving $R_A = \frac{h}{\varepsilon} \sim \left(\frac{\delta l}{L_0}\right)^{D} \sim N^{-\alpha D}$

this is in the opposite sense to fluid turbulence, N is maximal when $R_A \to 0$

How is SOC different to turbulence? consider...

Intermediate driving (or what happens as we change $R_A \sim h/\varepsilon$):

Suggest two conditions for avalanching transport:

$h\delta t \ll g\delta l$ - takes many timesteps δt to make a cell go unstable

$h\delta t \ll g\delta l \left(\frac{L_0}{\delta l} \right)^D$ -takes many timesteps to swamp the system

where g is average critical gradient, D is Euclidean dimension.

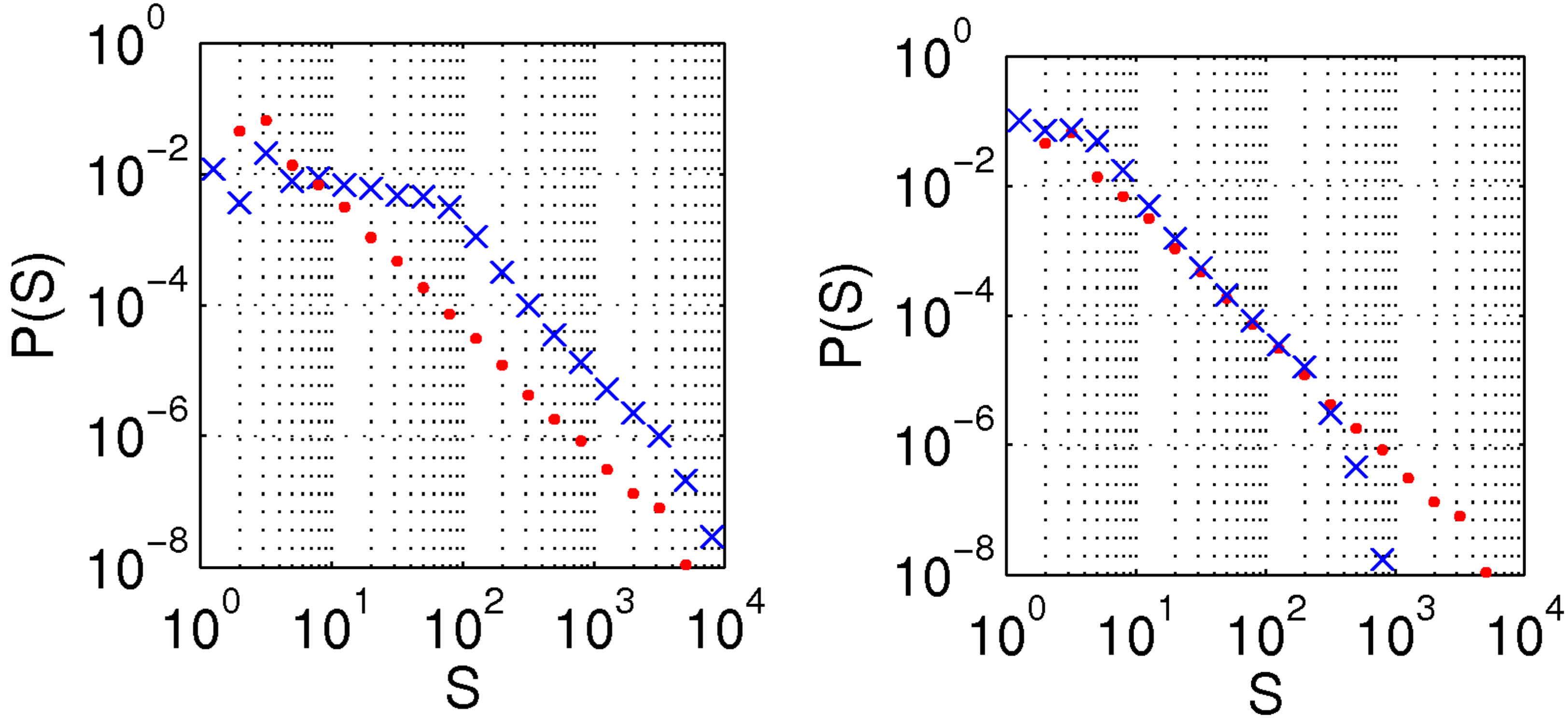
These are both satisfied for SDIDT $(h \to 0, \varepsilon \to 0)$

If $L_0 \gg \delta l$ we can consider intermediate behaviour $gL_0 \gg h\delta t > g\delta l$

where the smallest avalanches are swamped, but large avalanches persist.

Corresponds to:

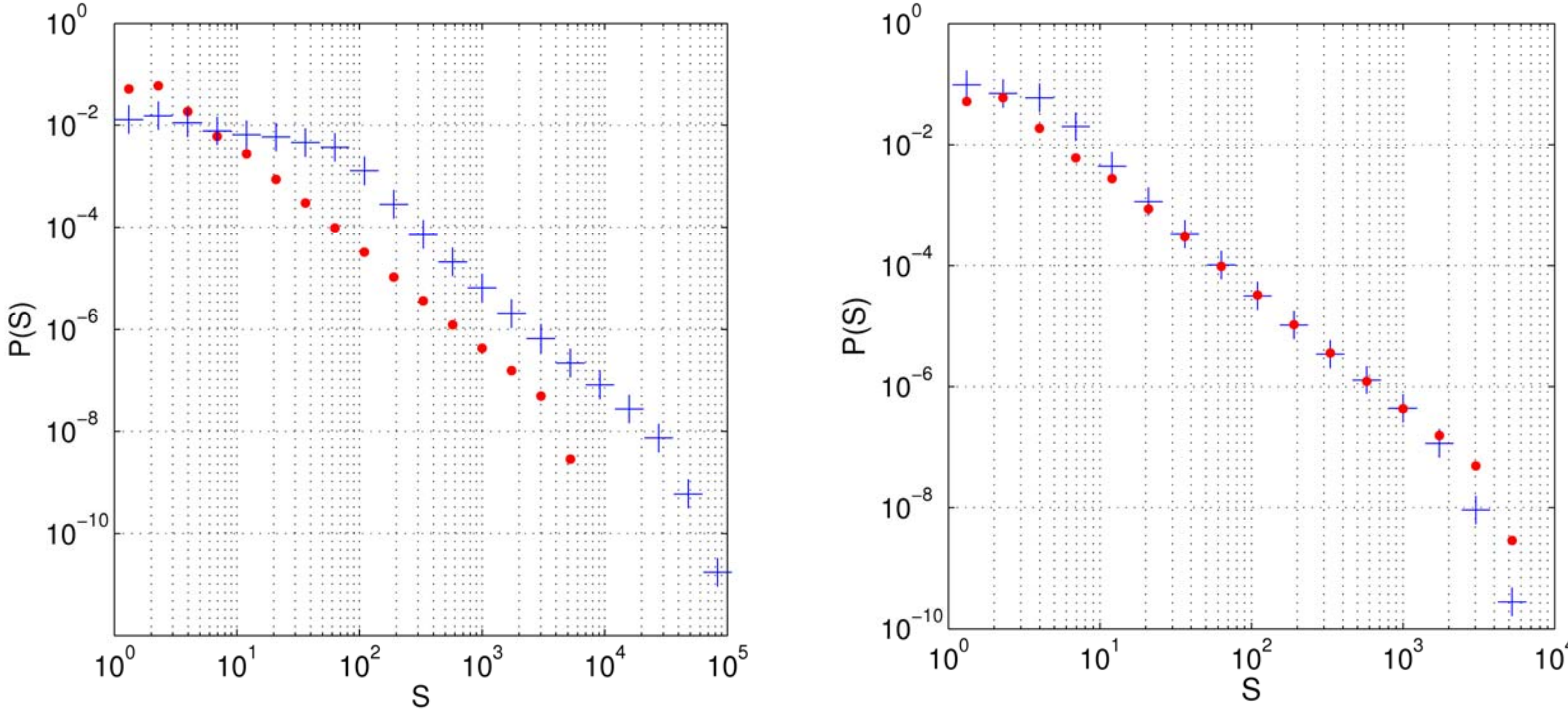
reducing the available d.o.f. by increasing h, and hence R_A



Two runs of the BTW (Bak et al, PRL, [1987]) sandpile box is *100×100* and *h=4* (●) and *16* (**X**).

Left: raw results; Right: the *h=16* run is rescaled *S→S/16*.

h=16 run has same scaling, smaller dynamic range than *h=4*



Two runs of the BTW (Bak et al, PRL, [1987]) sandpile
Box 100×100, $h=4$ (●); box 400×400 and $h=16$ (**X**).
Left: raw results; Right: the $h=16$ run is rescaled $S\rightarrow S/16$.
$h=16$, 400×400 run has same scaling, dynamic range as $h=4$, 100×100

Quantifying scaling I

Structure functions (c.f. wavelets)

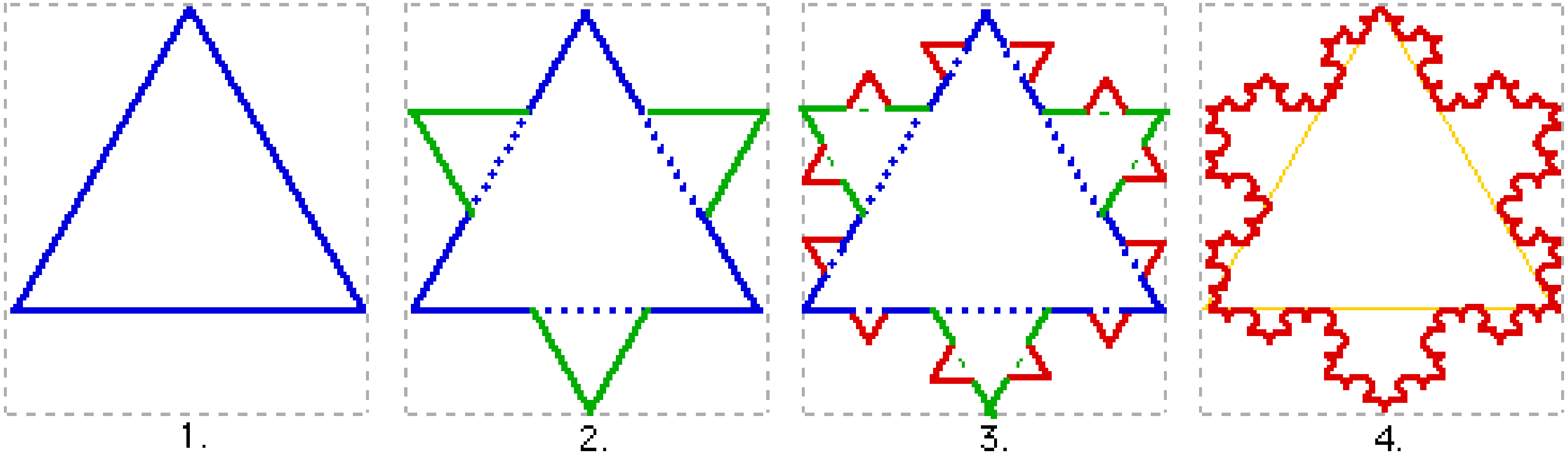
Uncertainties, finite size effects

Link to SDE models (self- affine processes)

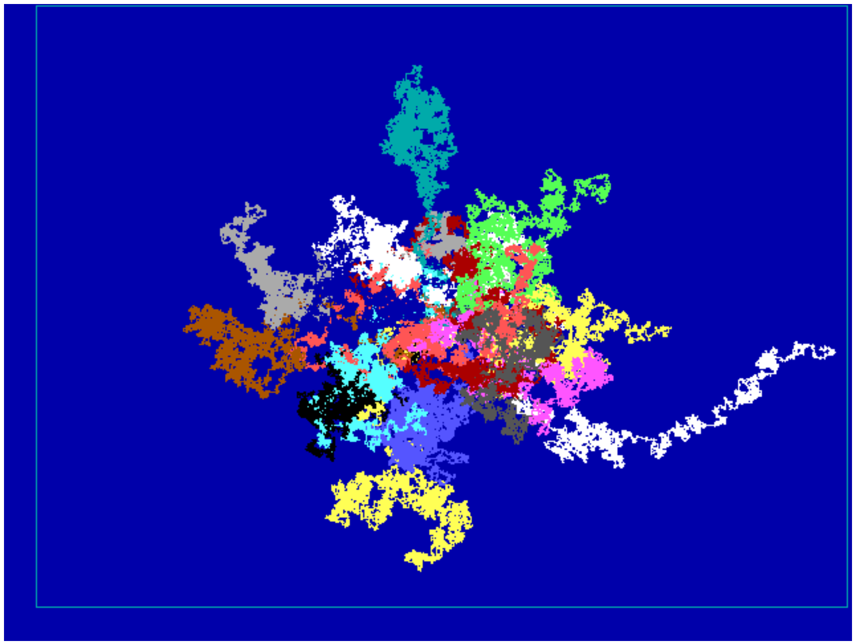
A regular fractal

Koch snowflake

line length $l \sim (4/3)^n$



A random fractal



16 particles- Brownian random walk

consider a random walk in 2D

$$\underline{r}(t_n) = \underline{r}_n = \underline{r}_{n-1} + \underline{l}$$

$$\underline{r}_n \bullet \underline{r}_n = r_{n-1}^2 + 2\underline{r}_{n-1} \bullet \underline{l} + l^2$$

$$\left\langle \underline{r}_n \bullet \underline{r}_n \right\rangle = \left\langle r_n^2 \right\rangle = \left\langle r_{n-1}^2 \right\rangle + l^2$$

$$\left\langle r_n^2 \right\rangle = nl^2$$

so if n steps take time t_n

$$\left\langle r_n^2 \right\rangle \sim t_n \text{ or } r \sim t^{1/2}$$

Quantifying scaling

structures on many length/timescales.

Reproducible, predictable in a *statistical* sense.

look at (time-space) differences:

$y(r,l) = x(r+l) - x(r)$

$y(t,\tau) = x(t+\tau) - x(t)$

for all available t_k of the timeseries $x(t_k)$

test for statistical scaling i.e

structure functions $S_p(r) = <|y(r,l)|^p> \propto l^{\zeta(p)}$

or $S_p(\tau) = <|y(t,\tau)|^p> \propto \tau^{\zeta(p)}$

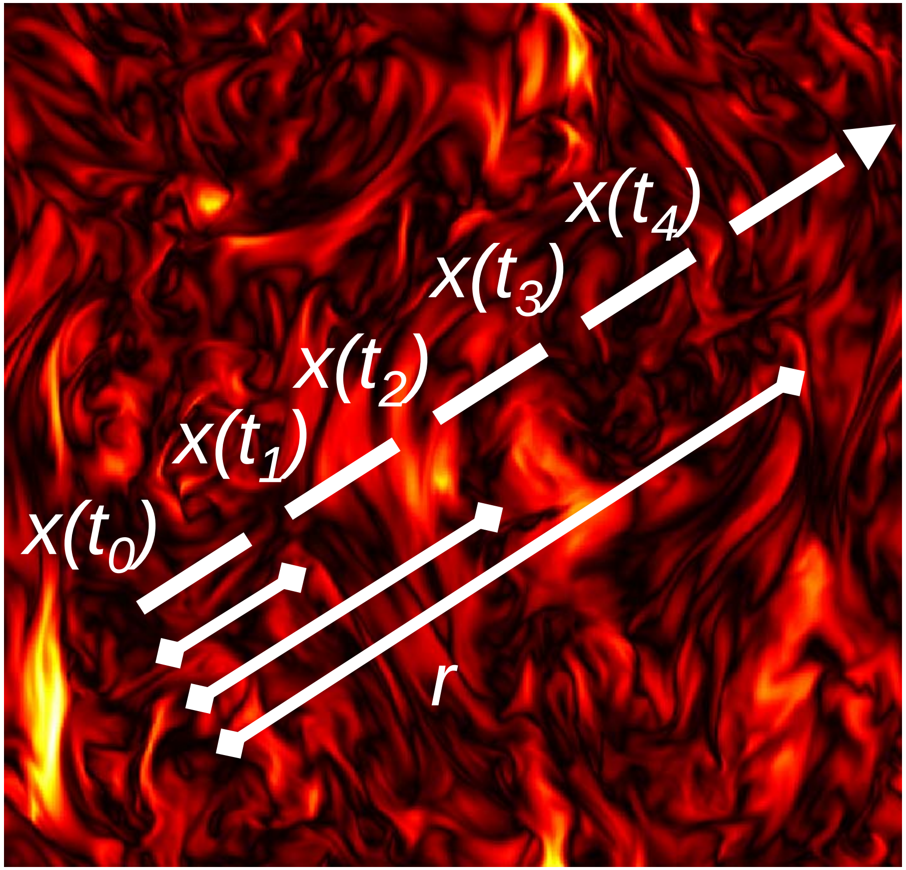
we want to measure the $\zeta(p)$

fractal (self- affine) $\zeta(p) \sim \alpha p$

multifractal $\zeta(p) \sim \alpha p - \beta p^2 + \ldots$

would like $<|y(r,l)|^p> = \int_{-\infty}^{\infty} |y|^p \, P(y,l) dy$

BUT finite system/data!



Data Renormalization

Consider a timeseries $x(t)$ sampled with precision Δ. We construct a *differenced* timeseries

$\delta x(t,\tau) = y(t,\tau) = x(t+\tau) - x(t)$ so

$x(t+\tau) = x(t) + y(t,\tau)$ and $y(t,\tau)$ is a random variable

then

$$x(t) = y(t_1,\Delta) + y(t_2,\Delta) + \ldots + y(t_k,\Delta) + y(t_{k+1},\Delta) + \ldots + y(t_N,\Delta)$$

$$= y^{(1)}(t_1, 2\Delta) + .. + y^{(1)}(t_k, 2\Delta) + .. + y^{(1)}(t_{N/2}, 2\Delta)$$

$$= y^{(n)}(t_1, 2^n\Delta) + \ldots + y^{(n)}(t_k, 2^n\Delta) + .. + y^{(n)}(t_{N/2^n}, 2^n\Delta)$$

we seek a self affine scaling

$y' = 2^\alpha y, \tau' = 2\tau, y^{(n)} = 2^{n\alpha} y,$ as $\tau = 2^n\Delta$

for arbitrary τ, normalize such that

$y'(t,\tau) = \tau^\alpha y(t,\Delta)$

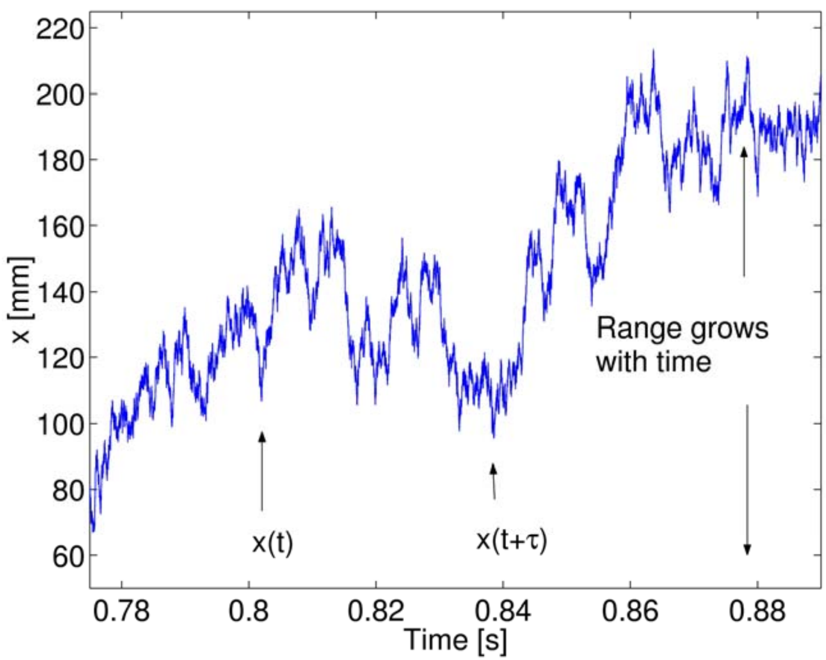
y is a random variable, so we have the same PDF under transformation:

$P(y'\tau^{-\alpha})\tau^{-\alpha} = P(y)$

the y are not Gaussian iid. We need to find α

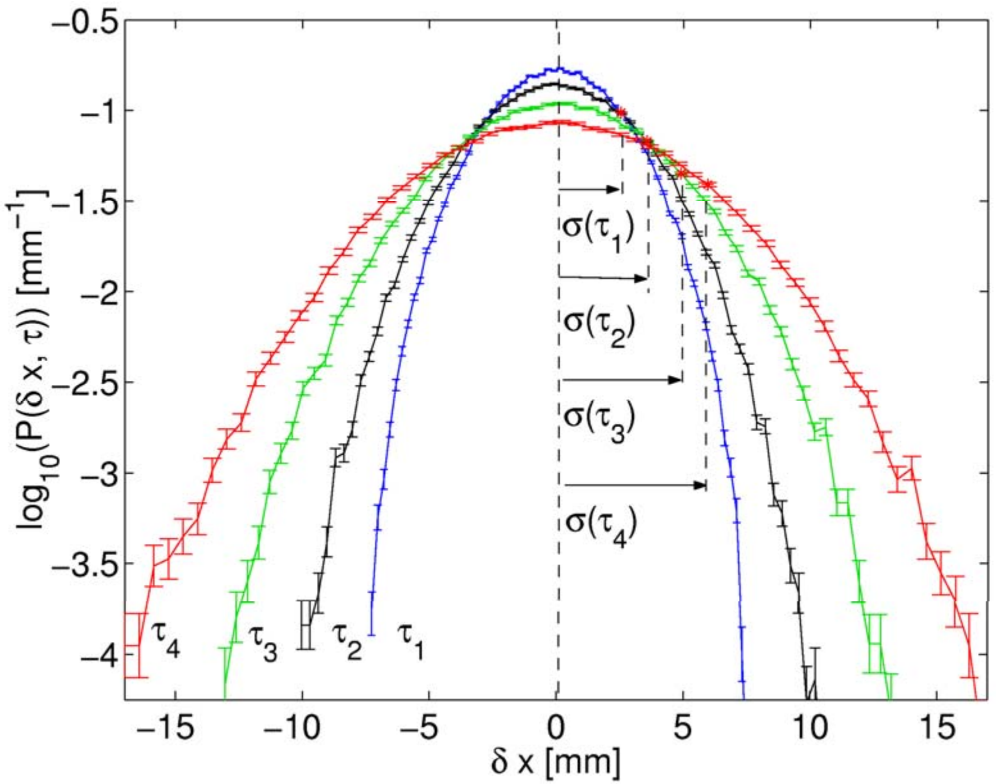
consider CLT case..

