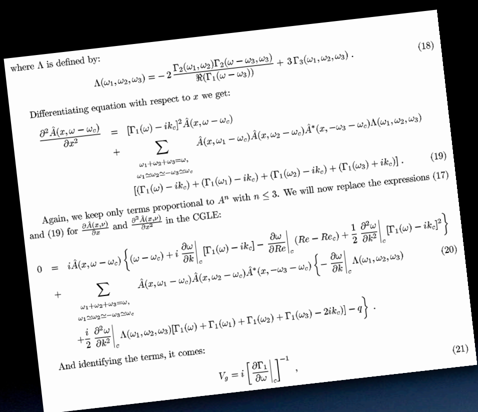
Thierry Dudok de Wit
University of Orléans, France

Nonlinear wave interactions and higher order spectra



where Λ is defined by:

$$\Lambda(\omega_1,\omega_2,\omega_3) = -2\frac{\Gamma_2(\omega_1,\omega_2)\Gamma_2(\omega-\omega_3,\omega_3)}{\Re(\Gamma_1(\omega-\omega_3))} + 3\Gamma_3(\omega_1,\omega_2,\omega_3) . \quad (18)$$

Differentiating equation with respect to x we get:

$$\begin{aligned}\frac{\partial^2 \hat{A}(x,\omega-\omega_c)}{\partial x^2} &= [\Gamma_1(\omega)-ik_c]^2\hat{A}(x,\omega-\omega_c)\\ &+ \sum_{\substack{\omega_1+\omega_2+\omega_3=\omega,\\ \omega_1\simeq\omega_2\simeq-\omega_3\simeq\omega_c}} \hat{A}(x,\omega_1-\omega_c)\hat{A}(x,\omega_2-\omega_c)\hat{A}^*(x,-\omega_3-\omega_c)\Lambda(\omega_1,\omega_2,\omega_3)\\ &[(\Gamma_1(\omega)-ik_c)+(\Gamma_1(\omega_1)-ik_c)+(\Gamma_1(\omega_2)-ik_c)+(\Gamma_1(\omega_3)+ik_c)] . \quad (19)\end{aligned}$$

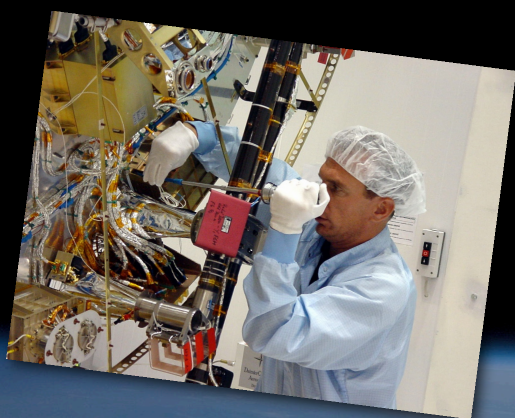
Again, we keep only terms proportional to A^n with $n \leq 3$. We will now replace the expressions (17) and (19) for $\frac{\partial \hat{A}(x,\nu)}{\partial x}$ and $\frac{\partial^2 \hat{A}(x,\nu)}{\partial x^2}$ in the CGLE:

$$\begin{aligned}0 &= i\hat{A}(x,\omega-\omega_c)\left\{(\omega-\omega_c)+i\left.\frac{\partial\omega}{\partial k}\right|_c[\Gamma_1(\omega)-ik_c]-\left.\frac{\partial\omega}{\partial Re}\right|_c(Re-Re_c)+\frac{1}{2}\left.\frac{\partial^2\omega}{\partial k^2}\right|_c[\Gamma_1(\omega)-ik_c]^2\right\} \quad (20)\\ &+ \sum_{\substack{\omega_1+\omega_2+\omega_3=\omega,\\ \omega_1\simeq\omega_2\simeq-\omega_3\simeq\omega_c}} \hat{A}(x,\omega_1-\omega_c)\hat{A}(x,\omega_2-\omega_c)\hat{A}^*(x,-\omega_3-\omega_c)\left\{-\left.\frac{\partial\omega}{\partial k}\right|_c\Lambda(\omega_1,\omega_2,\omega_3)\right.\\ &\left.+\frac{i}{2}\left.\frac{\partial^2\omega}{\partial k^2}\right|_c\Lambda(\omega_1,\omega_2,\omega_3)[\Gamma_1(\omega)+\Gamma_1(\omega_1)+\Gamma_1(\omega_2)+\Gamma_1(\omega_3)-2ik_c)]-q\right\} .\end{aligned}$$

And identifying the terms, it comes:

$$V_g = i\left[\left.\frac{\partial\Gamma_1}{\partial\omega}\right|_c\right]^{-1} , \quad (21)$$

?



Outline

- Higher Order Spectra (HOS)
  - → Their properties
  - → Application to wave-wave interactions
  - → Spectral energy transfers

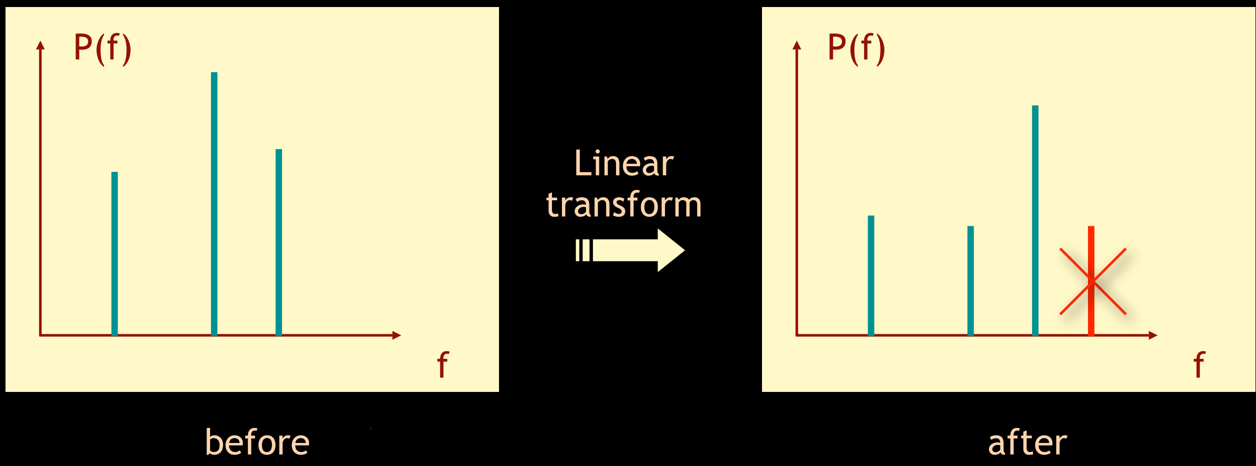
- Why is the Fourier transform ubiquitous ?