Self –affine ('fractal') scaling in timeseries



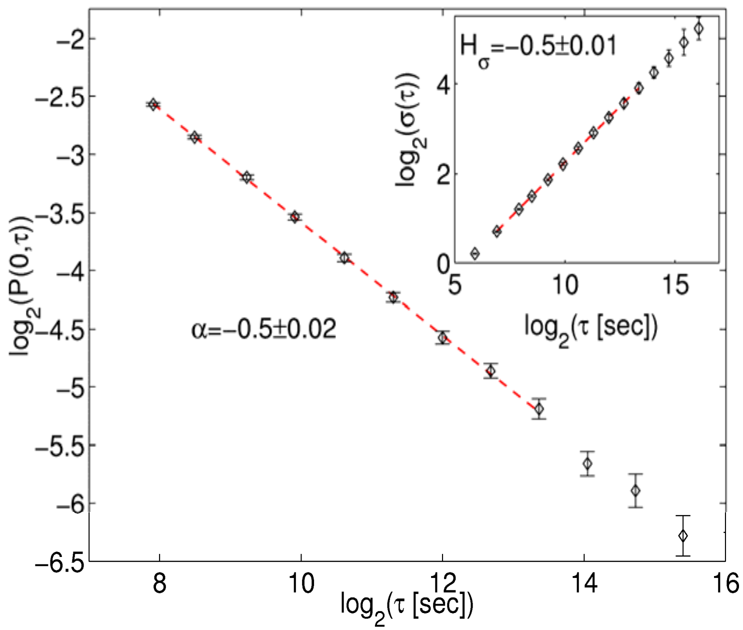
Example-Brownian walk

Fluctuations: $\delta x(t,\tau) = x(t+\tau) - x(t)$

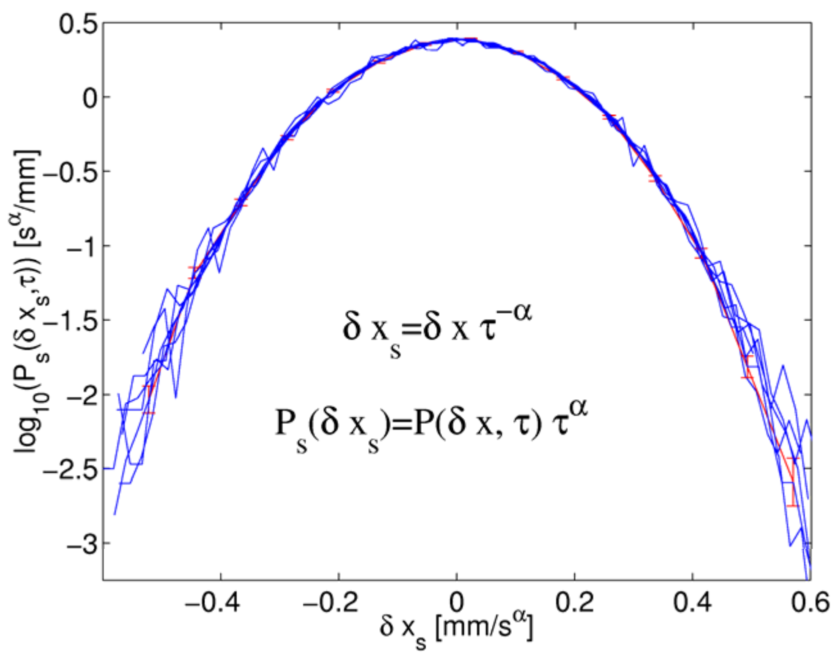


Probability of wandering different distances in a given time (Gaussian)

Rescale



The height of the peaks is power law- a single factor rescales them

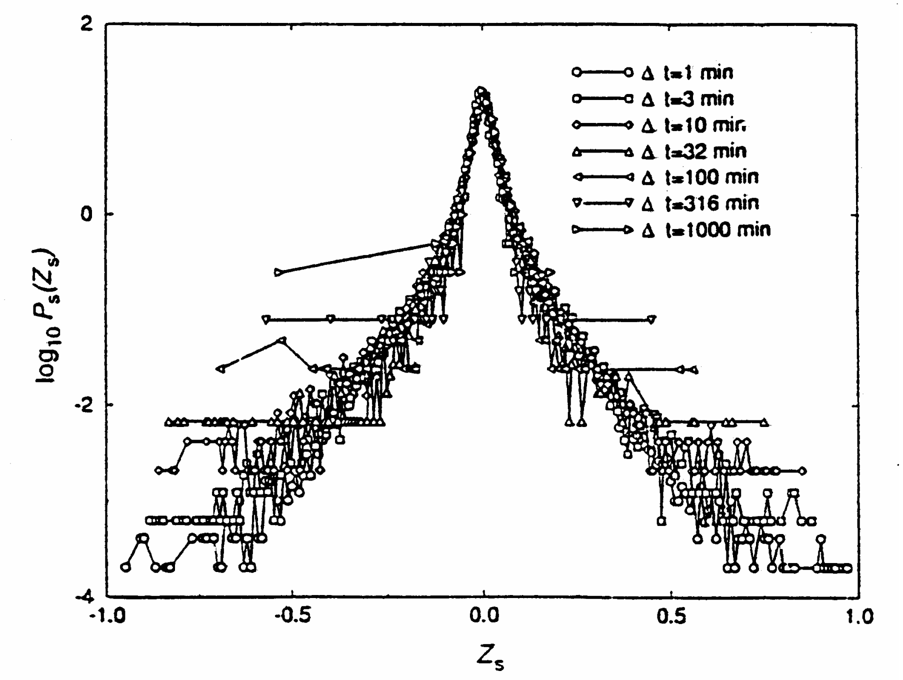


The same factor rescales all the curves- α=1/2

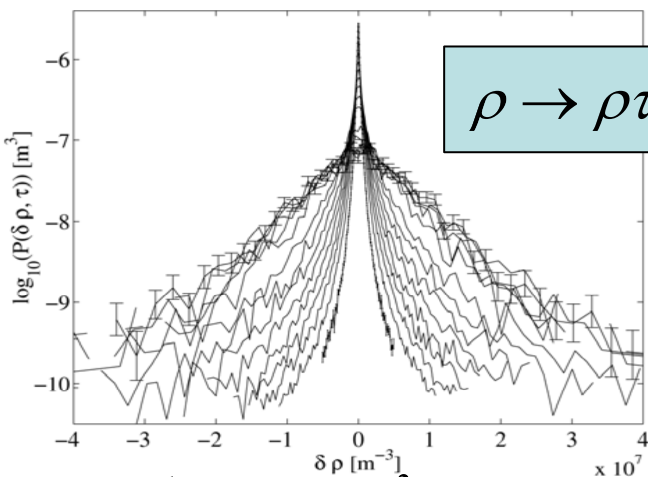
Self-similarity

Example- financial markets

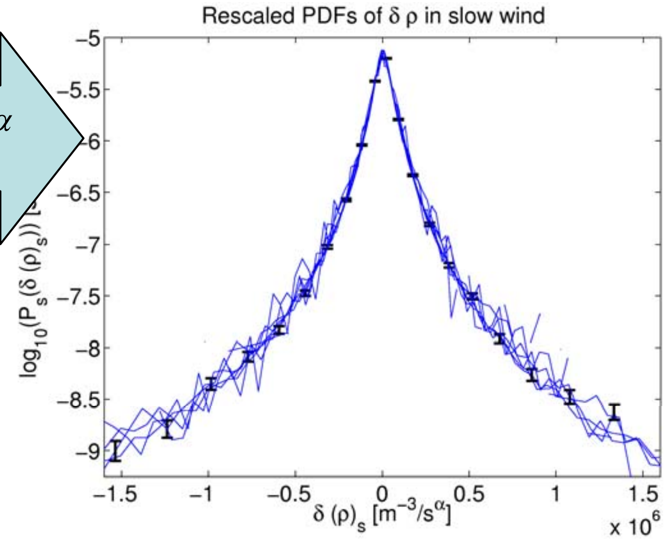
- Mantegna and Stanley- Nature ’95
- S+P500 index
- ‘heavy tailed’ distributions



example- ρ,B^2 in the solar wind



$$\rho \to \rho\tau^{-\alpha}$$



slow sw shown, ρ, B^2

selfsimilar scaling up to τ~few hrs

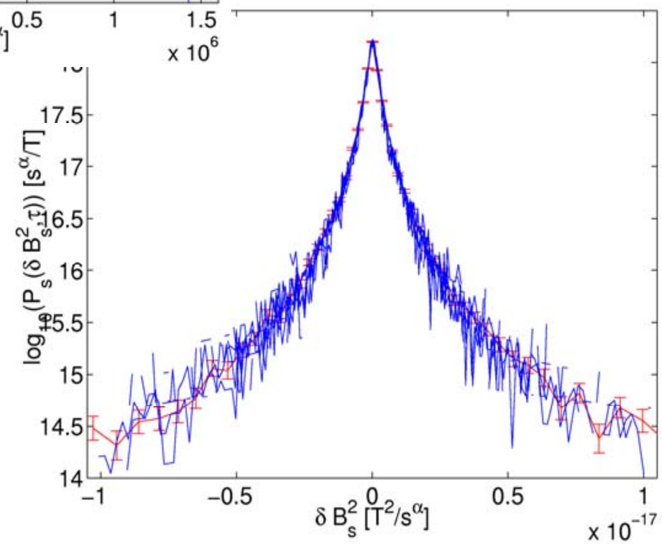
WIND 46/98s

Key Parameters ’95-’98

Approx 10^6 samples

Verified with ACE

Hnat, SCC et al GRL,2002, POP 2004



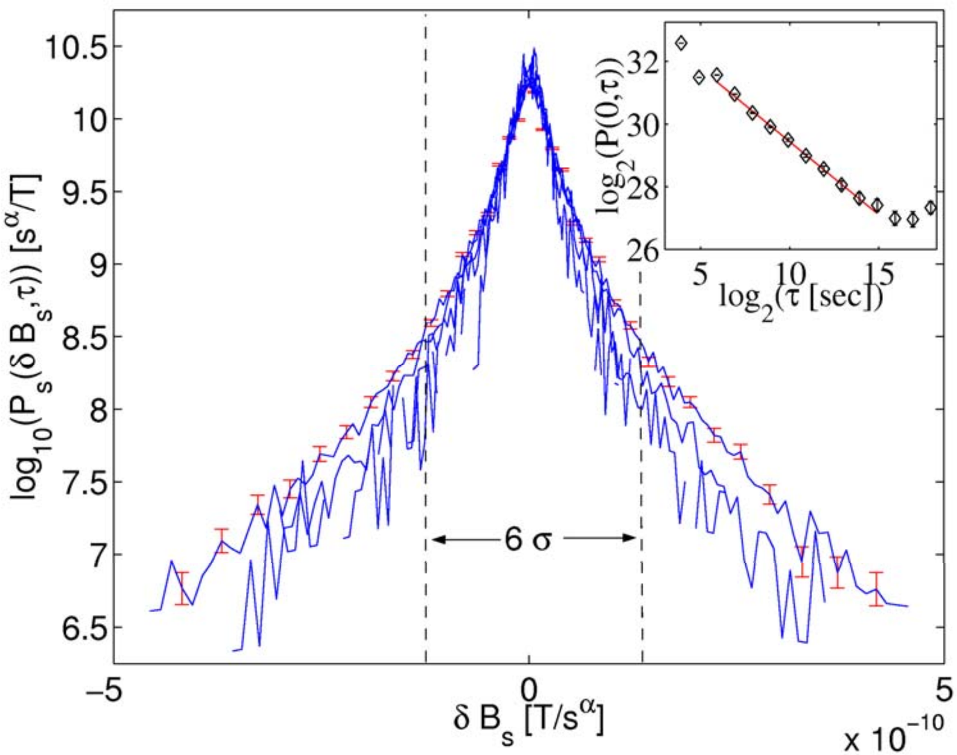
Example- strong multifractal

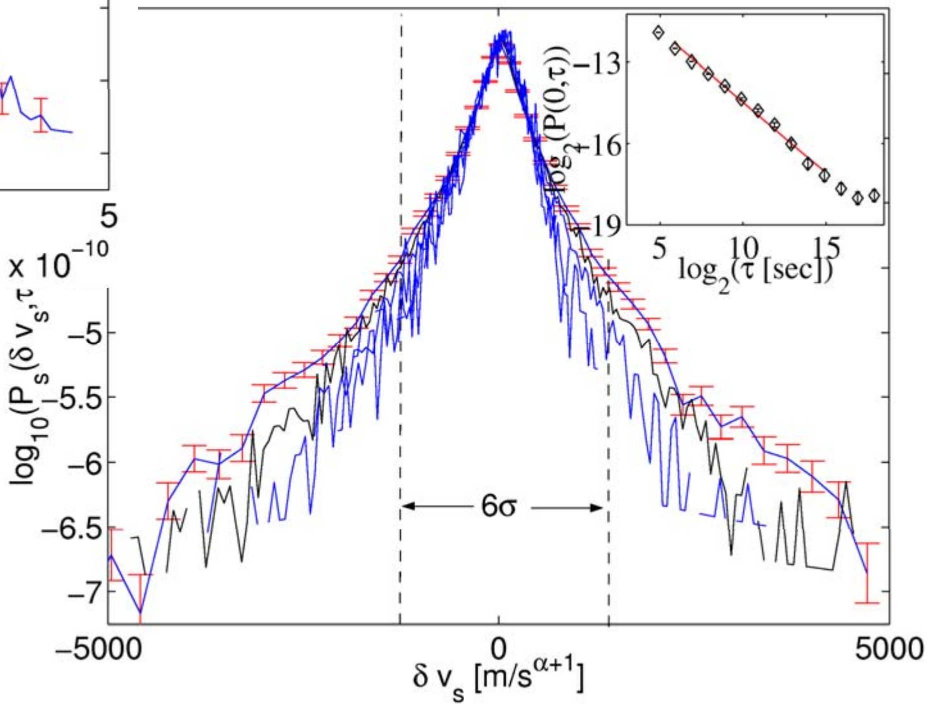
solar wind **v,B**

moments

$$S^m = <\delta x^m> \sim \tau^{\zeta(m)}$$

$\zeta(m)$ quadratic in m





Diffusion- random walk

Brownian random walk

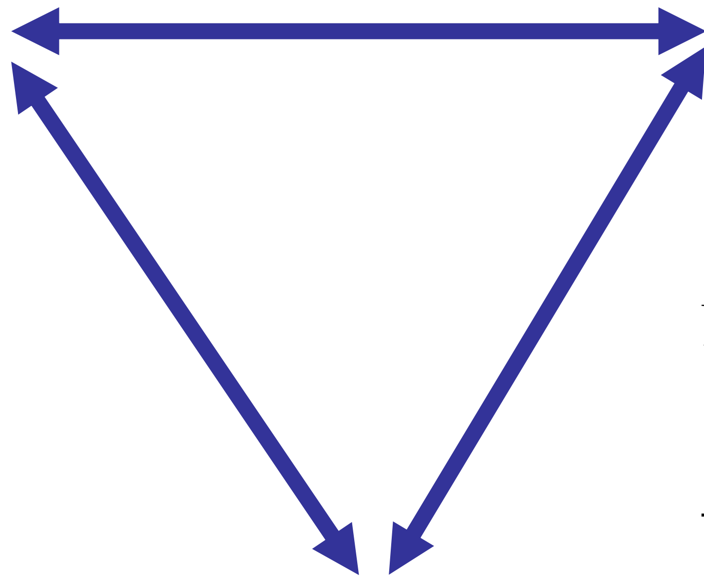
$$\frac{dx}{dt} = \eta$$

η is stochastic iid

diffusion equation

$$\frac{\partial P(y,t)}{\partial t} = D\nabla^2 P(y,t)$$

$\Rightarrow P(y,t)$ is Gaussian



Note: $y(t)$ is distance travelled in interval $t = \tau$ –a differenced variable

Renormalization-scaling system looks the same under

$$t' = \frac{t}{\tau},\ y' = \frac{y}{\tau^{\alpha}} \text{ and } \alpha = \frac{1}{2} \text{..........which implies } P(y',t') = \tau^{\alpha} P(y,t)$$

$\Rightarrow P(y,t)$ is Gaussian, the fixed point under RG

Fokker- Planck models
(see also fractional kinetics and Lévy flights)

Langevin equation

$$\frac{dx}{dt} = \beta(x) + \gamma(x)\eta$$

η stochastic iid

Fokker- Planck equation

$$\frac{\partial P(y,t)}{\partial t} = \nabla(A(y)P(y,t) + B(y)\nabla P(y,t))$$

can solve for $P(y,t)$

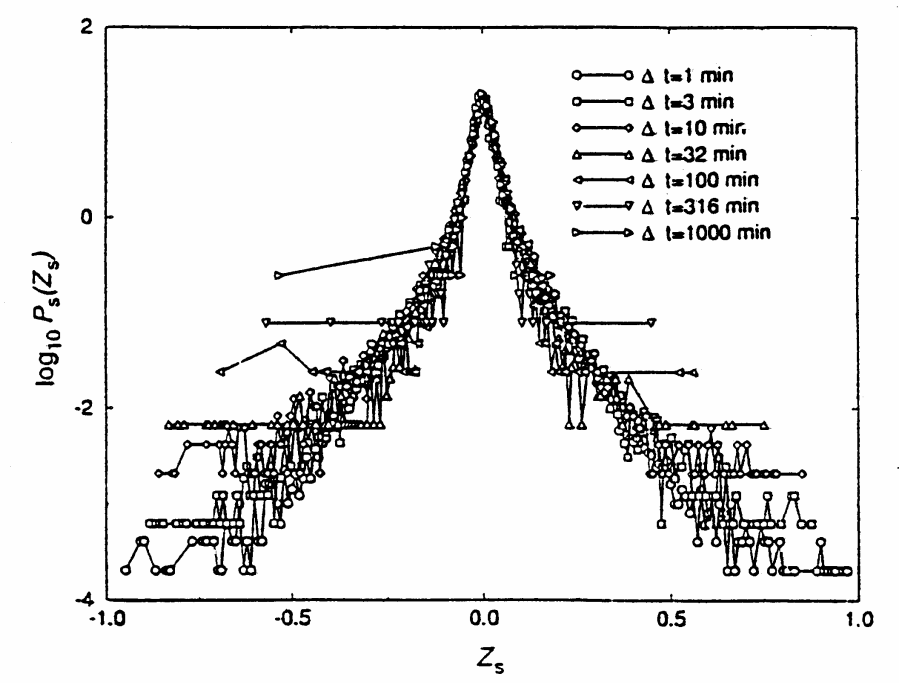
Note: $y(t)$ is distance travelled in interval $t = \tau$ –a differenced variable

Renormalization-scaling system looks the same under

$$t' = \frac{t}{\tau},\ y' = \frac{y}{\tau^{\alpha}} \text{ and } \alpha \neq \frac{1}{2} \text{..........which implies } P(y',t') = \tau^{\alpha} P(y,t)$$

financial markets and SDE models

- Mantegna and Stanley- Nature '95
- S+P500 index
- 'heavy tailed' distributions



The efficient market

- Efficient- arbitrageurs constantly trade to exploit differences in price
- As a consequence any price differences are very short lived
- The market is a ‘fair game’

Implies

- Fluctuations are uncorrelated
- Fluctuations aggregate many (*N*) trades, thus an equilibrium, large *N* model implies Gaussian statistics (CLT)
- Change in price *S*, *dS* in *t*-*t*+*dt* governed by:

$$\frac{dS}{S} = \sigma dX + \mu dt$$

Black-Scholes and all that..