- Because Fourier modes are the eigenmodes of linear differential systems, which occur so frequently

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + a_2 \frac{d^2 y(t)}{dt^2} + \cdots = x(t) + b_1 \frac{dx(t)}{dt} + b_2 \frac{d^2 x(t)}{dt^2} + \cdots$$

Why Fourier spectra ?

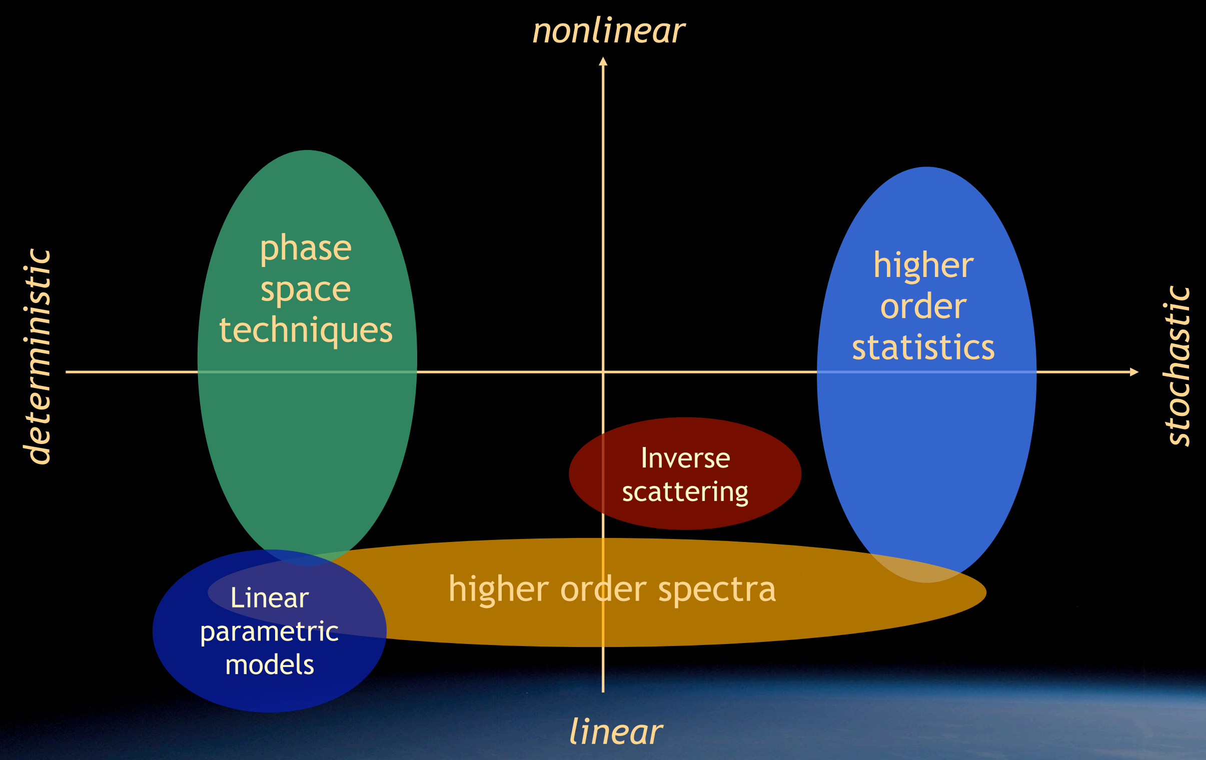
- Frequencies are therefore natural ***invariants*** of linear stationary differential systems



Linear vs nonlinear

- In linear systems, all the pertinent information is contained in the power spectral density
- In nonlinear systems, Fourier modes may get coupled
  ➔ their phases also contain pertinent information
- ***higher order spectral analysis*** precisely exploits this phase information

Techniques for nonlinear systems



Definition of HOS

- Take a nonlinear system that is described by

$$\frac{\partial u(x,t)}{\partial x} = f\left(u(x,t)\right)$$

$f(u)$: a continuous, nonlinear and time-independent function

$u(x,t)$: plasma density, magnetic field, …

- With mild assumptions, one may decompose f into a series (Wiener, 1958)

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} &= \int g(\tau_1)\, u(x,t-\tau_1)\, d\tau_1 \\ &+ \int\!\!\int g(\tau_1,\tau_2)\, u(x,t-\tau_1)\, u(x,t-\tau_2)\, d\tau_1\, d\tau_2 \\ &+ \int\!\!\int\!\!\int g(\tau_1,\tau_2,\tau_3)\, u(x,t-\tau_1)\, u(x,t-\tau_2)\, u(x,t-\tau_3)\, d\tau_1\, d\tau_2\, d\tau_3 \\ &+ \cdots\end{aligned}$$

Definition of HOS

- Taking the discrete Fourier transform, we get a Volterra series

$$\frac{\partial u_p}{\partial x} = \Gamma_p u_p + \sum_{k,l} \Gamma_{kl} u_k u_l \delta_{k+l,p} + \sum_{k,l,m} \Gamma_{klm} u_k u_l u_m \delta_{k+l+m,p} + \dots$$

with $u_p = u(x, \omega_p)$

- The **kernels** Γ embody the physical information of the process. In a Hamiltonian framework, they are directly connected to known physical processes (Zakharov, 1970)

- In plasmas,only low order kernels are expected to be significant (Galeev, 1980)

Three-wave couplings

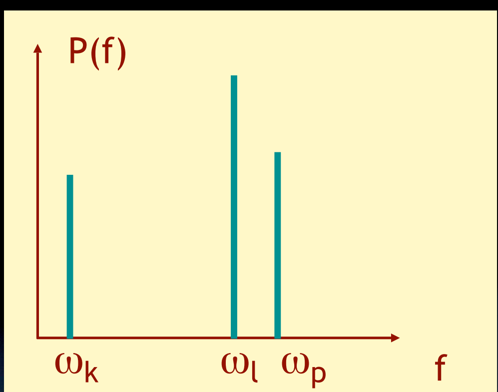
- A Fourier mode can only couple to other ones in a specific way

- For quadratic nonlinearities, the resonance condition reads

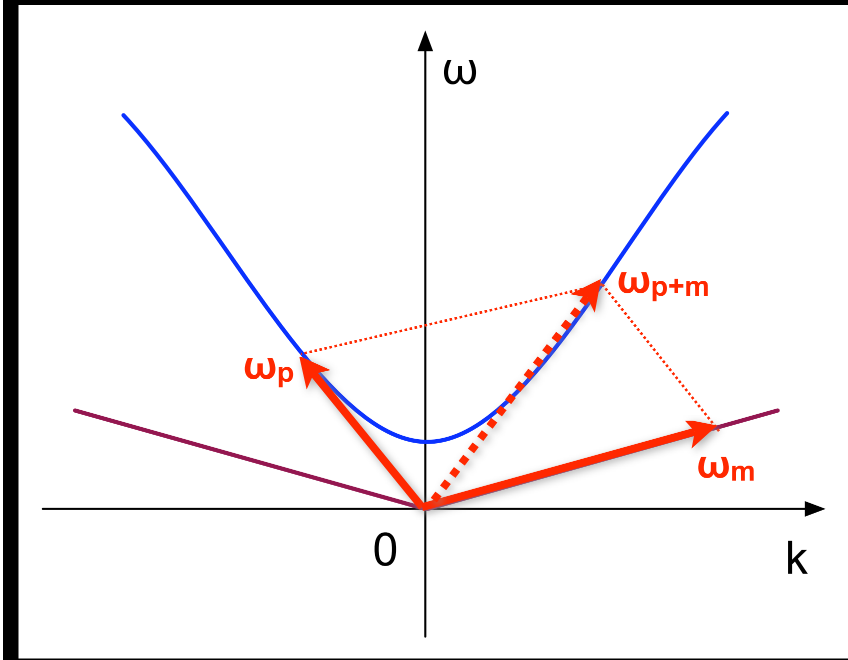
$$\omega_k + \omega_l = \omega_p$$

This describes three-wave interactions

Examples : harmonic generation, decay instability (L→S+L')



Three-wave couplings



energy conservation

$$\omega_p + \omega_m = \omega_{p+m}$$

momentum conservation

$$\vec{k}_p + \vec{k}_m = \vec{k}_{p+m}$$

Four-wave couplings

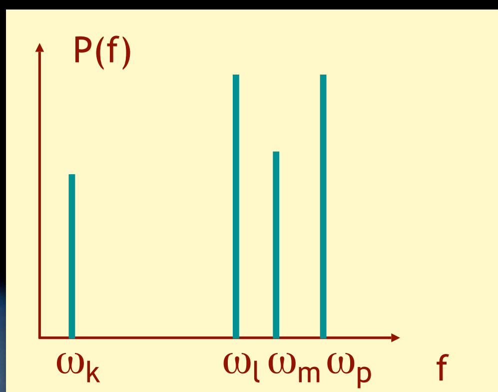
$$\frac{\partial u_p}{\partial x} = \Gamma_p u_p + \sum_{k,l} \Gamma_{kl} u_k u_l \delta_{k+l,p} + \sum_{k,l,m} \Gamma_{klm} u_k u_l u_m \delta_{k+l+m,p} + \dots$$

- For cubic nonlinearities, the resonance condition is

$$\omega_k + \omega_l + \omega_m = \omega_p$$

This describes four-wave interactions

Example : modulational instability (L+S = L'+L'')



Definition of HOS

- Rearranging the Volterra series and taking the expectation for a homogeneous plasma ($\partial/\partial x = 0$), we have

$$\Gamma_p \langle u_p^* u_p \rangle + \sum_{k+l=p} \Gamma_{kl} \langle u_k u_l u_{k+l}^* \rangle + \sum_{k+l+m=p} \Gamma_{klm} \langle u_k u_l u_m u_{k+l+m}^* \rangle + \cdots = 0$$

power spectrum: $P(\omega_p) = \langle u_p u_p^* \rangle$

bispectrum: $B(\omega_k, \omega_l) = \langle u_k u_l u_{k+l}^* \rangle$

trispectrum: $T(\omega_k, \omega_l, \omega_m) = \langle u_k u_l u_m u_{k+l+m}^* \rangle$

The power spectrum $P(\omega_p)$ is not an invariant quantity anymore !