Anticipate a Diffusion equation for $\log(S)$ -since $\frac{dS}{S} = \sigma dX + \mu dt$

provided we have the self- similar scaling for

the stochastic variable dX

I $< dX^2 > \sim dt$

we can write an equation for price evolution

II $dS = A(S,t)dX + B(S,t)dt$

can then write a Taylor expansion for any $f(S)$ using I.

This leads to the B-S SDE for the price of options...

Riskless portfolio $\pi = f(S) + \beta S$, $f(S)$ is an option on stock S

key phenomenology is that of scaling

Nonlinear F-P model for self similar fluctuations- asymptotic result (alternative- fractional kinetics)

If the PDF of fluctuations $y = x(t+\tau) - x(t)$ on timescale τ is selfsimilar:

$P(y,\tau) = \tau^{-\alpha} P_s(y\tau^{-\alpha})$

P is then a solution of a Fokker- Planck equation:

$\frac{\partial P}{\partial \tau} = \nabla\left[AP + B\nabla P\right]$, where transport coefficients $A = A(y), B = B(y)$

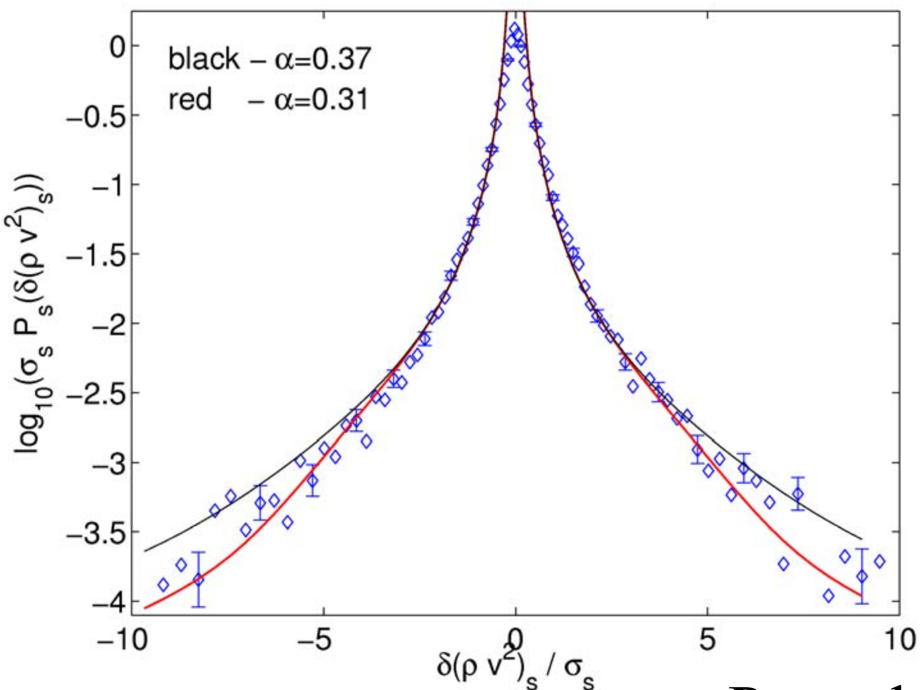
with $A \propto y^{1-1/\alpha}, B \propto y^{2-1/\alpha}$ we solve the Fokker- Planck for P_s

This corresponds to a Langevin equation: $\frac{dx}{dt} = \beta(x) + \gamma(x)\xi(t)$

and we can obtain β, γ via the Fokker- Planck coefficients

see Hnat, SCC et al. Phys. Rev. E (2003), Chapman et al, NPG (2005)

Fokker Planck fit to PDFs



Procedure:

1) Measure exponent
2) Solve FP for PDF functional form
3) Check this fits the observed PDF

Quantifying scaling II

Uncertainties, extreme events, finite size effects

Will discuss structure functions but remarks relate to other measures of scaling

Quantifying scaling II

Calculating exponents- the problems

Structure functions-estimating the $\zeta(p)$ from data

Define structure function (generalized variogram) S_p for differenced timeseries:

$$y(t,\tau) = x(t+\tau) - x(t)$$

$$S_p(\tau) = <|y(t,\tau)|^p> \propto \tau^{\zeta(p)}$$ if scaling

We would like to calculate $S_p(\tau) = <|y(t,\tau)|^p> = \int_{-\infty}^{\infty} |y|^p \, P(y,\tau) dy$

then $S_p(\tau) = \tau^{\zeta(p)} \int_{-\infty}^{\infty} y_s^p P_s dy_s$

Conditioning- an estimate is:

$<|y|^p> = \int_{-A}^{A} |y|^p \, P(y,\tau) dy$ where $A = [10-20]\sigma(\tau)$

strictly ok if selfsimilar: $y \rightarrow y_s \tau^{\alpha}, P \rightarrow P_s \tau^{-\alpha}$, $\zeta(p) = p\alpha$

if ξ(p) is quadratic in p (multifractal)- weaker estimate

Theory-data comparisons- examples

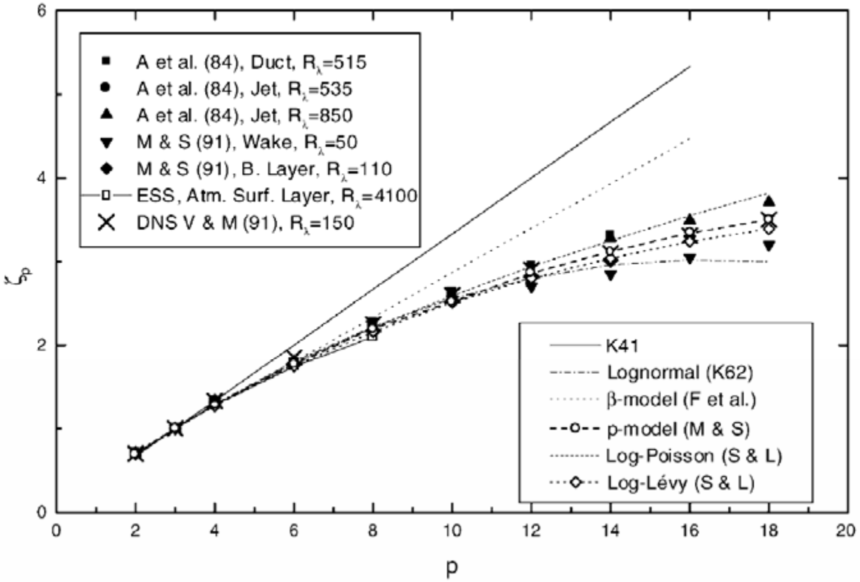


Fig. 11. Power-law exponents ζ_p of the structure functions as a function of the order p, together with the values predicted by K41 and the various intermittency models of Table 1.

Fluid experiments,
Anselmet et al, PSS, 2001

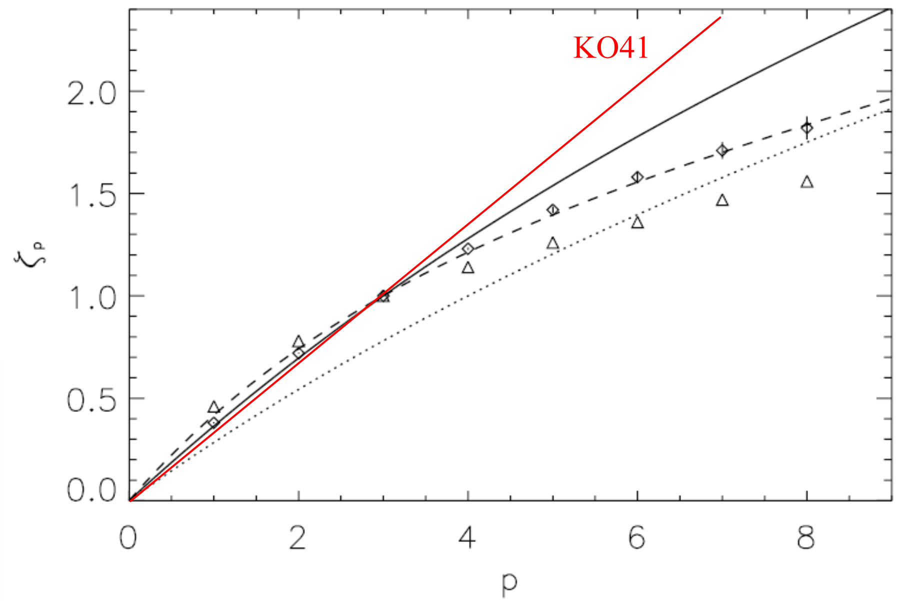


FIG. 4. Scaling exponents ζ_p^+ for 3D MHD turbulence (diamonds) and relative exponents ζ_p^+/ζ_3^+ for 2D MHD turbulence (triangles). The continuous curve is the She-Leveque model ζ_p^{SL}, the dashed curve the modified model ζ_p^{MHD} (7), and the dotted line the IK model ζ_p^{IK}.

2 and 3D MHD simulations
Muller & Biskamp PRL 2000

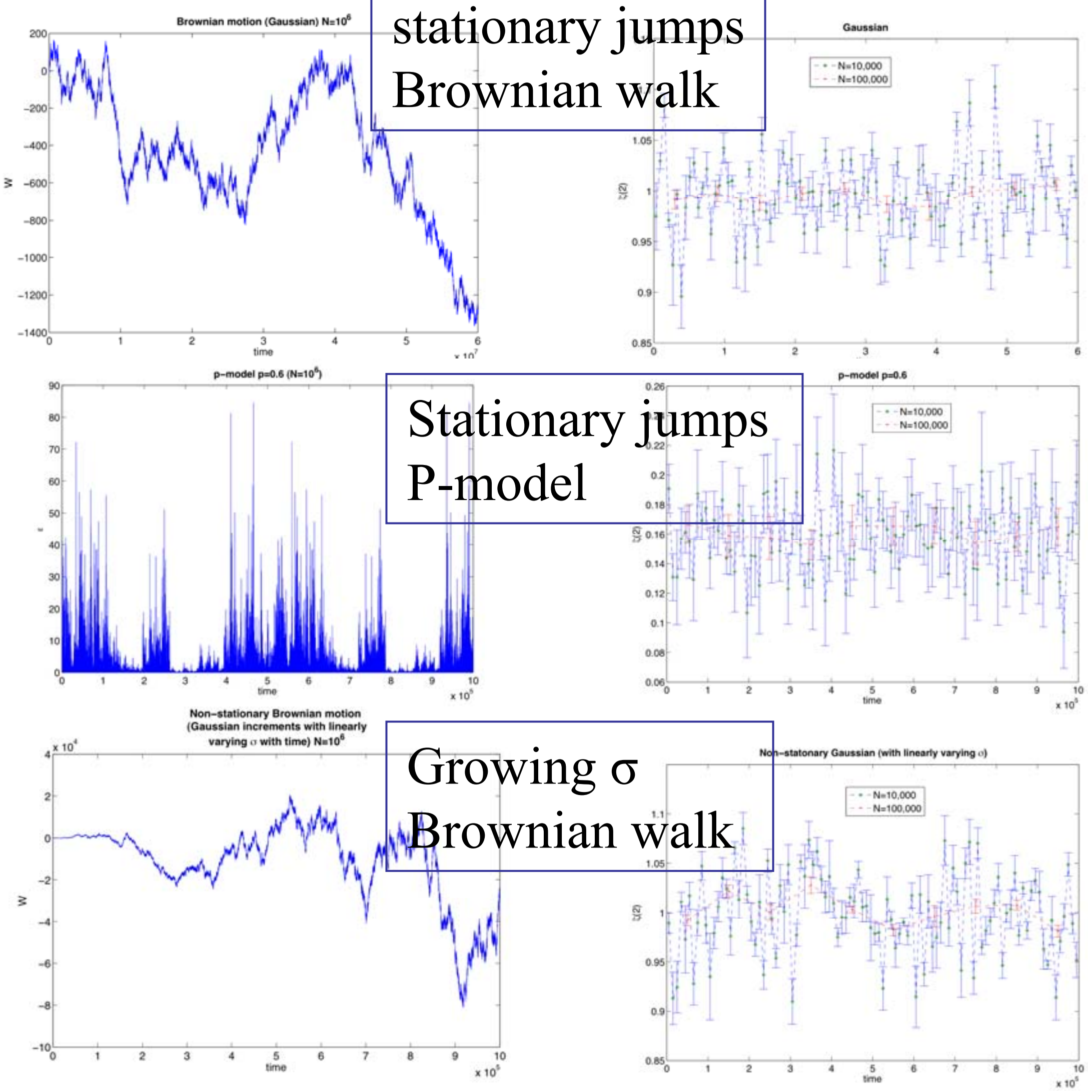
How large can we take p? *See eg Dudok De Wit, PRE, 2004*

Finite sample effect- Brownian walk and p- model

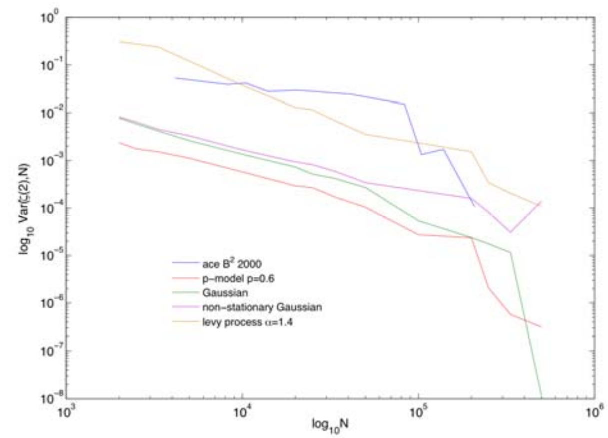
stationary jumps
Brownian walk

Stationary jumps
P-model

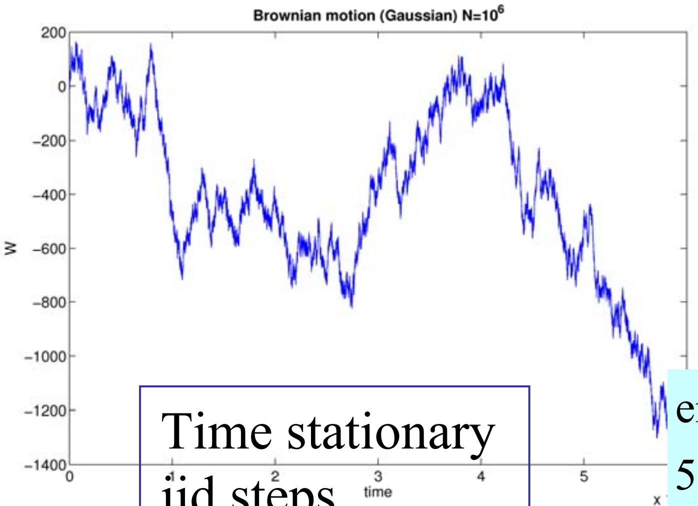
Growing σ
Brownian walk

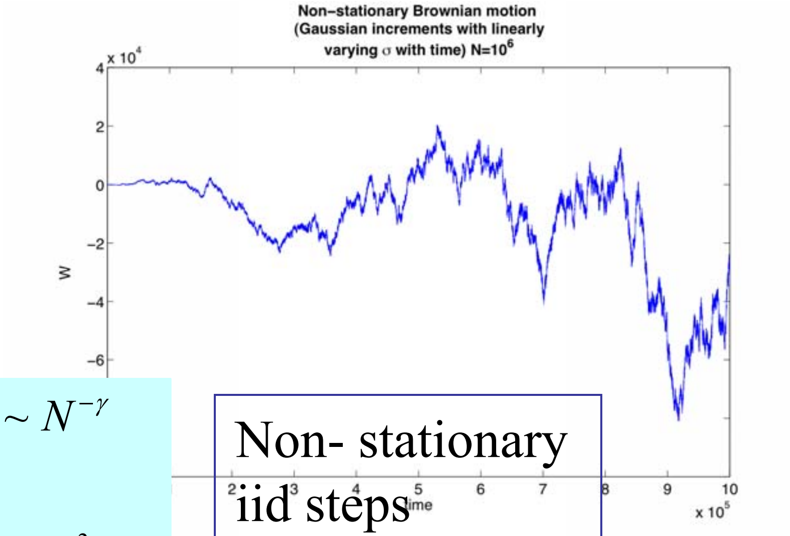


Shown- $\zeta(2)$
from consecutive
intervals, $N=10^5, 10^6$



Finite sample effect- error on exponent ζ(2) as a function of sample size N





Time stationary
iid steps

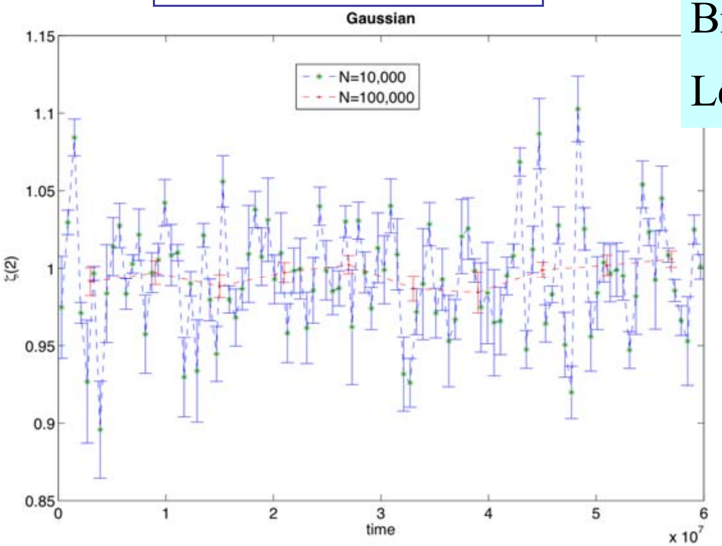
error on $\zeta(2) \sim N^{-\gamma}$

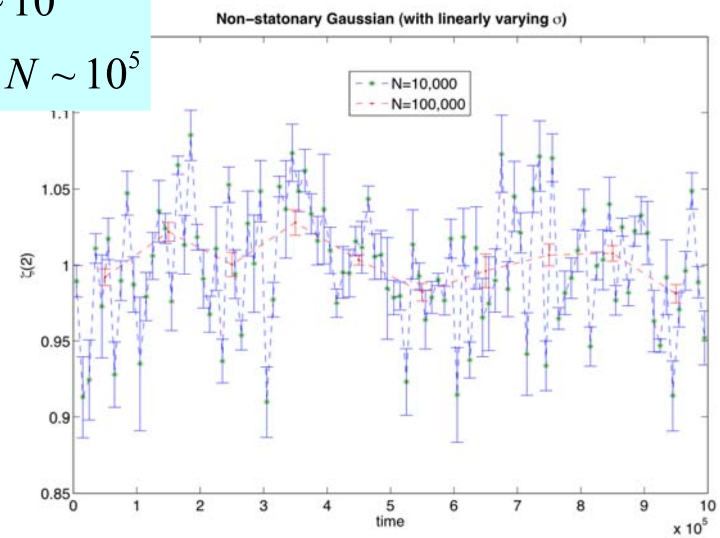
5% error:

Brownian: $N \sim 10^3$

Levy, p-model $N \sim 10^5$

Non- stationary
iid steps

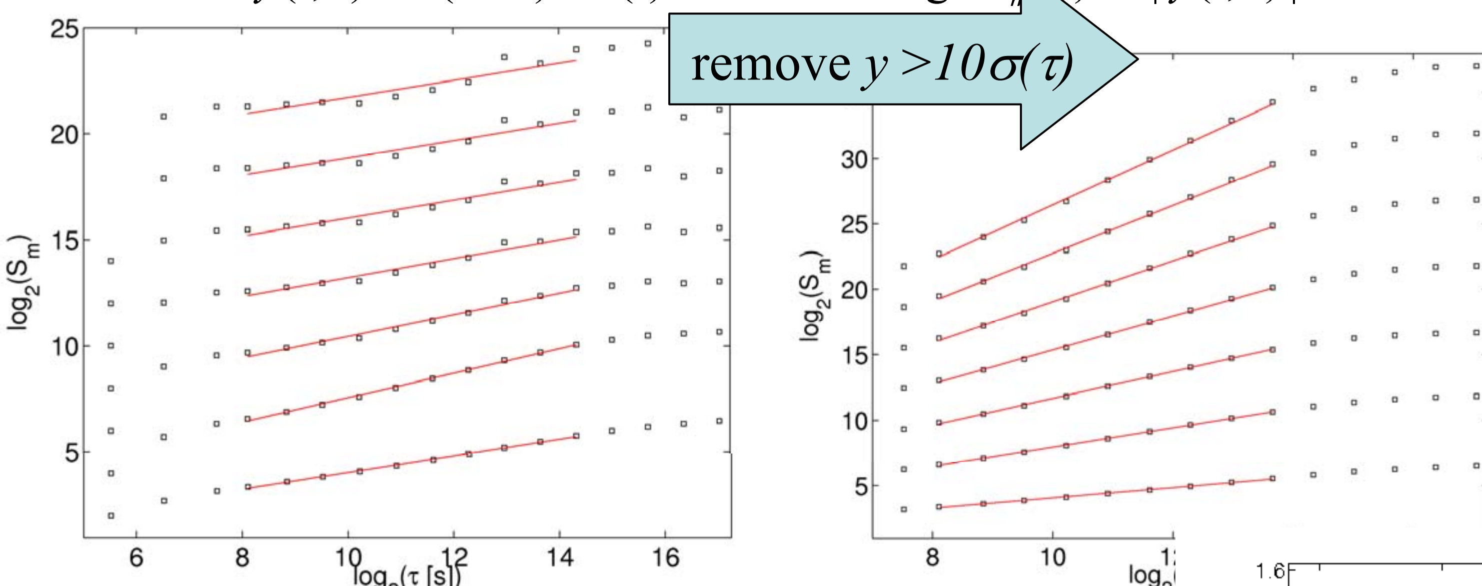




Kiyani, SCC et al, PRE, 2009. See also Dudok De Wit, PRE, 2004

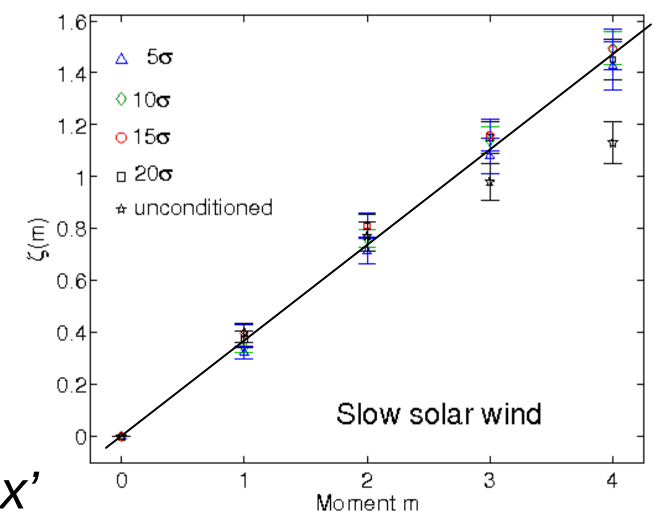
Structure functions- sensitive to undersampling of largest events (example - ρ in slow sw)

$y(t,\tau) = x(t+\tau) - x(t)$ test for scaling - $S_m(\tau) = \langle | y(t,\tau) |^m \rangle \propto \tau^{\zeta(m)}$



2 sources of uncertainty in exponent

1) Fitting error of lines (error bar estimates)
2) Outliers- Shown: removed < 1% of the data
   ACE 98-01 (4years)-10^6 samples.
   Threshold 450 km/sec.



fractal or multifractal?

fractal (self- affine) $\zeta(p) \sim \alpha p$

multifractal $\zeta(p) \sim \alpha p - \beta p^2 + \ldots$

cf Fogedby et al PRE 'anomalous diffusion in a box'

Quantifying scaling II

Things we can calculate (to some precision with finite datasets)-

Some tricks

Trick

Pose the question such that it is (relatively) easy to answer- low order moments, values far apart…

Exponents are hard to measure- ask questions that don't need precise measurements!

Example:

Bershadskii and Sreenivasan PRL '04 argued that in MHD turbulence |B| is passive scalar..

Appeal to ***universality*** in scaling exponents (same physics, same scaling)

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + v \bullet \nabla Q = 0$$

e.g.

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho v) = 0 = \frac{\partial \rho}{\partial t} + v \bullet \nabla \rho$$

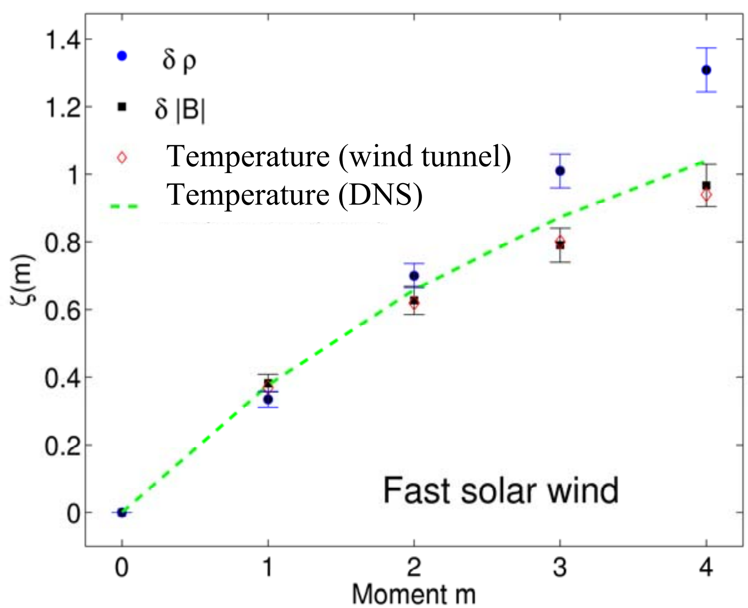
with $\nabla \bullet v = 0$ incompressible flow

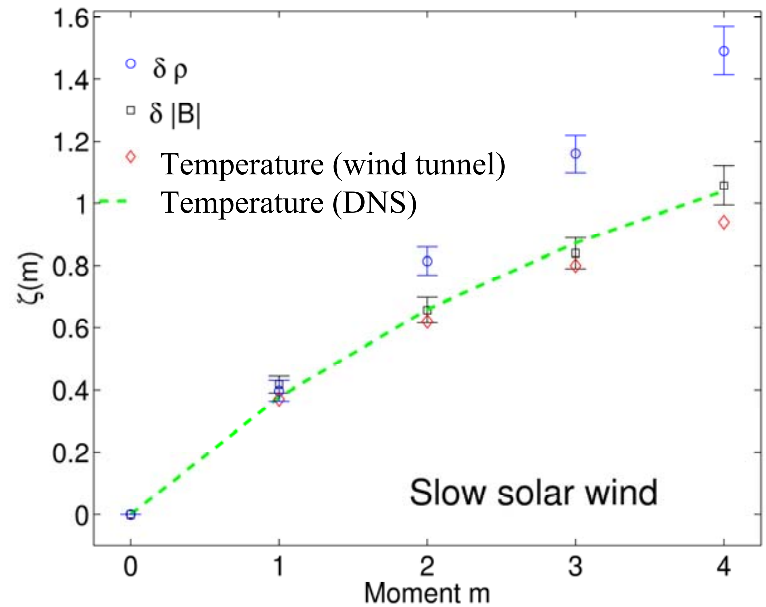
if the flow is incompressible- ρ must be a passive scalar-

question- does ρ have the same exponents as B?

Passive scalars comparison

does not need to be so precise..





ρ is not passively advected with the flow?

Hnat, SCC et al PRL '05

1 year ACE data (1998)

Compare ρ with passive scalars:

Conditioned |B| (same dataset), + others

Argued that |B| is passive scalar..

Bershadskii and Sreenivasan PRL '04

intermittency free parameters in cascades- determination of anomalous scaling exponents

example: Kolmogorov vz MHD scaling

velocity difference $d_r v = v(l+r) - v(l)$, energy transfer rate $\varepsilon_r \sim \dfrac{d_r v^2}{T}$

Kolmogorov: simply have T as the eddy turnover time $T \sim r/d_r v$ so that $\varepsilon_r \sim \dfrac{d_r v^3}{r}$

MHD: now T is due to (say) Alfvenic collisions $T \sim \dfrac{r}{d_r v}\left(\dfrac{v_0}{d_r v}\right)^{\alpha}$ giving $\varepsilon_r \sim \dfrac{d_r v^{3+\alpha}}{r}$

intermittency $\langle \varepsilon_r^{\ p} \rangle \sim \bar{\varepsilon}^{\,p} \left(r/L \right)^{\tau(p)}$

$\Rightarrow$ Kolmogorov: $\langle d_r v^p \rangle \sim r^{p/3} \bar{\varepsilon}^{\,p/3} \left(L/r \right)^{\tau(p/3)} \sim r^{\zeta(p)}$

$\Rightarrow$ MHD: same with $p/3 \rightarrow p/(3+\alpha)$ intermittency free $E(k) \sim \langle dv^2 \rangle / k \sim k^{-(5+\alpha)/(3+\alpha)}$

$\langle \varepsilon_r \rangle = \bar{\varepsilon}$ independent of r (steady state) so $\tau(1) = 0$ and $\zeta(\alpha+3) = 1$

what is α?

Kolmogorov Obukhov (1941) hydrodynamic: α=0

Irosnikov Kraichnan (1964) weak isotropic MHD α=1,

Goldreich Sridhar (1994-5) strong MHD $\alpha_{\perp} = 0$

Boldyrev (2005) strong, background field anisotropic MHD $\alpha_{\perp} = 1$

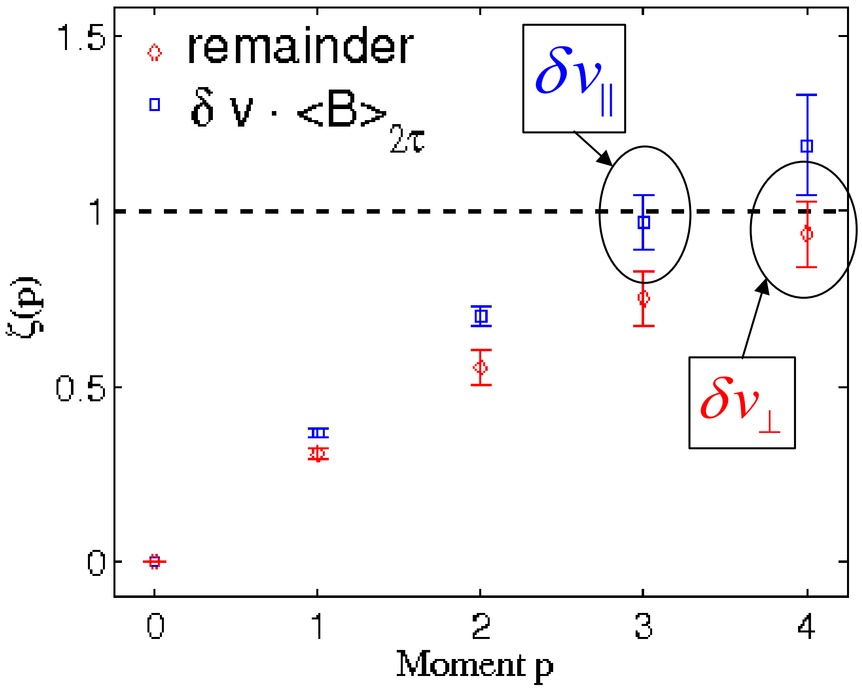
Solar wind example: Velocity fluctuations parallel and perpendicular to the *local* B field direction

Exponents $\zeta(p)$ for $<\left|\delta v_{\parallel,\perp}\right|^p>\sim \tau^{\zeta(p)}$ for

$\delta v_{\parallel} = \delta \mathbf{v}.\hat{\mathbf{b}}$ and its remainder $\delta v_{\perp} = \sqrt{\delta\mathbf{v}\bullet\delta\mathbf{v} - \left(\delta\mathbf{v}.\hat{\mathbf{b}}\right)^2}$

$\zeta(3+\alpha)=1$ determines phenomenology

$\overline{\mathbf{B}} = \mathbf{B}(t) + \ldots + \mathbf{B}(t+\tau')$, $\hat{\mathbf{b}} = \dfrac{\overline{\mathbf{B}}}{\left|\overline{\mathbf{B}}\right|}$,here $\tau' = 2\tau$ and $\delta\mathbf{v} = \mathbf{v}(t+\tau) - \mathbf{v}(t)$



ACE 64s av. 1998-2001 *Chapman et al GRL (2007)*

Trick

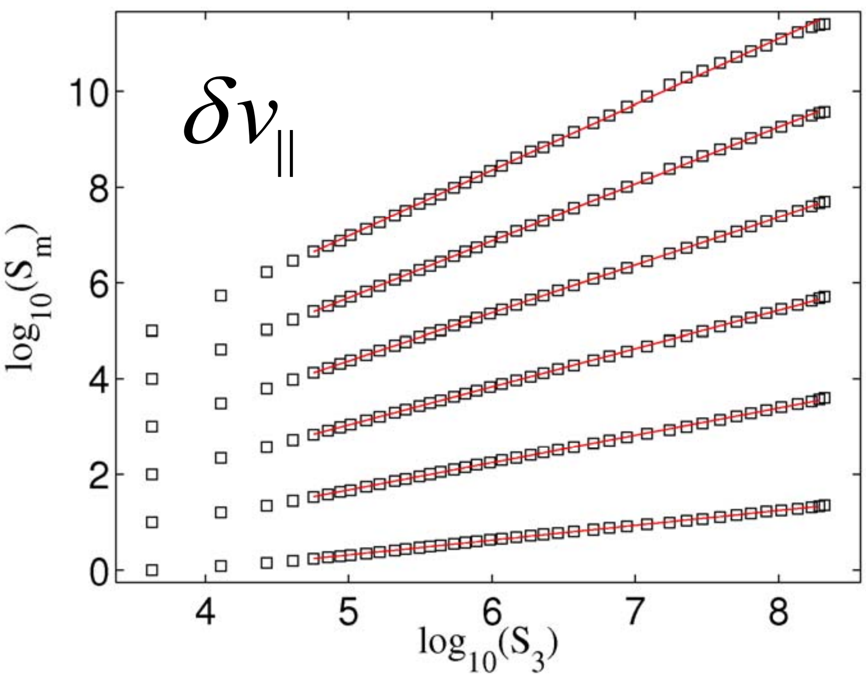
Extended Self Similarity (ESS)

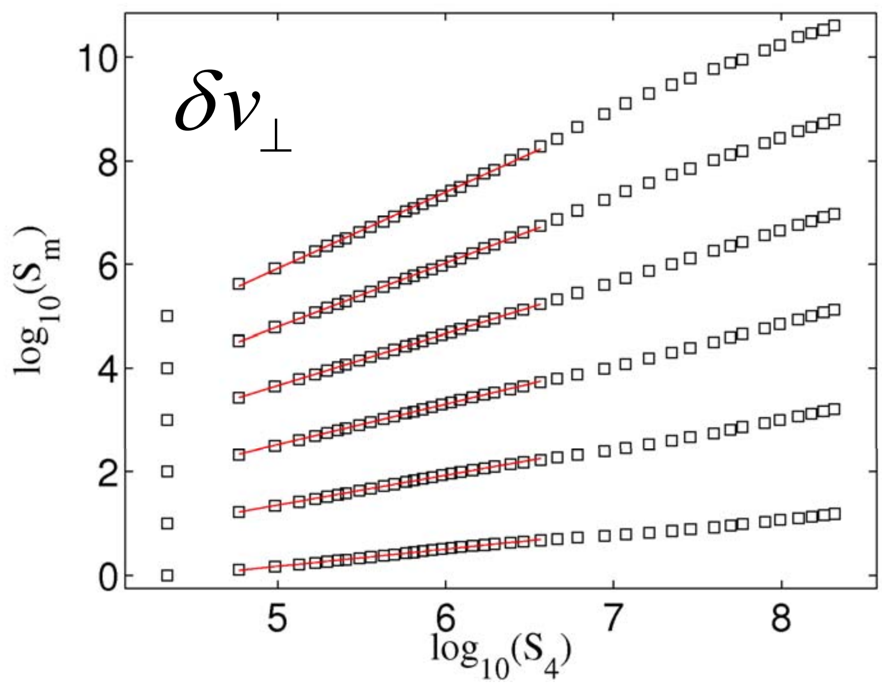
Generalized or extended self simlarity- ESS plots:

$S_p = <\left|\delta \mathbf{v} \cdot \hat{\mathbf{b}}\right|^p>$ and its remainder versus S_3, S_4

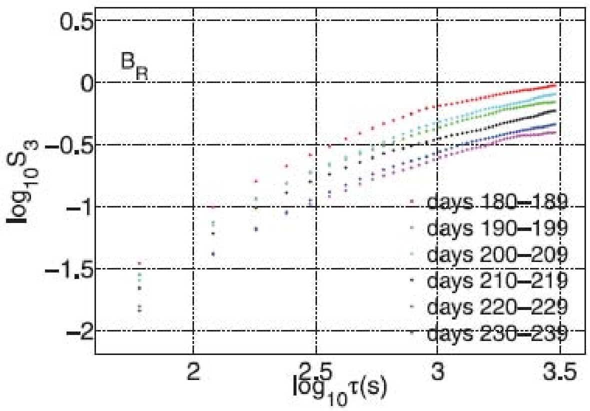
ESS tests $S_p = S_q^{\zeta(p)/\zeta(q)}$ i.e. $S_p \sim G(\tau)^{\zeta(p)}$

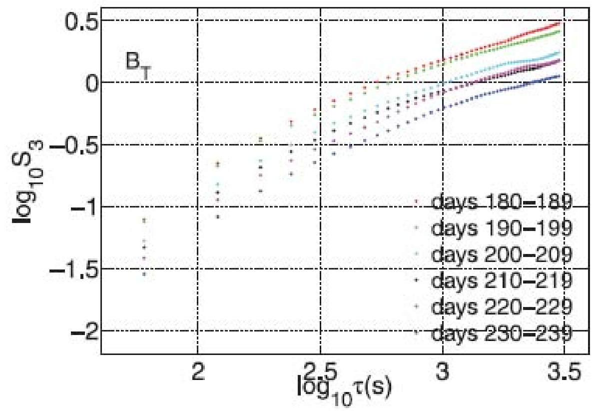
gives exponents when e.g. $\zeta(3) \approx 1$ or $\zeta(4) \approx 1$

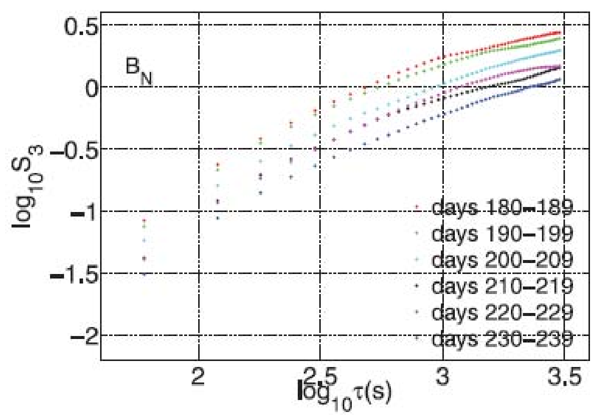




Quiet, fast solar wind-ULLYSES polar passes- evolving MHD turbulence







IR turbulence- expect

$S_3 \sim \tau^{\zeta(3)}$

i.e. straight line on log-log plot

not quite seen here!