The bicoherence

- The normalised bispectrum gives the ***bicoherence***

$$B(\omega_k, \omega_l) = \langle u_k u_l u_{k+L}^* \rangle \rightarrow b^2(\omega_k, \omega_l) = \frac{|\langle u_k u_l u_{k+l}^* \rangle|^2}{\langle |u_k u_l|^2 \rangle \langle |u_{k+l}|^2 \rangle}$$

- The bicoherence is bounded : $0 < b^2 < 1$
- It measures the amount of signal energy at bifrequency (ω_k, ω_l) that is quadratically phase coupled to ω_{k+l}
- bicoherence =0 ⇔ no phase coupling

The tricoherence

- Similarly define the ***tricoherence***

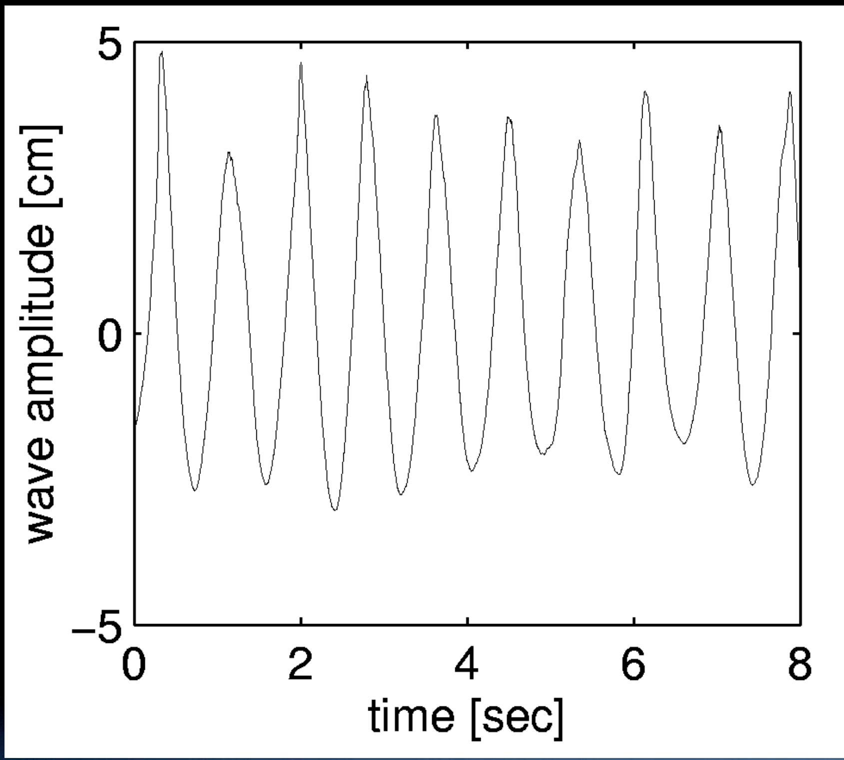
$$t^2(\omega_k, \omega_l, \omega_m) = \frac{|\langle u_k u_l u_m u^*_{k+l+m} \rangle|^2}{\langle |u_k u_l u_m|^2 \rangle \langle |u_{k+l+m}|^2 \rangle}$$

- It measures the amount of signal energy at trifrequency (ω_k, ω_l , ω_m) that is quadratically phase coupled to ω_{k+l+m}

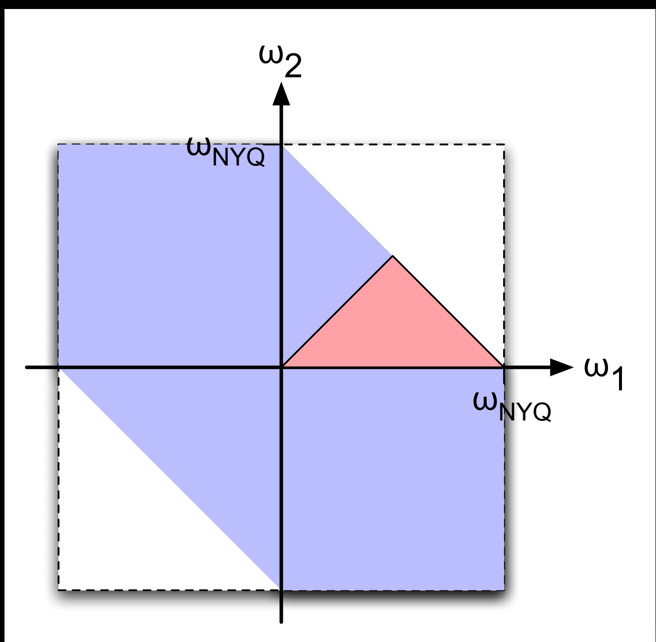
Example :

swell in a water basin

Example : water waves

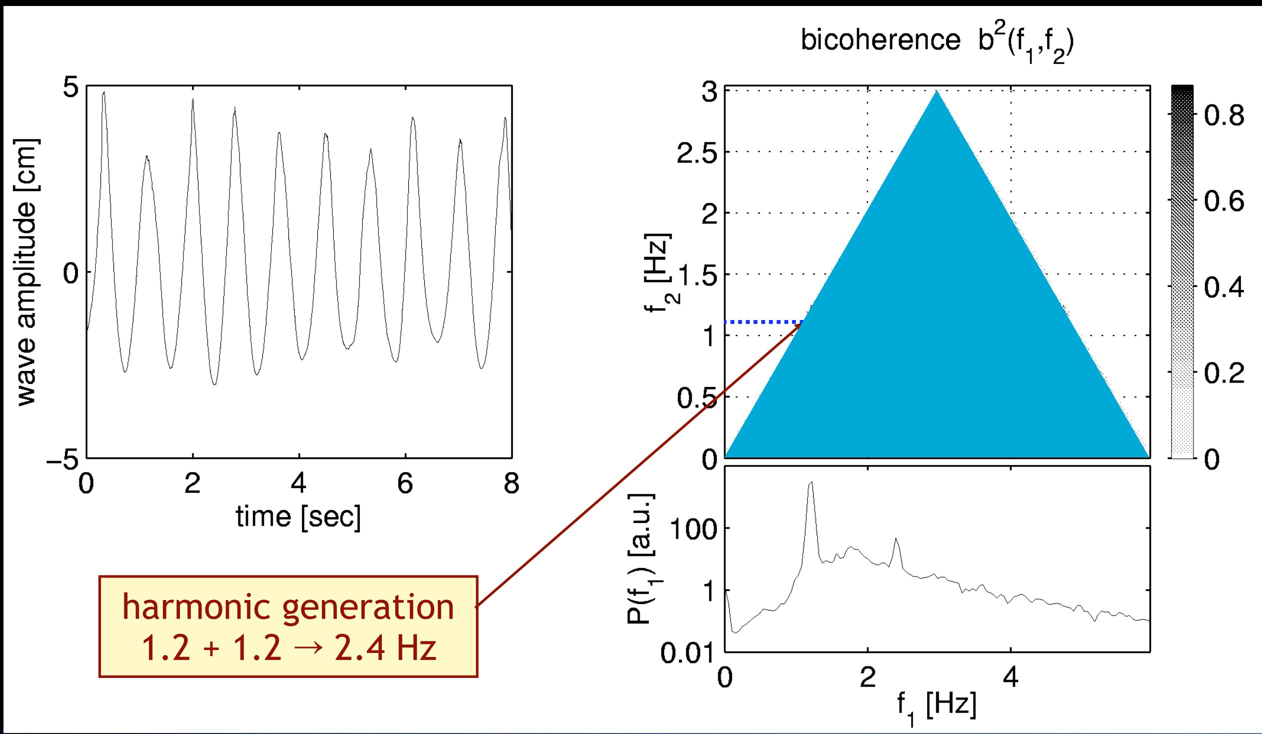


Principal domain of bicoherence



Because of symmetries, the principal domain reduces to a triangle

Example : water waves

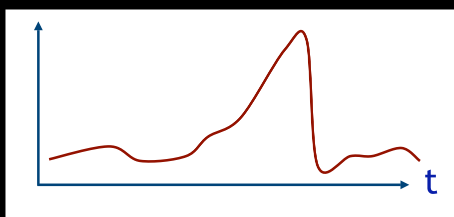


Bicoherence : interesting properties

- Signals that are asymmetric vs time reversal (*u(t)* ↔ *u(−t)*) give rise to imaginary bispectra

$$\langle u_k u_l u^*_{k+l} \rangle \text{ imaginary}$$

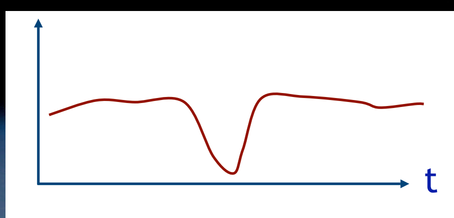
  → typically occurs with wave steepening



- Signals that are up-down asymmetric (*u(t)* ↔ *− u(t)*) give rise to real bispectra

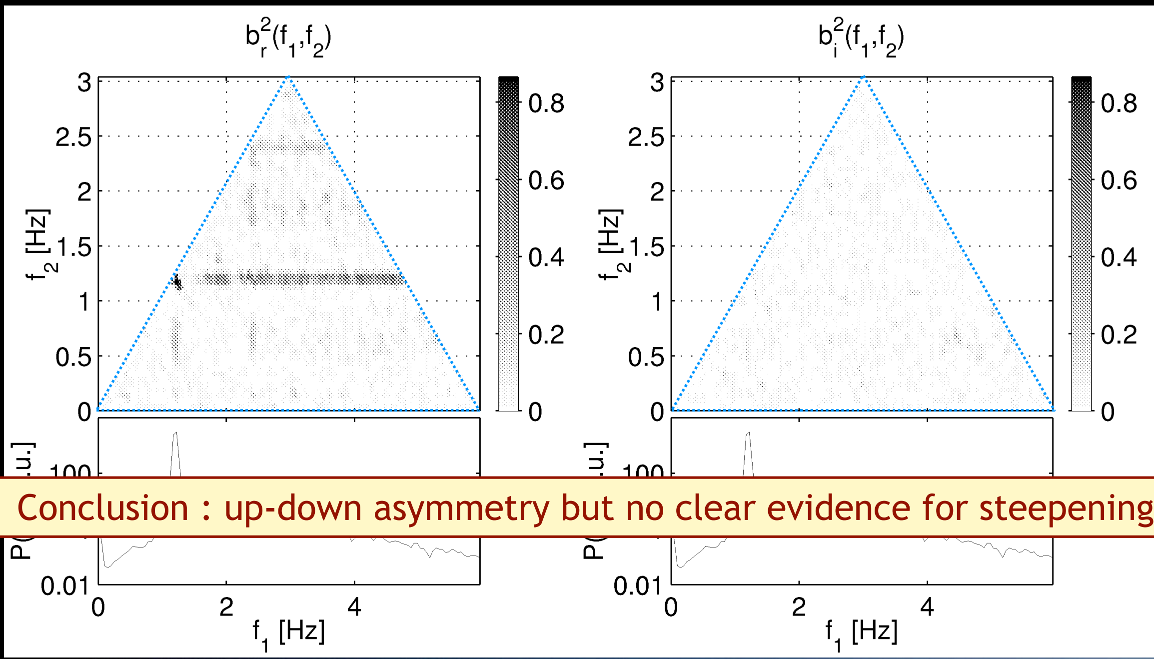
$$\langle u_k u_l u^*_{k+l} \rangle \text{ real}$$