Nicol, SCC et al, Ap J., (2008)

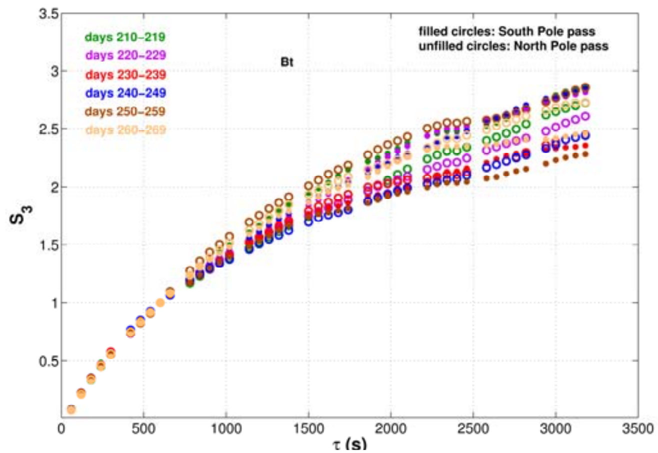
Generalized similarity (scaling)- turbulence at the outer scale

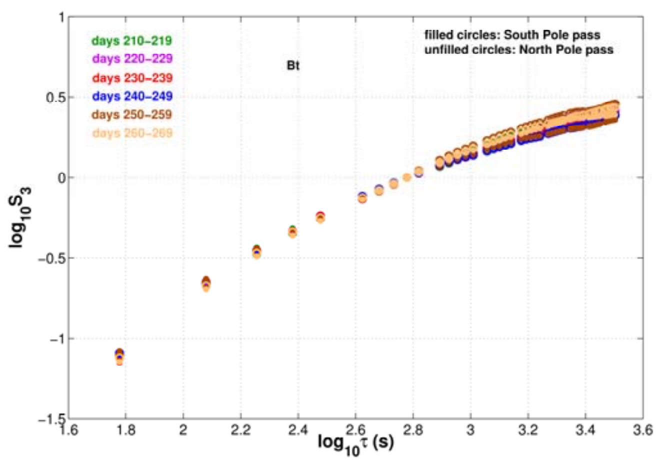
South pass 1994, North pass 1995, solar min

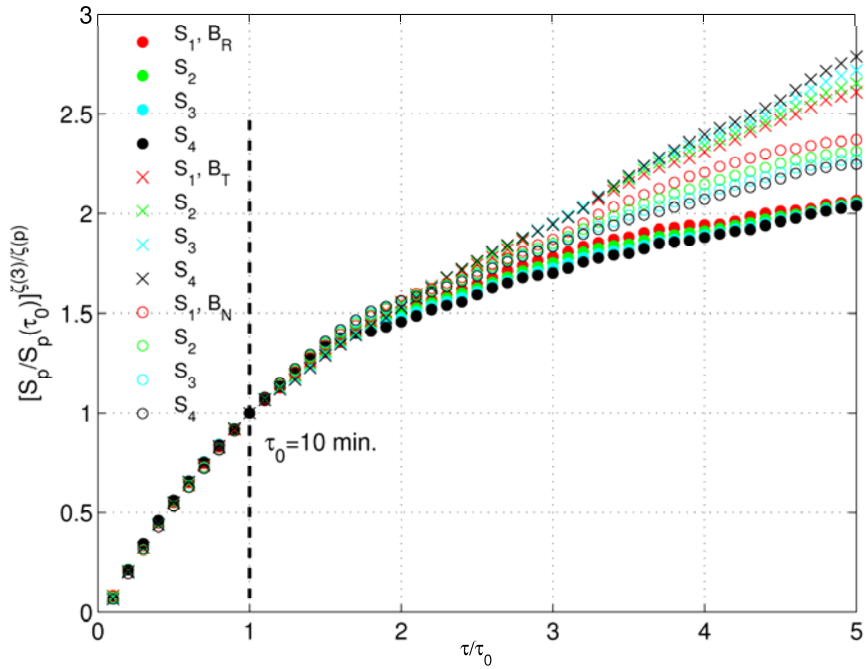
$S_p \sim g(\tau)^{\xi(p)}$

invert to obtain $g(\tau)$

same $g(\tau)$ seen







Trick-

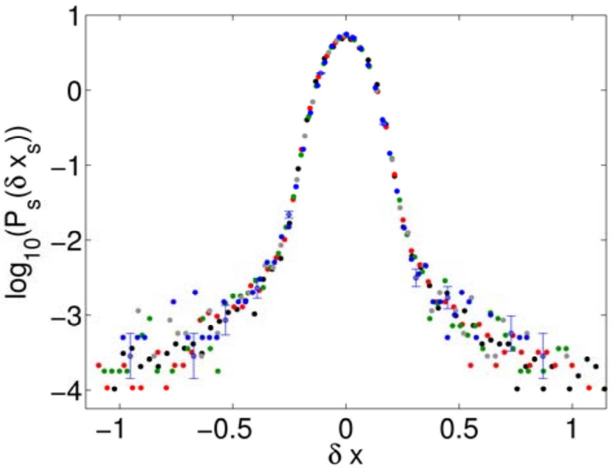
Use the fact that self- affine process only requires one exponent to rescale the PDF..

A more precise test for fractality- the effect of extremes: example-Lèvy flight

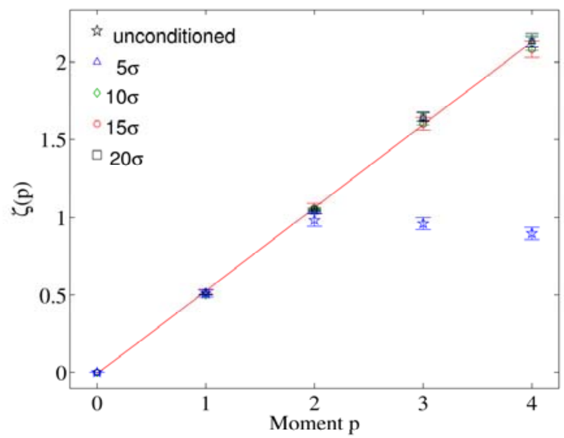
$P(x) \sim \frac{C}{x^{1+\mu}}, x \to \pm\infty, 1 < \mu < 2$ power law tails, self similar

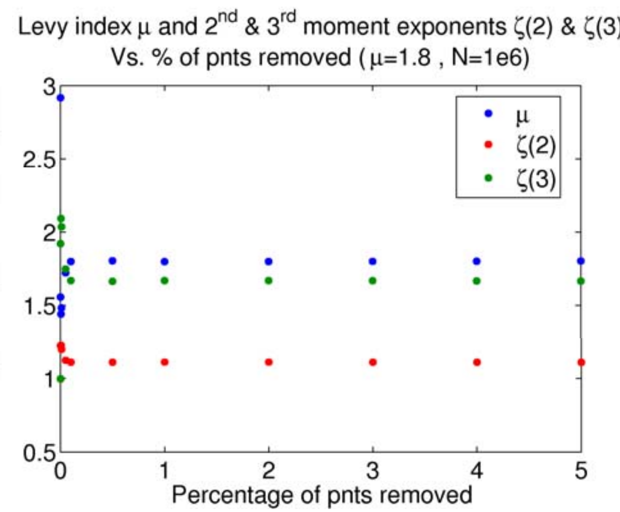
for a finite length flight $(x - \langle x \rangle)^2 \sim t^{2/\mu}$

so $\mu = 2$ is Gaussian distributed, Brownian walk



PDF rescaling $x \to x_s \tau^{\alpha}, P \to P_s \tau^{-\alpha}$





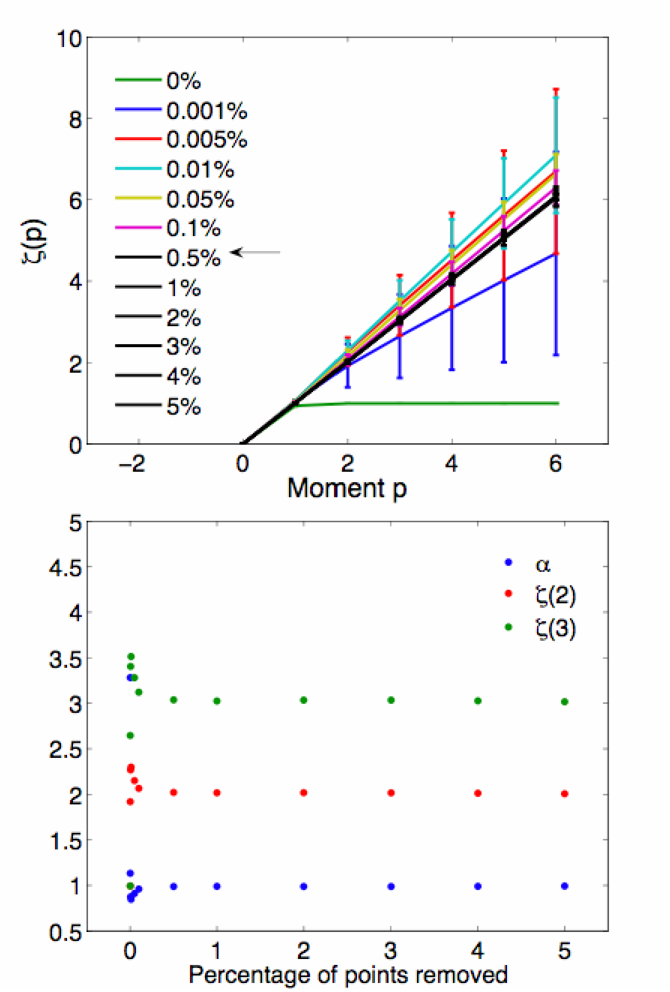
Structure functions: $S_p(\tau) = \langle | x(t,\tau) |^p \rangle \propto \tau^{\zeta(p)}$

expect $\zeta(p) \sim \alpha p, \alpha = 1/\mu$

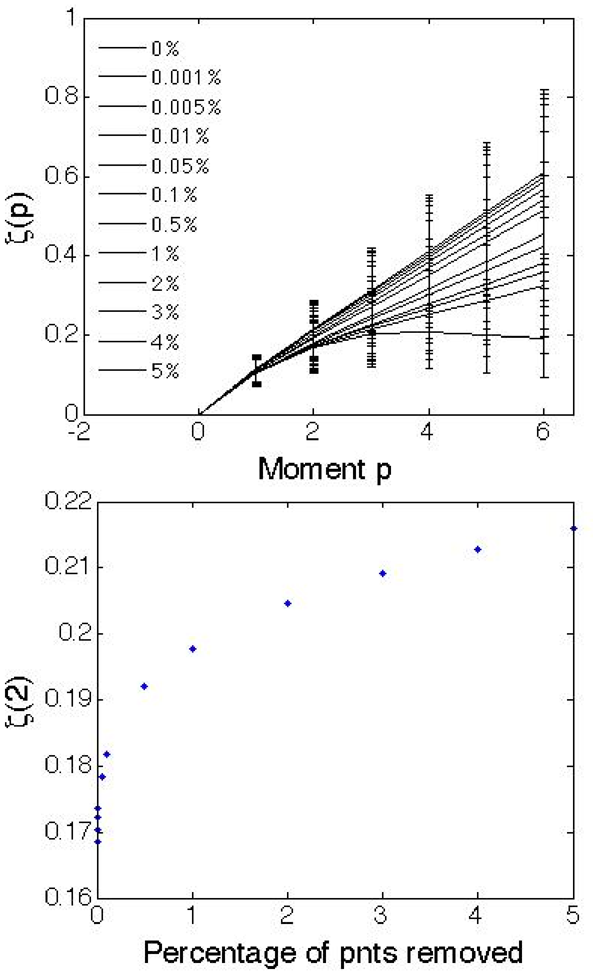
Chapman et al, NPG, 2005,Kiyani, SCC et al PRE (2006)

Distinguishing self- affinity (fractality) and multifractality

Levy flight -- Fractal



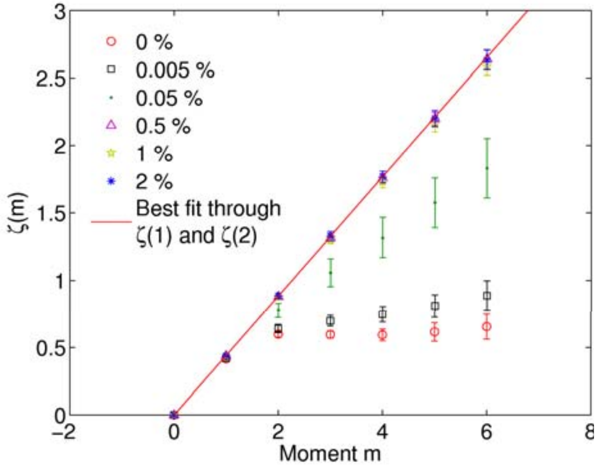
P-model -- Multifractal



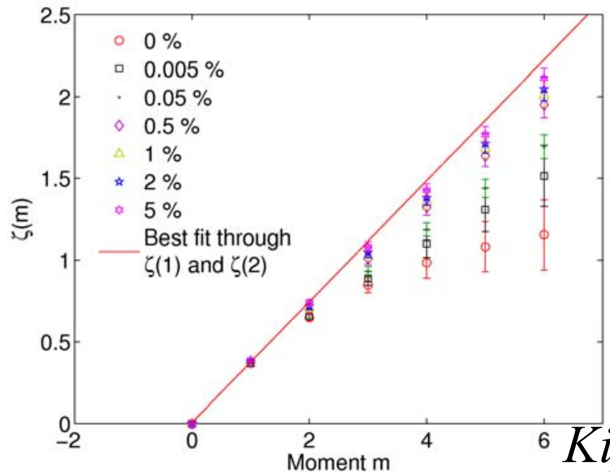
Kiyani, SCC et al, PRL (2007)

Example: solar wind solar cycle variation WIND -- $|B|^2$

2000 - Solar max

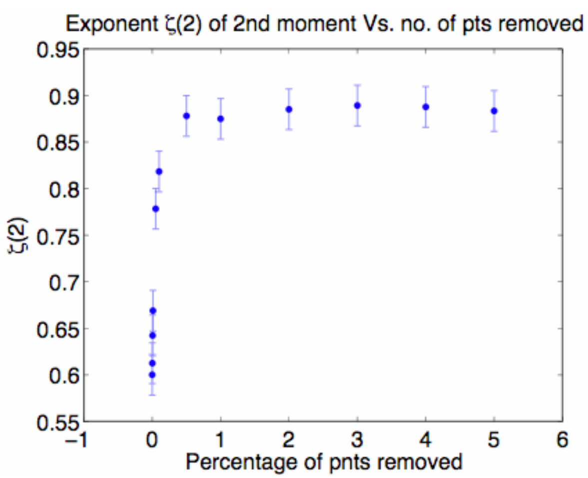


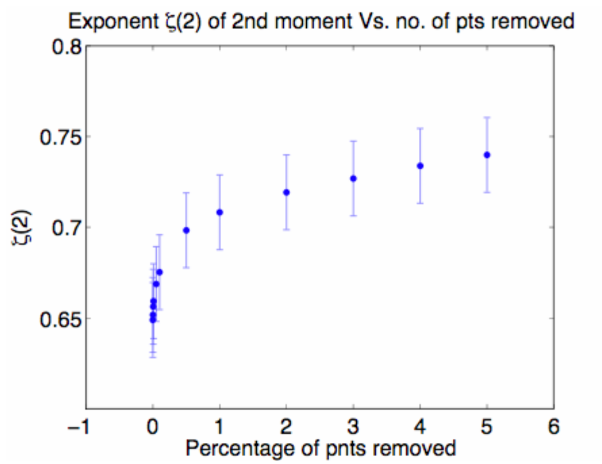
1996 - Solar min



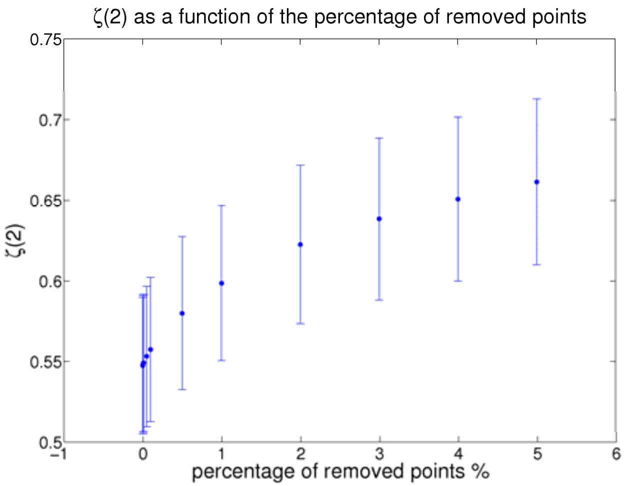
Fractal signature ‘embedded’ in (multifractal) solar wind inertial range turbulence -coincident with complex coronal magnetic topology

Kiyani, SCC et al, PRL (2007), Hnat, SCC et al, GRL, (2007)

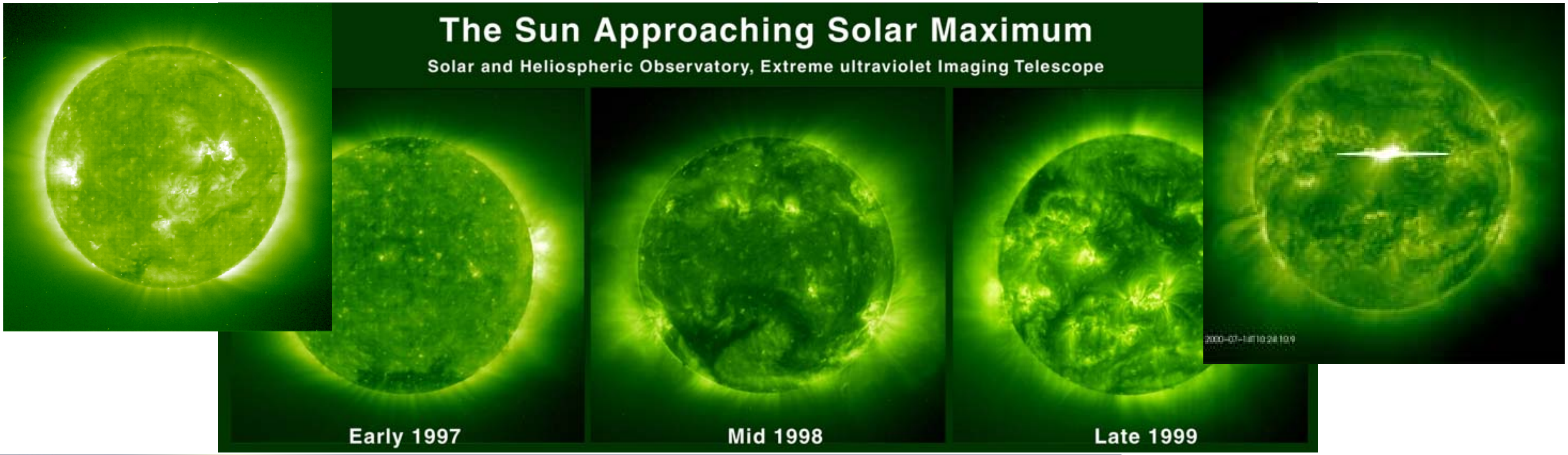




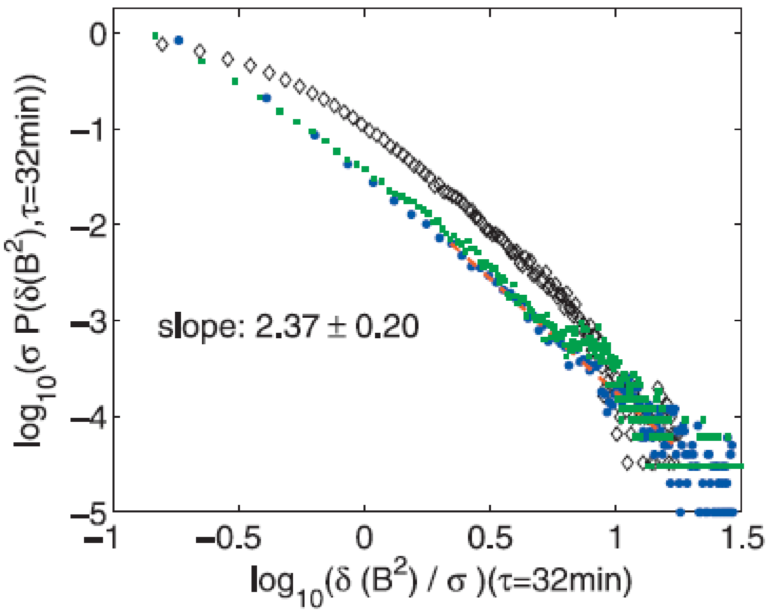
ULYSSES- north polar pass at solar minimum

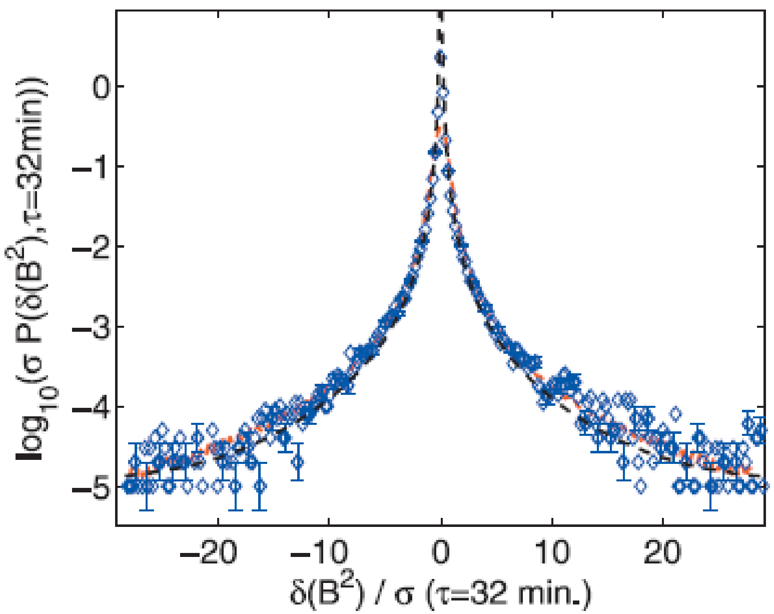


ULYSSES 60s averages
July-Aug 1995, ~8.5×10^4 points, selected as a quiet interval
-Multifractal
-Fractality coincides with topologically complex coronal fields?



Left: B² fluctuation PDF solar max and solar min
Right: solar max, FP and Lévy fit





WIND 1996 min (◊), 2000 max (◦), ACE 2000 max (□)

Hnat, SCC et al, GRL, (2007)

Statistics of ‘bursts’

Avalanche distributions, waiting times

Avalanching systems and scaling behaviour

Avalanche models: add grains slowly, redistribute only if local gradients exceeds a critical value

Suggested as a model for bursty transport and energy release in plasmas- solar corona, magnetotail, edge turbulence in tokamaks (L-H), accretion disks

Avalanching systems

- Threshold for avalanching
- Avalanches are much faster than feeding rate
- Avalanches on all sizes, no characteristic size
- Feeding rate=outflow rate on average only
- System moves through many metastable states- rather than toward an equilibrium

Measures of ‘burstiness’

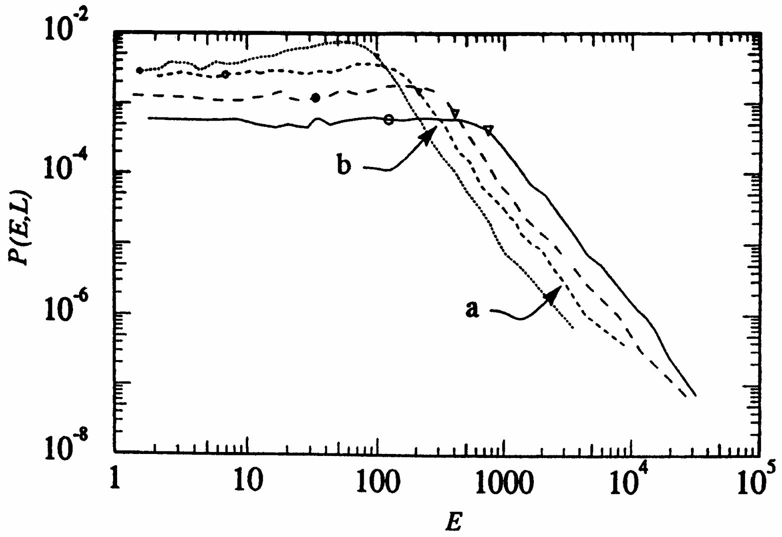
Statistics of:

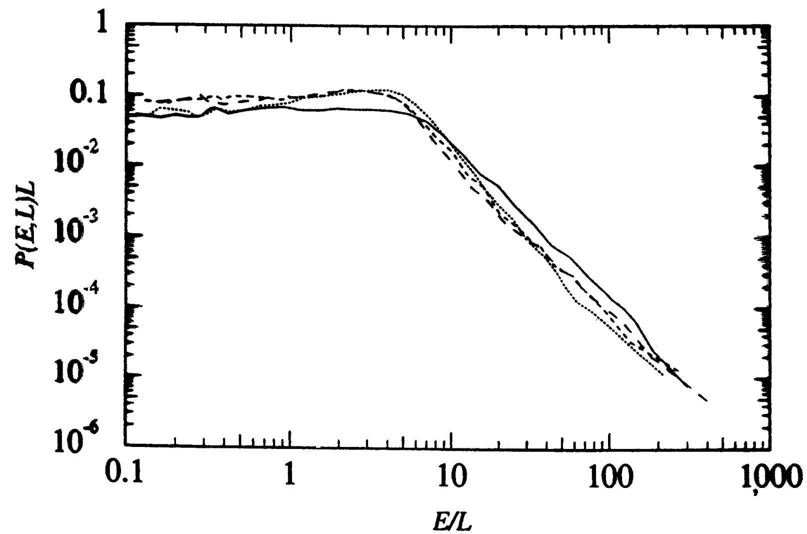
- Waiting time between events
- Energy dissipated
- Peak size
- Duration

Questions:

- Scaling? PDF, CDF, rank order plots etc
- Finite size scaling?

Statistics of avalanches (rice)





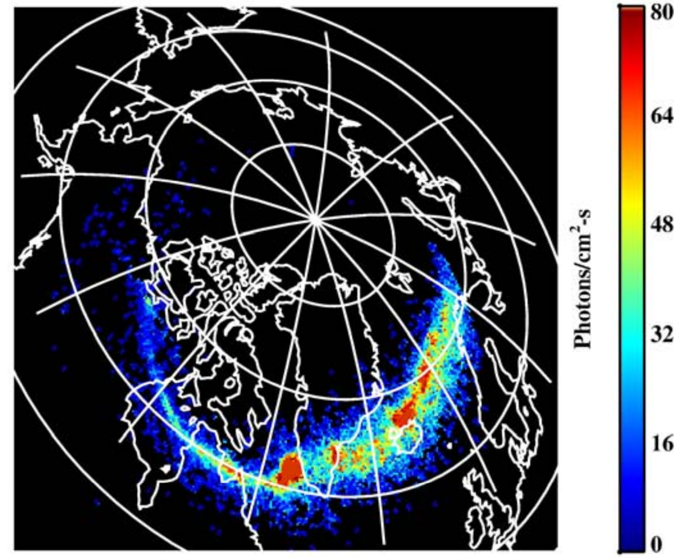
Shown: Statistics of energy dissipated per avalanche

- Power law- no characteristic event size: scaling
- 'finite size scaling'- Normalize to the size of the box

Frette et al, Nature (1996)

- Dynamical quantity- rice
- Flux is conserved
- d.o.f. are the possible avalanche (sizes/topplings)

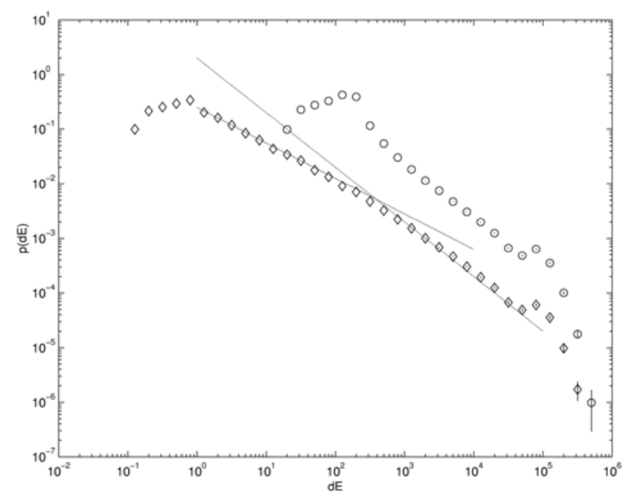
The dynamic aurora- a window on an avalanching system?



Shown, POLAR UVI image of the earths' aurora

Has been proposed as a candidate avalanching system

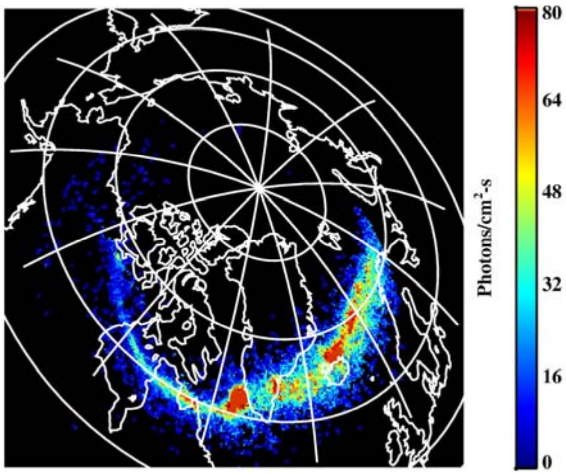
SCC et al GRL 1998

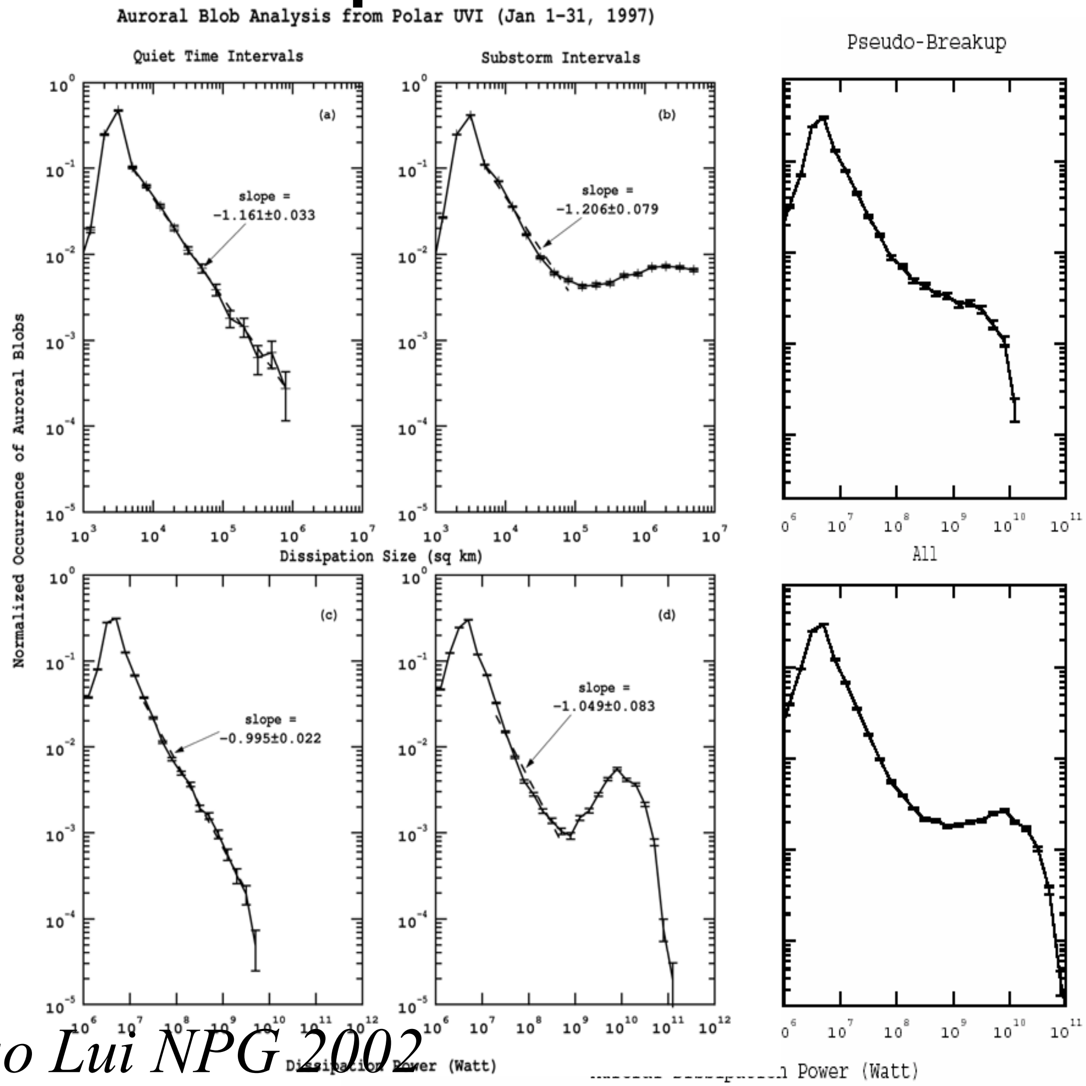


BUT there is also magnetotail turbulence..

Counting auroral snapshot 'blobs'

- 1 month of POLAR UVI data=200,000 'blobs'
- Quiet and active times
- Robust power law(?)
- +substorms



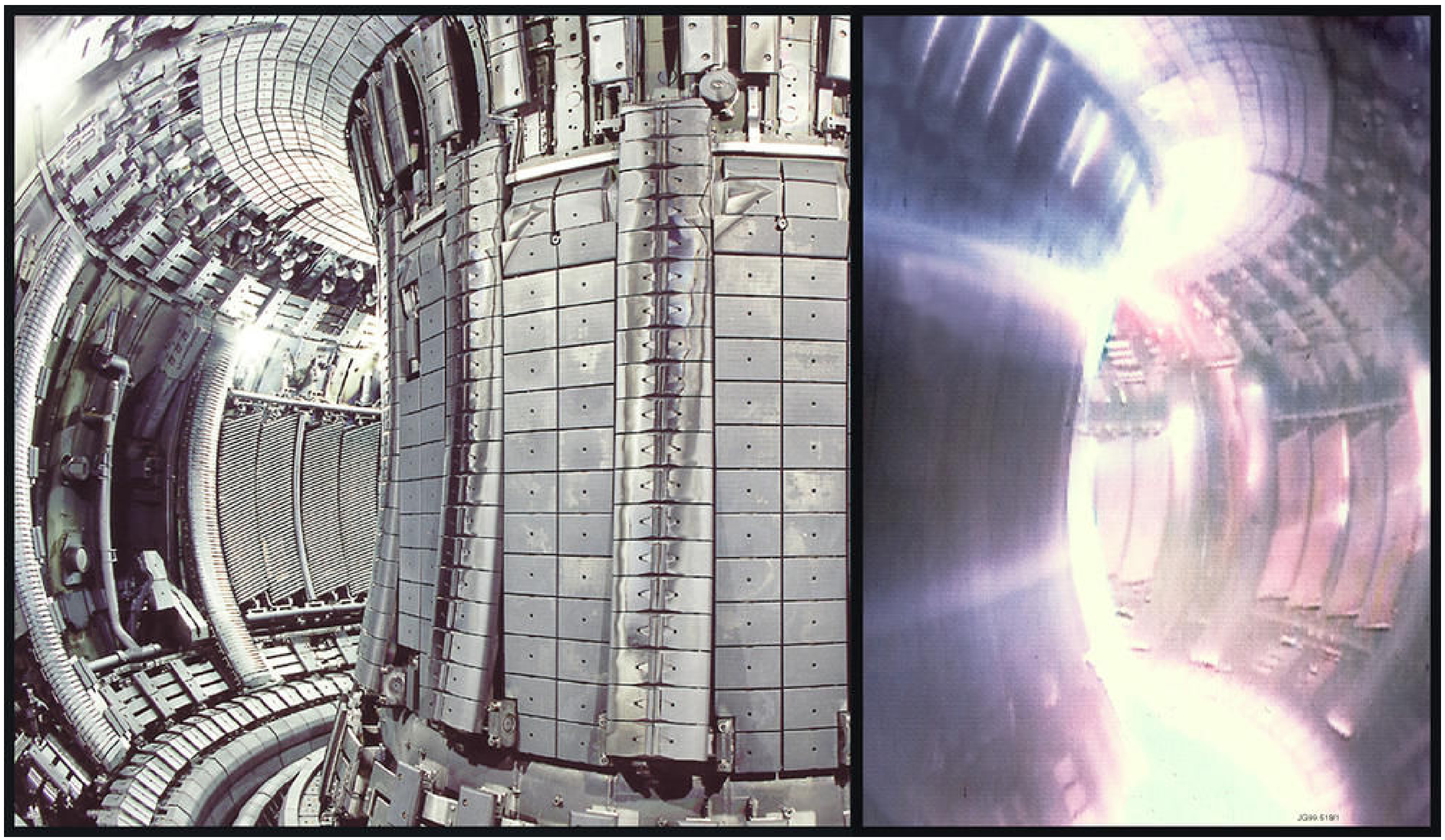


Lui et al GRL, 2000, see also Lui NPG 2002

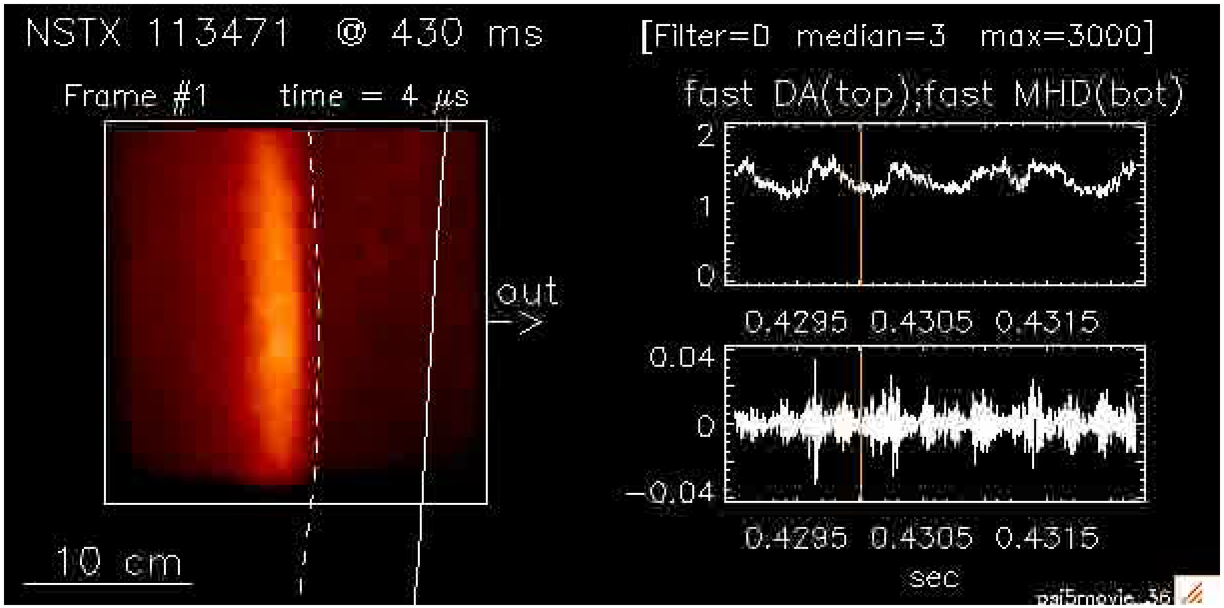
In the Laboratory

Anomalous plasma transport- an avalanche process?

L, H mode (confinement states)- a transition?