  → typically occurs with cavitons



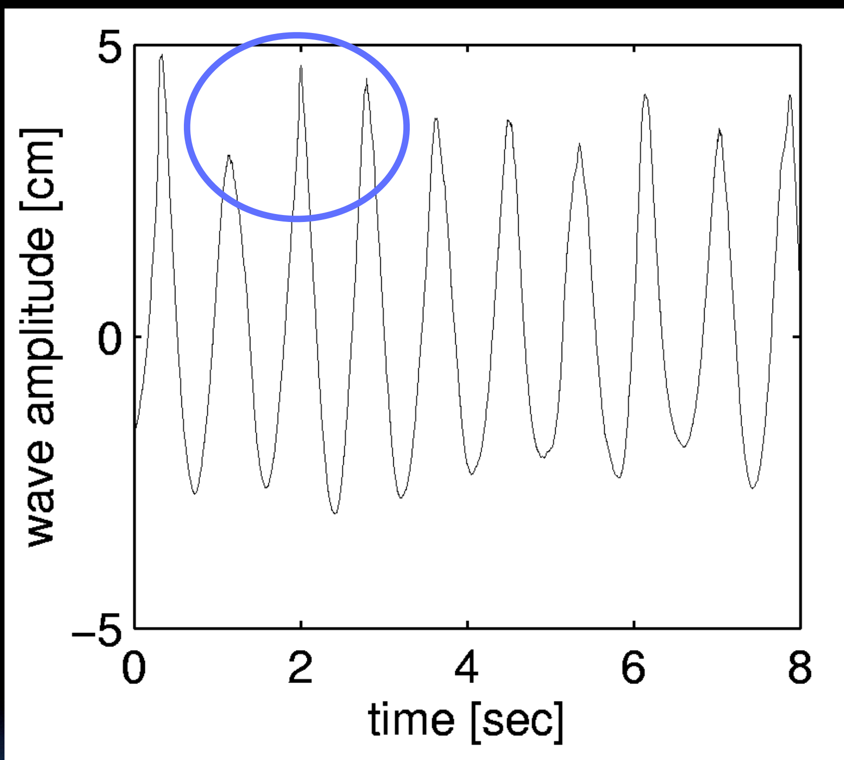
Asymmetry : ocean waves example

bicoherence from **real part** only — bicoherence from **imaginary** part only



Conclusion : up-down asymmetry but no clear evidence for steepening

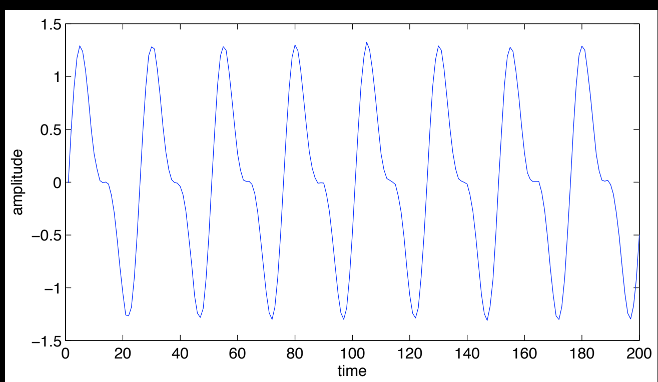
Physical picture



High bicoherence ≠ Nonlinear interactions

Example : sine wave + first harmonic

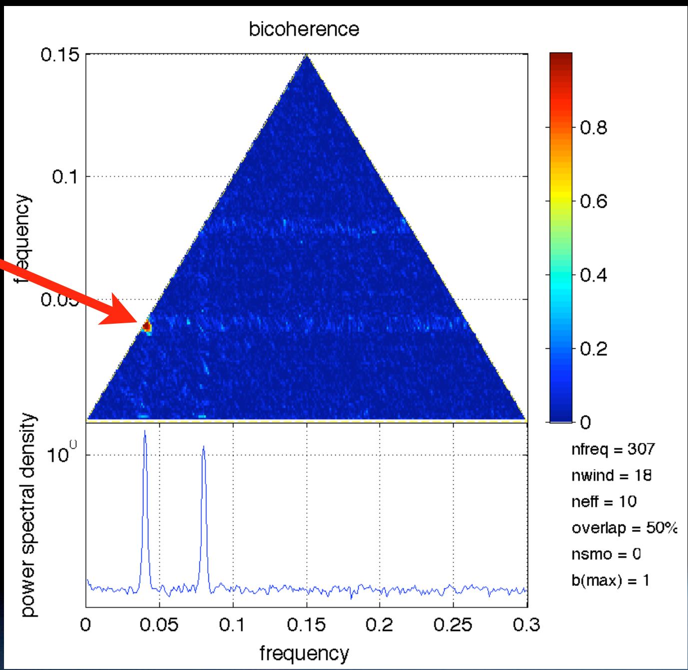
We generate a sine wave + its first harmonic + some noise



There is no nonlinear coupling here !

Example : sine wave + first harmonic

- Even though there is no nonlinear coupling at work, the bicoherence is huge, simply because the phases are coupled



- Sometimes, a nonlinear instrumental gain may explain the phase coupling

example

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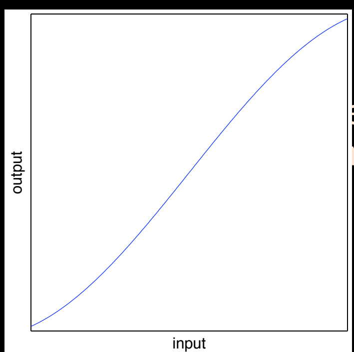
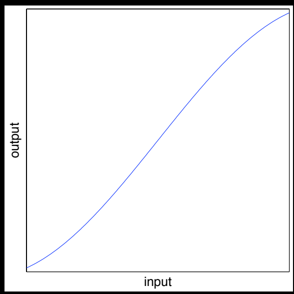


es, we forget that we need a large statistics to nificant level of bicoherence (at least 30 wave

These two examples both reveal the existence of significant phase couplings between specific Fourier modes. We must stress, however, that a phase coupling does not necessarily imply the existence of nonlinear wave interactions per se. In the first example, the ridge could be interpreted both as a decay ($0.6 \to f_1 + f_2$) or as an inverse decay ($f_1 + f_2 \to 0.6$) process. At this stage we cannot tell whether the observed phase coupling is accompanied by an energy transfer between Fourier modes (i.e. the wavefield is dynamically evolving) or whether it is just the remnant of some nonlinear effect that took place in the past or maybe