Bursty plasma ‘turbulent’ transport- magnetically confined plasmas



Movies of edge turbulence on NSTX in 2004.
S.J. Zweben et al, (2004) ,R.J. Maqueda, (2003)
Avalanche model with L-H transition- *SCC et al PRL (2001)*
See SCC et al Phys Plasmas (2009) for a comprehensive list of refs…

Blob statistics- Edwards Wilkinson- dynamics

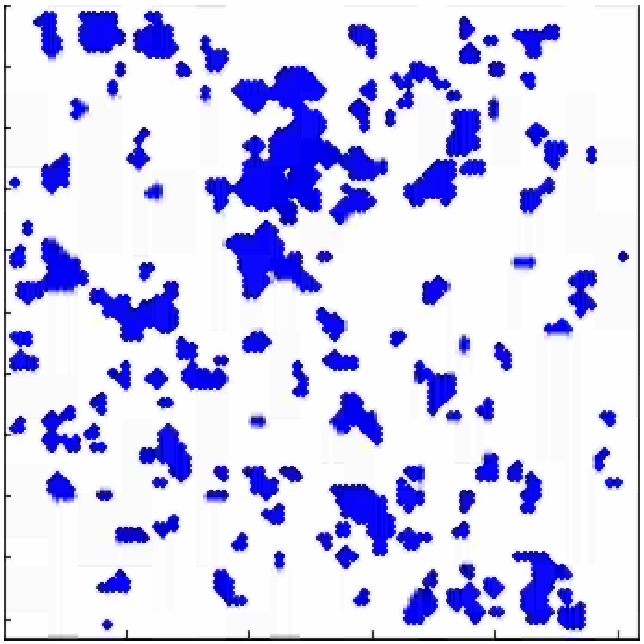
A *linear* model
Shown: 100^2 grid D=0.3
Solves:

$$\frac{\partial \bar{h}}{\partial t} = D\nabla^2 \bar{h} + \eta$$

where η is iid 'white' random source of grains

'height' $\bar{h} = h - \langle h \rangle$

blue patches are $\bar{h} > h_0$



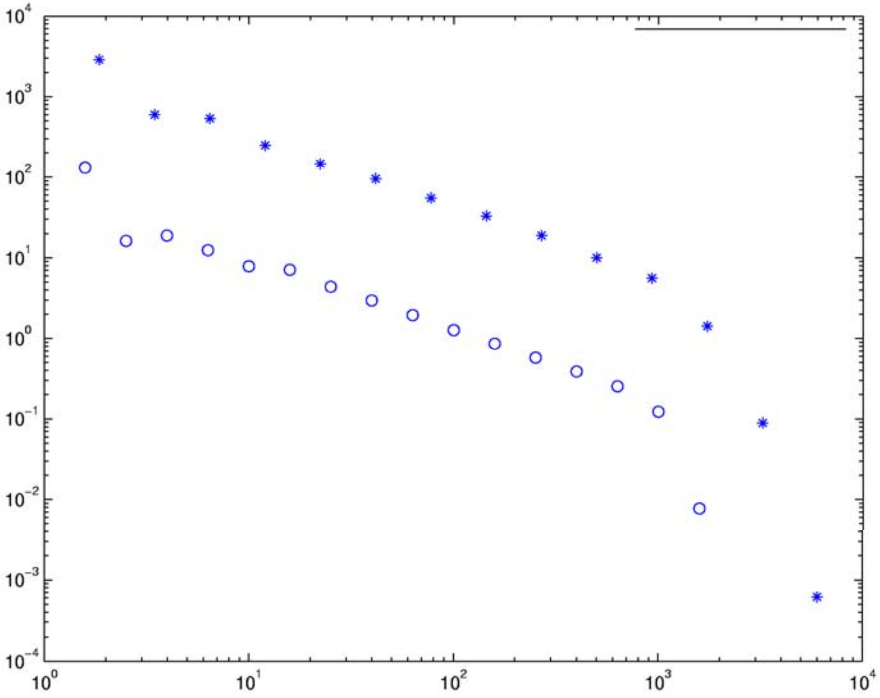
Chapman et al PPCF 2004

Edwards Wilkinson- statistics

Statistics of instantaneous patch size are power law

Linear model- driver (random rain of particles) has inherent fractal scaling (Brownian surface) +selfsimilar diffusion which preserves scaling

•No robustness- scaling exponent *depends* on drive.
•No transport of patches



Chapman et al PPCF 2004

Power laws and blobs?

- Linear systems e.g. EW model give 'blobs' with power law statistics
- Missing element is 'bursty' (intermittent) *transport* via avalanches. Requires threshold (nonlinear diffusion)- breaks symmetry
- It matters what the exponent is

$$\frac{\partial \bar{h}}{\partial t} = D(\bar{h})\nabla^2 \bar{h} + \eta$$

$D(\bar{h}) \propto \mathrm{H}(\nabla \bar{h} - \bar{h}_0)$ - avalanche models

$D(\bar{h}) \propto (\nabla \bar{h})^2$ KPZ - transforms to Burgers eqn.

Information Entropy and Correlation

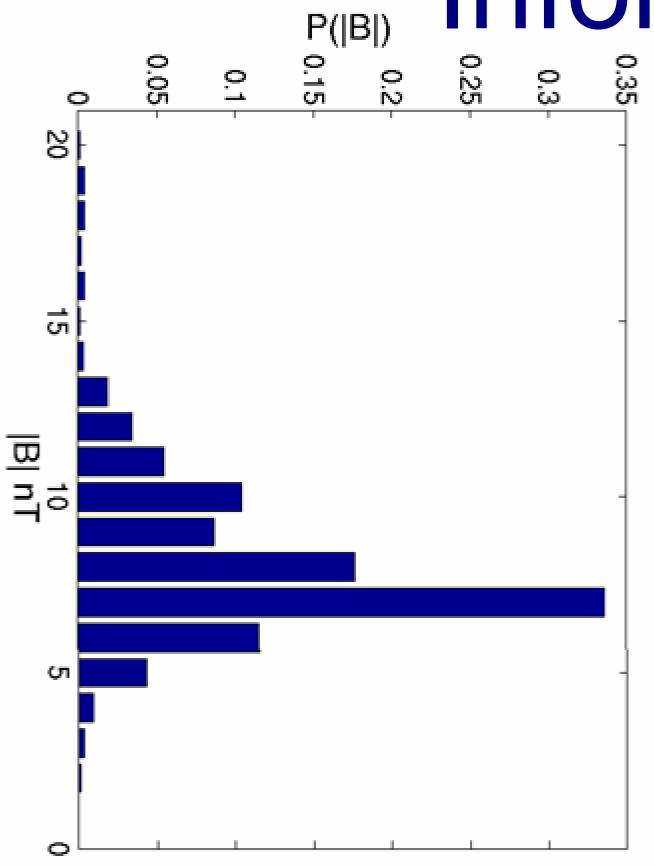
Mutual Information- principles and practice

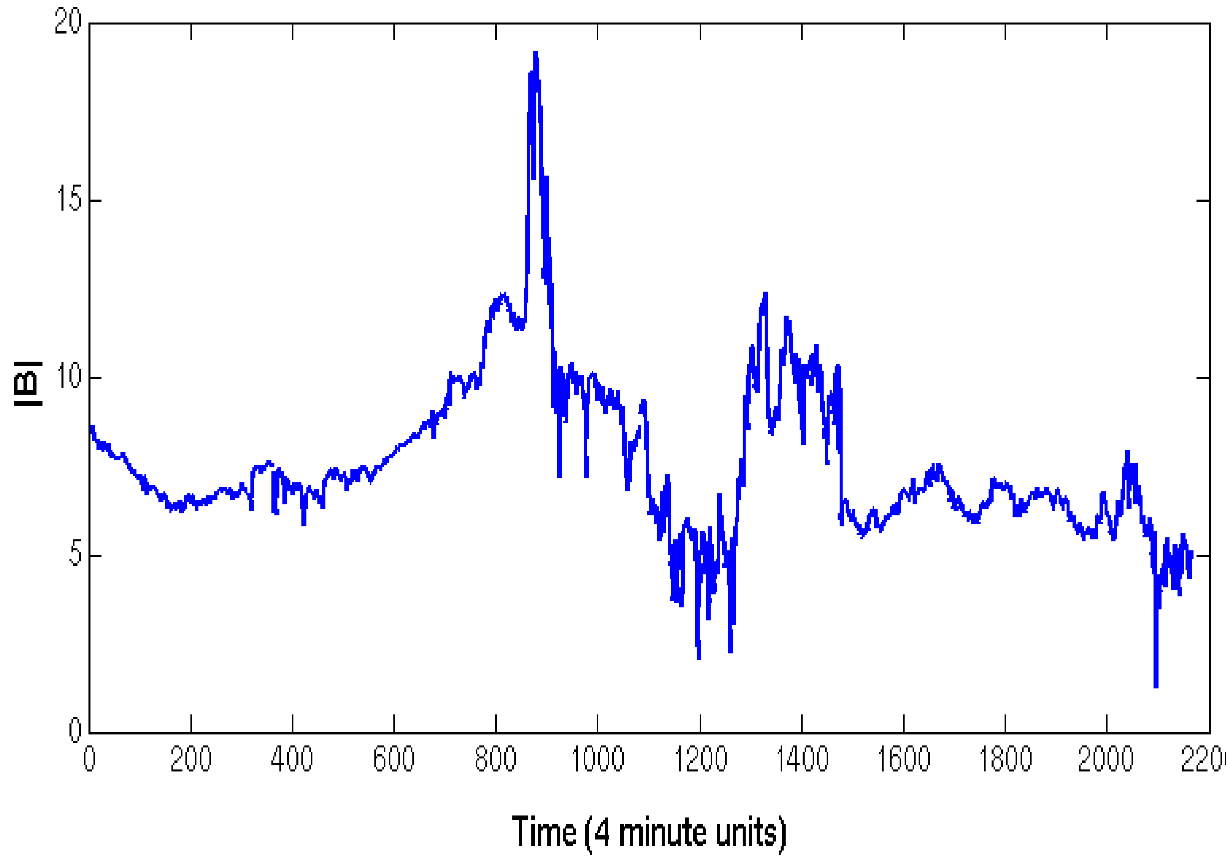
Information and Mutual Information

- A given signal can be thought of as a sequence of symbols that form an alphabet.
- Signal has alphabet $X=\{x_1, x_2 \ldots x_i\}$
- Each symbol in the alphabet has a probability of occurrence

$$P(x_i) = \frac{n_{x_i}}{N}$$

Information entropy





Information and entropy

- A signal (X) carries a certain amount of information expressed as an entropy H(X) in the order of its symbols $\{x_i\}$

$$H(X) = -\sum_i P(x_i)\log_2\left(P(x_i)\right)$$

- Log_2 => binary units
- We assume the relation $$0 \times \log_2 0 = 0$$

Mutual Information

- Entropy can also be defined for joint probability distributions

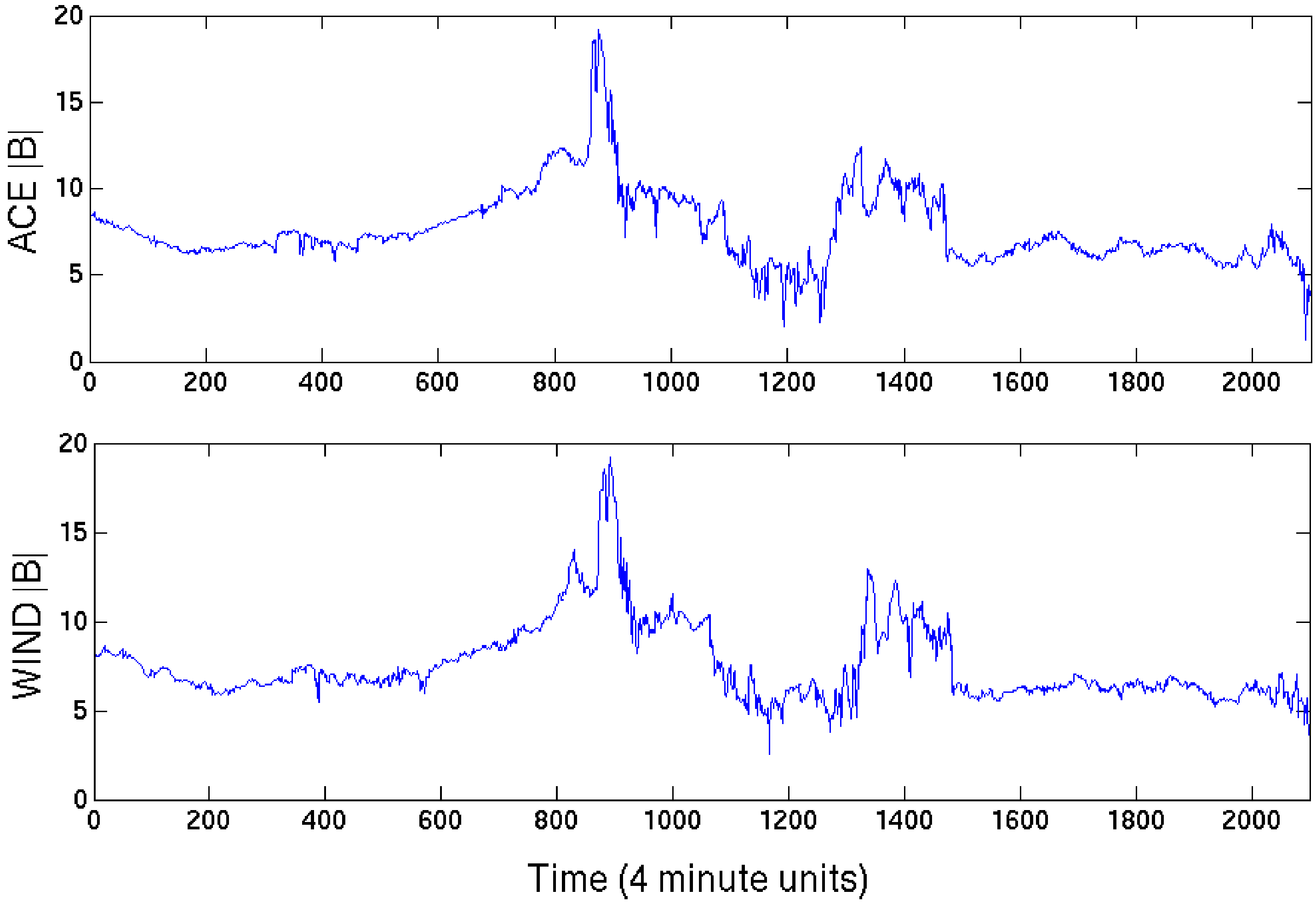
$$H(X,Y) = -\sum_{ij} P(x_i, y_j) \log_2 \left(P(x_i, y_j) \right)$$

- Mutual Information compares the information content of two signals

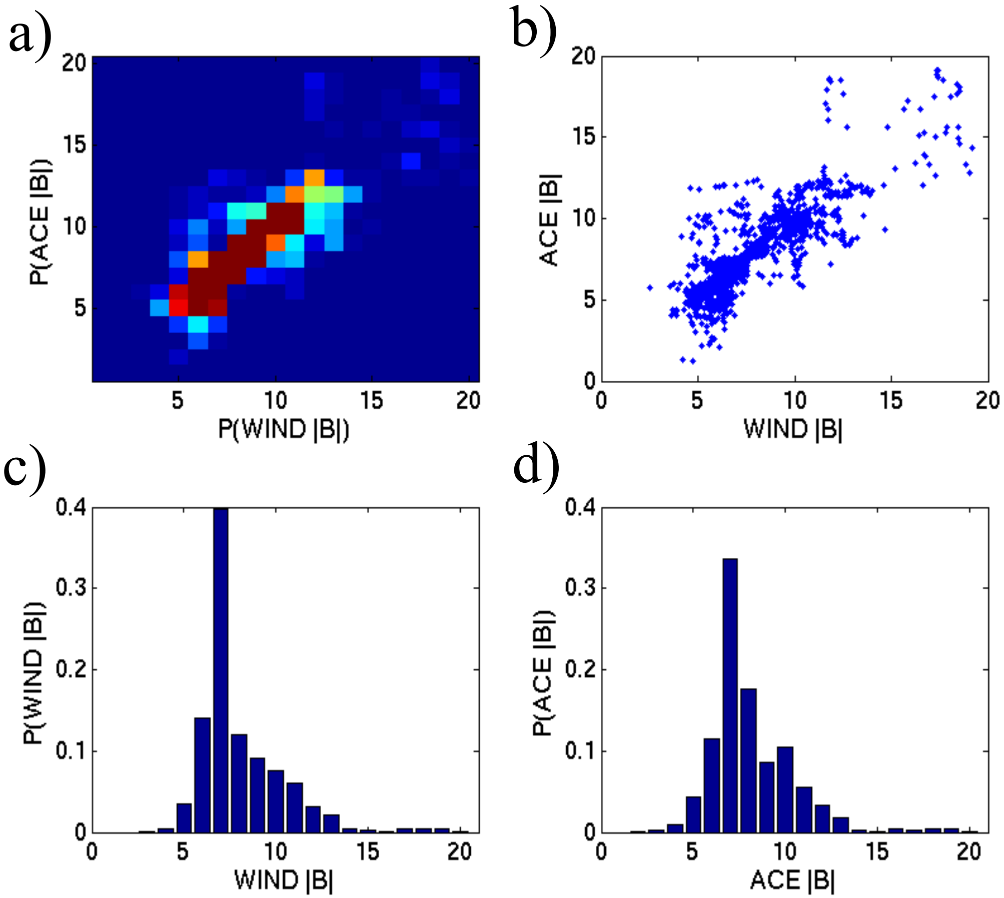
$$I(X;Y) = \sum_{ij} P(x_i, y_j) \log_2 \left[P(x_i, y_j) / P(x_i) P(y_j) \right]$$

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

Timeseries


ACE |B|
WIND |B|
Time (4 minute units)
0
200
400
600
800
1000
1200
1400
1600
1800
2000
0
5
10
15
20

Mutual Information



a) P(WIND |B|, ACE |B|)

b) Raw data WIND |B| vs ACE |B|

c) P(WIND |B|)

d) P(ACE |B|)

MI = 1.09 bits

Ratio of MI to H = 0.39

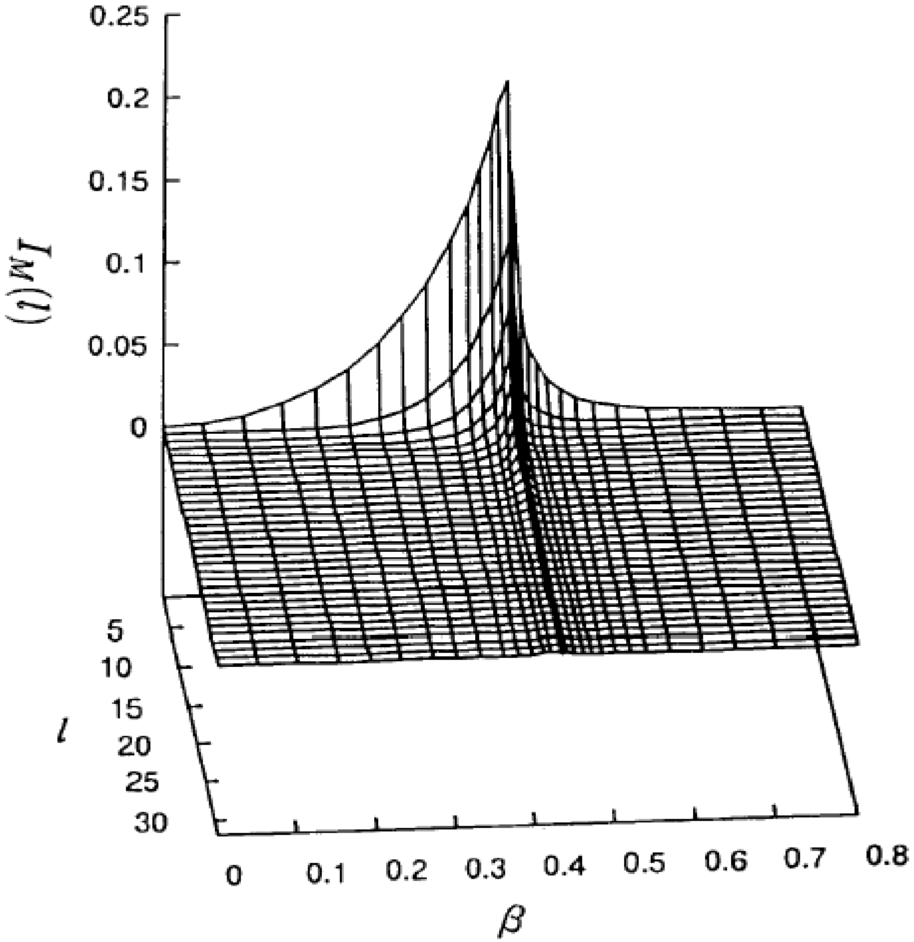
Simple systems with phase transitions

- Initial interest in MI and phase transitions given by Matsuda *et al* treatment of the Ising model