Example

- Sometimes, a nonlinear instrumental gain may explain the phase coupling
- 

- But in most cases, we forget that we need a large statistics to determine a significant level of bicoherence (at least 30 wave periods)

$$\text{Bias } \hat{b}^2 \approx \frac{4\sqrt{3}}{M}$$

$$\text{Var } \hat{b}^2 \approx \frac{4\hat{b}^2}{M}\left(1 - \hat{b}^2\right)$$

$$\hat{B}(\omega_k, \omega_l) = \frac{1}{M}\sum_{i=1}^{M} u_k^{(i)} u_l^{(i)} u_{k+l}^{*(i)}\,, \qquad (13)$$

M : number of independent ensembles

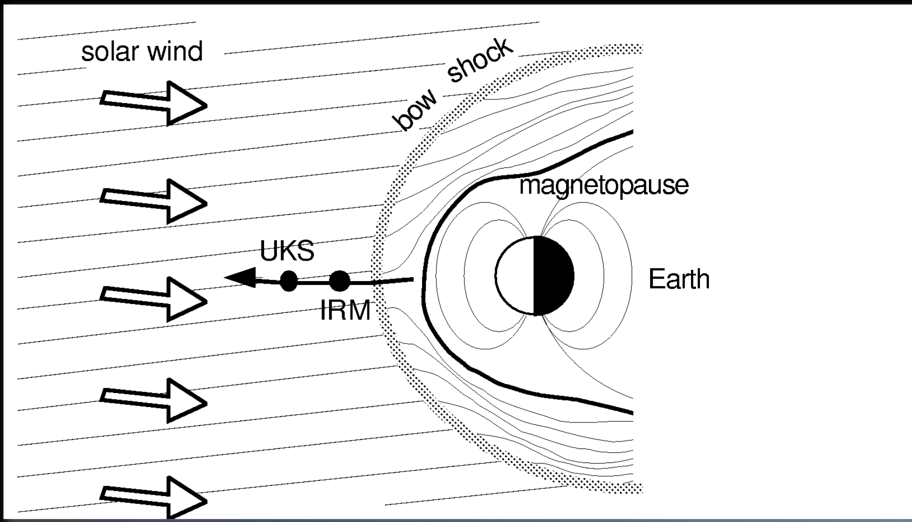
The empirical estimate of the bicoherence becomes

Another example :

magnetic field measurements by two satellites

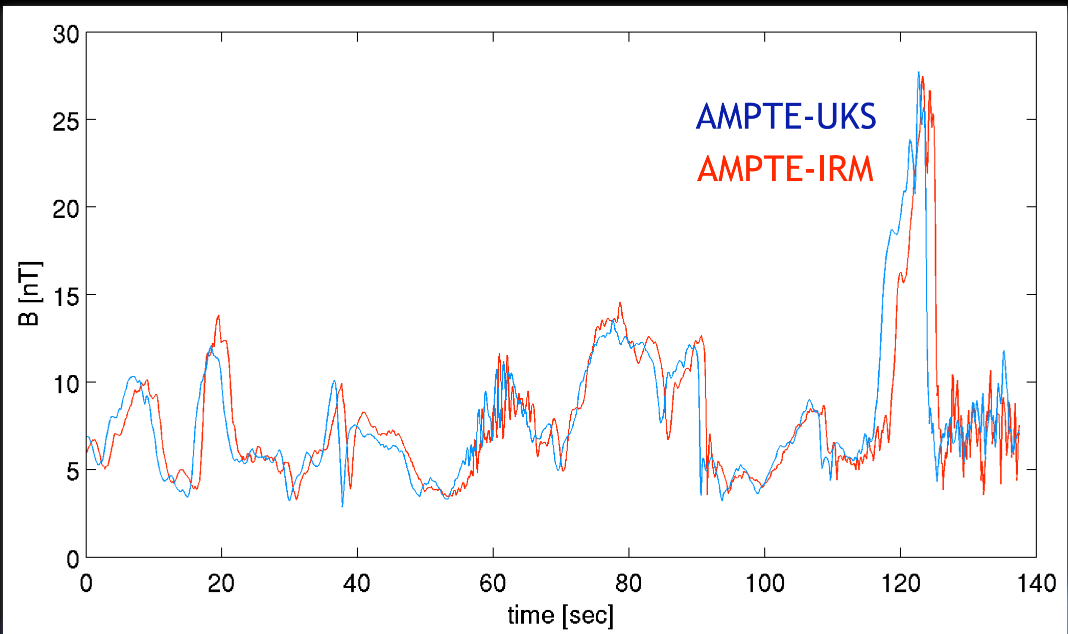
Example : magnetic field data

- The dual AMPTE-UKS and AMPTE-IRM satellites measure **B** just upstream the Earth's quasiparallel bow shock