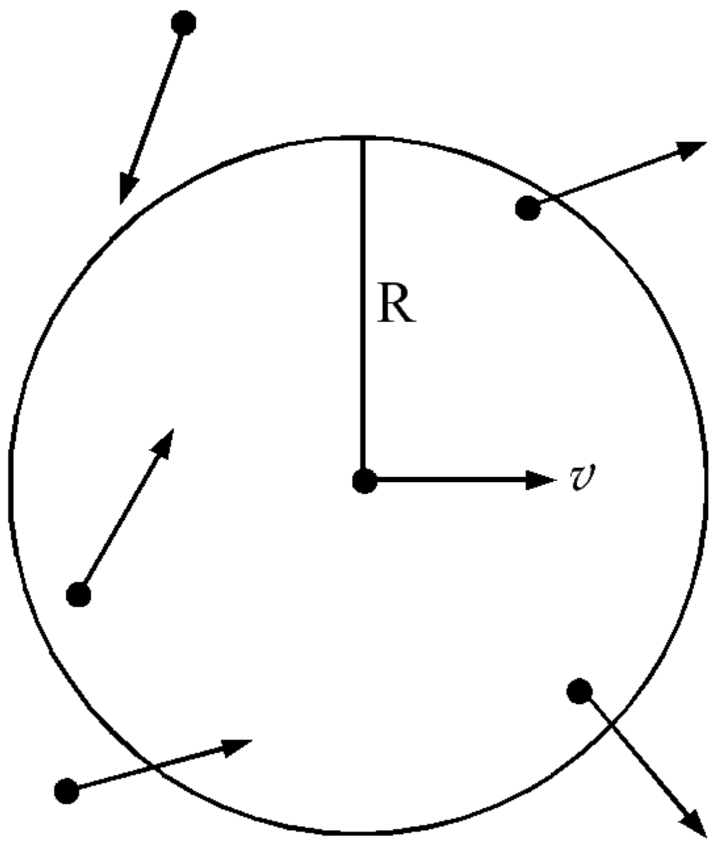
- Study showed that MI peaks at the phase transition and is robust to coarse graining

The Ising Model

- Matsuda *et al*:



The Vicsek Model



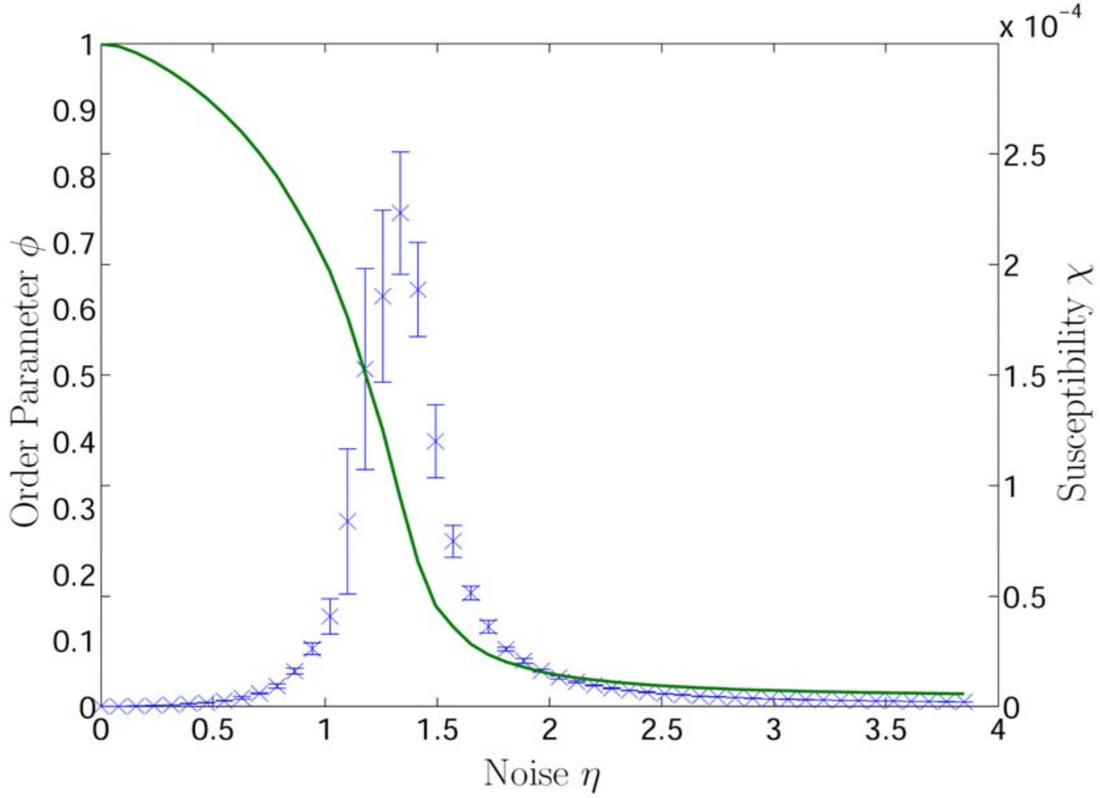
Dynamical rules for each particle:

$$x_{n+1} = x_n + \vec{v}\,\delta t$$
$$\theta_{n+1} = \langle \theta_n \rangle_R + \delta\theta_n$$

Order parameter and susceptibility:

$$\phi = \frac{1}{N v_0} \left| \sum_{i=1}^{N} \underline{v}_i \right|$$
$$\chi = \sigma^2(\phi) = \frac{1}{N} \left(\langle \phi^2 \rangle - \langle \phi \rangle^2 \right)$$

The Vicsek Model

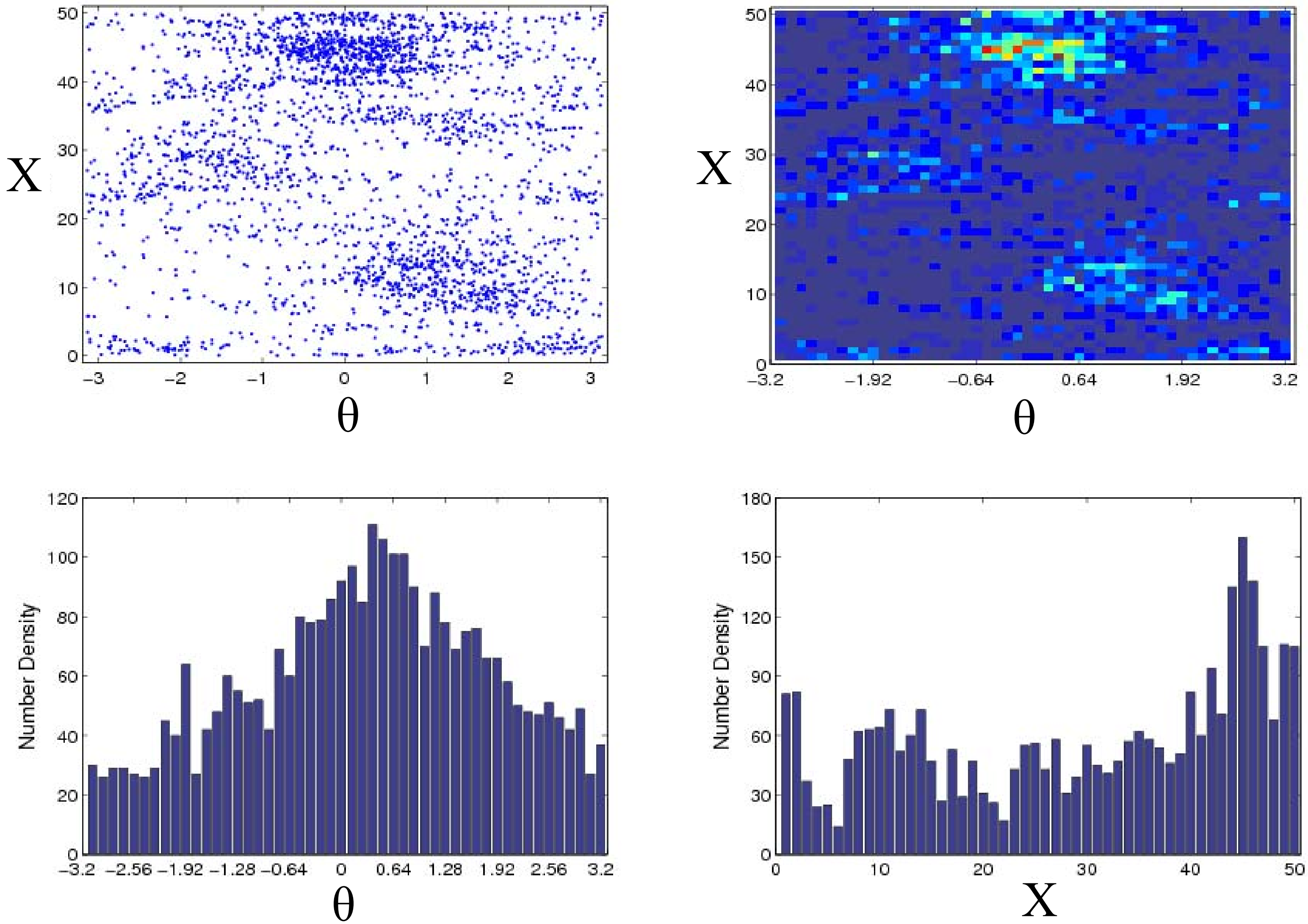


The Vicsek Model

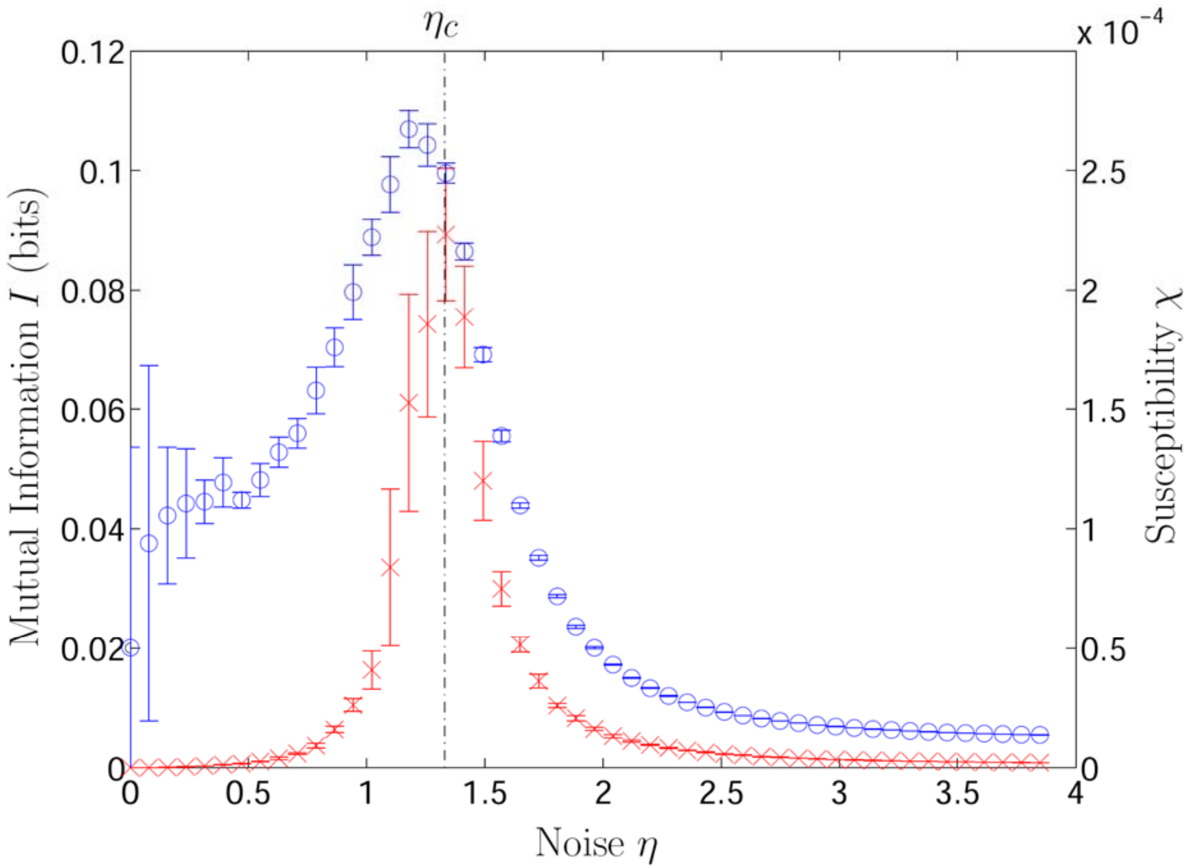
- Mutual information is calculated between position and angle of motion for a snapshot.
- MI for each dimension is the averaged to give total.
- This is done for 50 realisations of the model.

$$
\begin{aligned}
I(X,\Theta) &= \sum_{i,j} P(X_i,\Theta_j)\log_2\left(\frac{P(X_i,\Theta_j)}{P(X_i)P(\Theta_j)}\right)\\
I(Y,\Theta) &= \sum_{i,j} P(Y_i,\Theta_j)\log_2\left(\frac{P(Y_i,\Theta_j)}{P(Y_i)P(\Theta_j)}\right)\\
I &= \frac{I(X,\Theta)+I(Y,\Theta)}{2}
\end{aligned}
$$

The Vicsek Model



The Vicsek Model

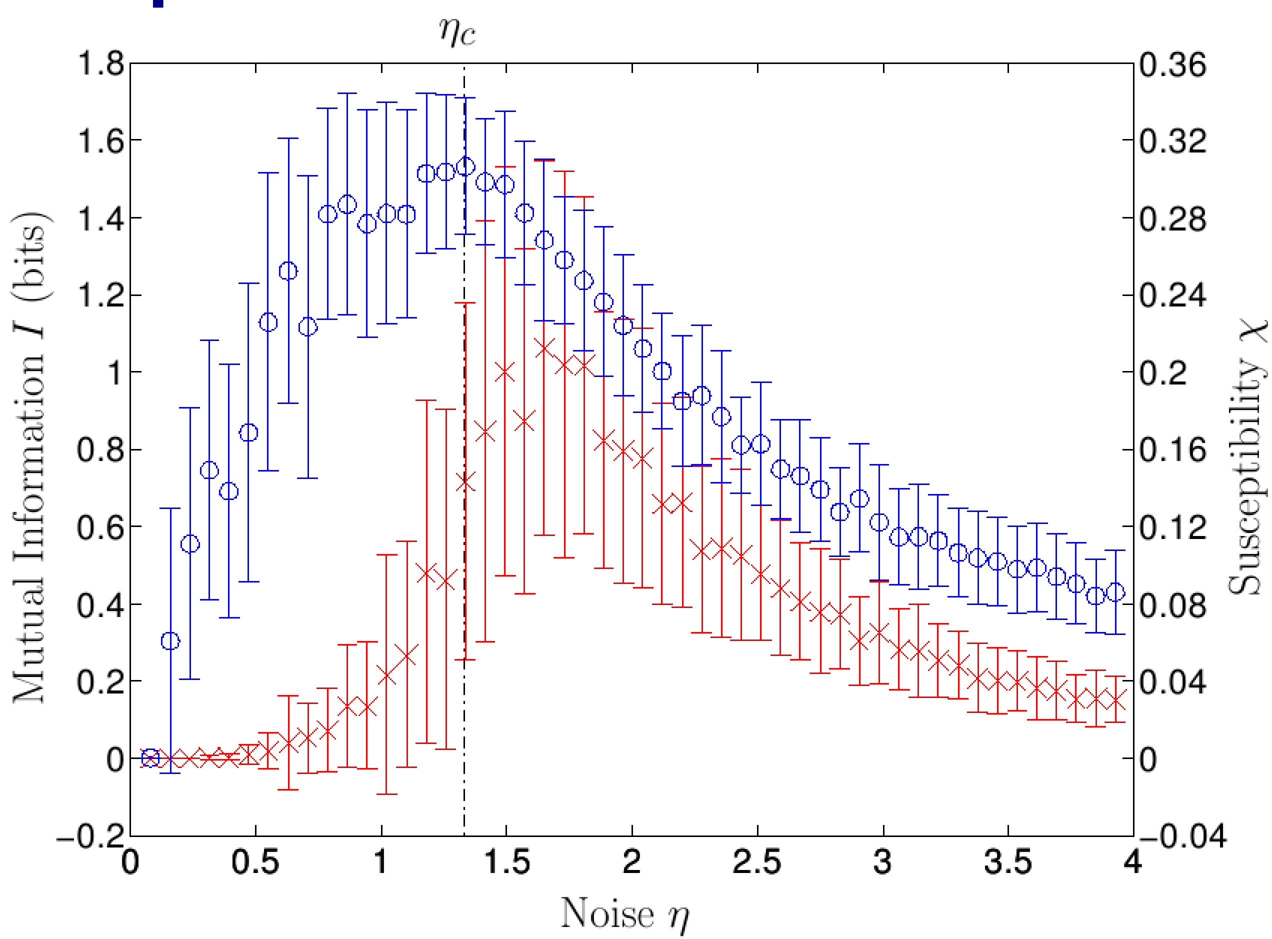


Wicks, SCC et al PRE (2007)

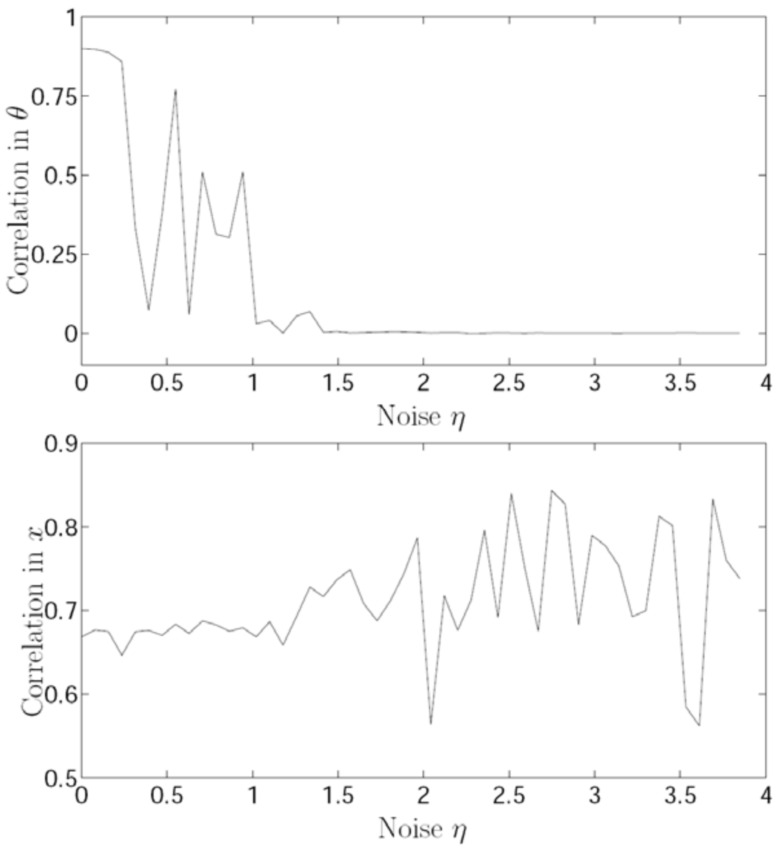
‘real world’- follow only a few particles

- 10 particles chosen at random.
- Time series of 5000 steps used.
- MI calculated between each particle's X position and X velocity for 500 step sections
- Compared to susceptibility for same sections.

(assumption: Vicsek model is ergodic)



Follow only a few particles- linear measure



- Average cross correlation between the same 10 particles.
- No obvious help in determining critical noise

End

See the C0907 web site for more reading…