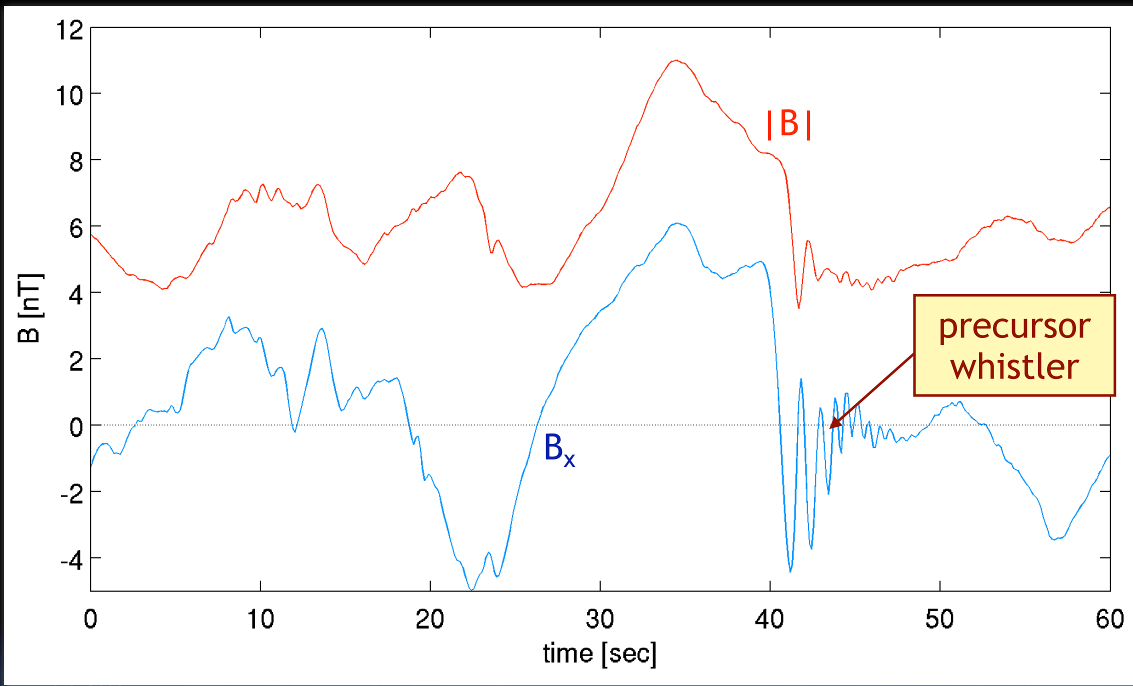
Excerpt of magnetic field

- Structures seen by UKS are observed about 1 sec later by IRM



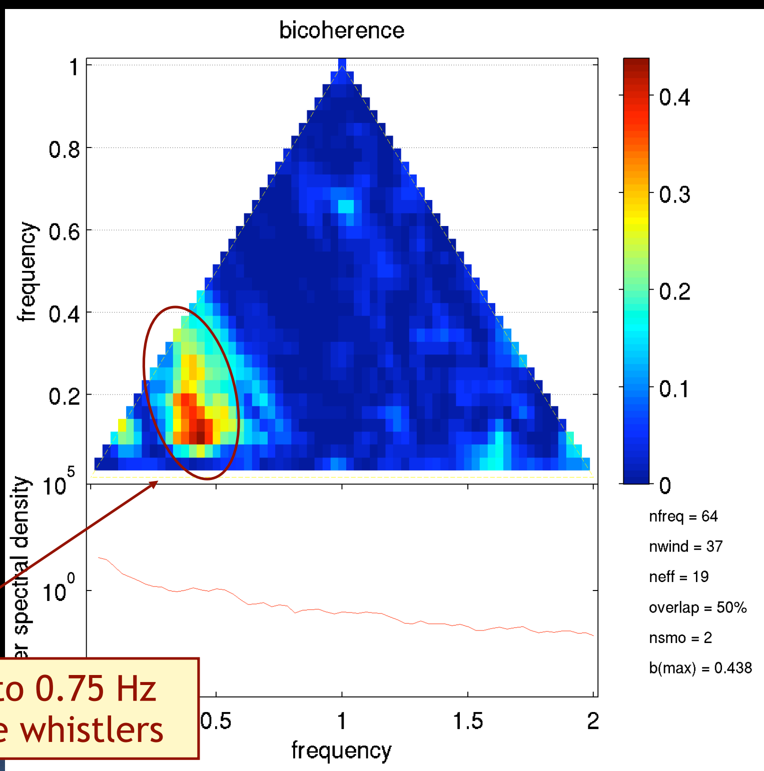
Excerpt of magnetic field

- Some structures show clear evidence for steepening (SLAMS)



Bicoherence of AMPTE data

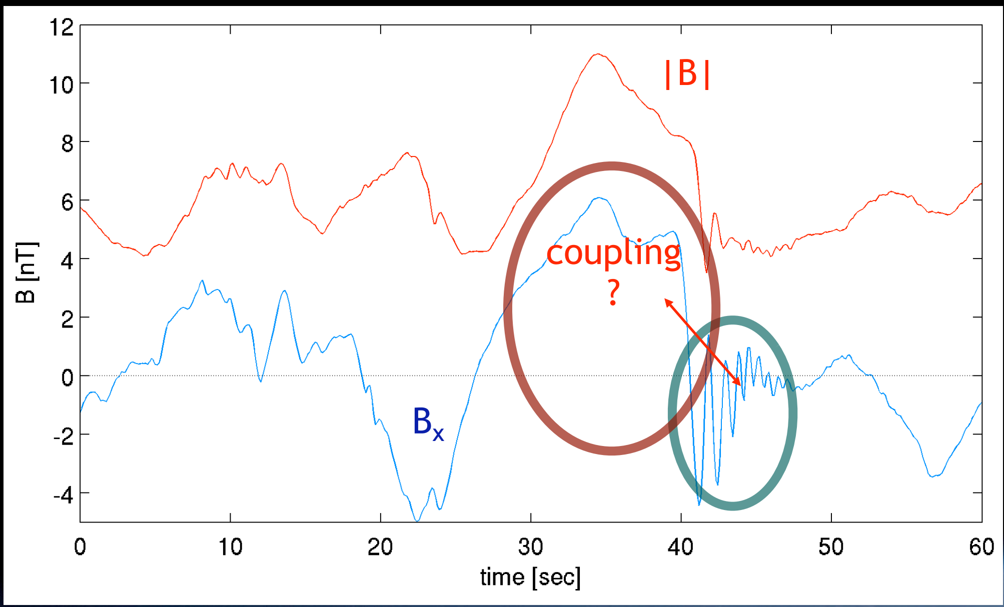
- Existence of quadratic wave interactions is attested by bicoherence analysis
- The tricoherence is weak, suggesting that four-wave interactions are not at play



0.15 + (0.4 to 0.6) = 0.55 to 0.75 Hz
SLAMS + whistlers = more whistlers

Physical picture

- There is a phase coupling between the precursor whistlers and the SLAMS



Physical picture

HOS tell us there is a phase coupling

But they don't say what caused this coupling

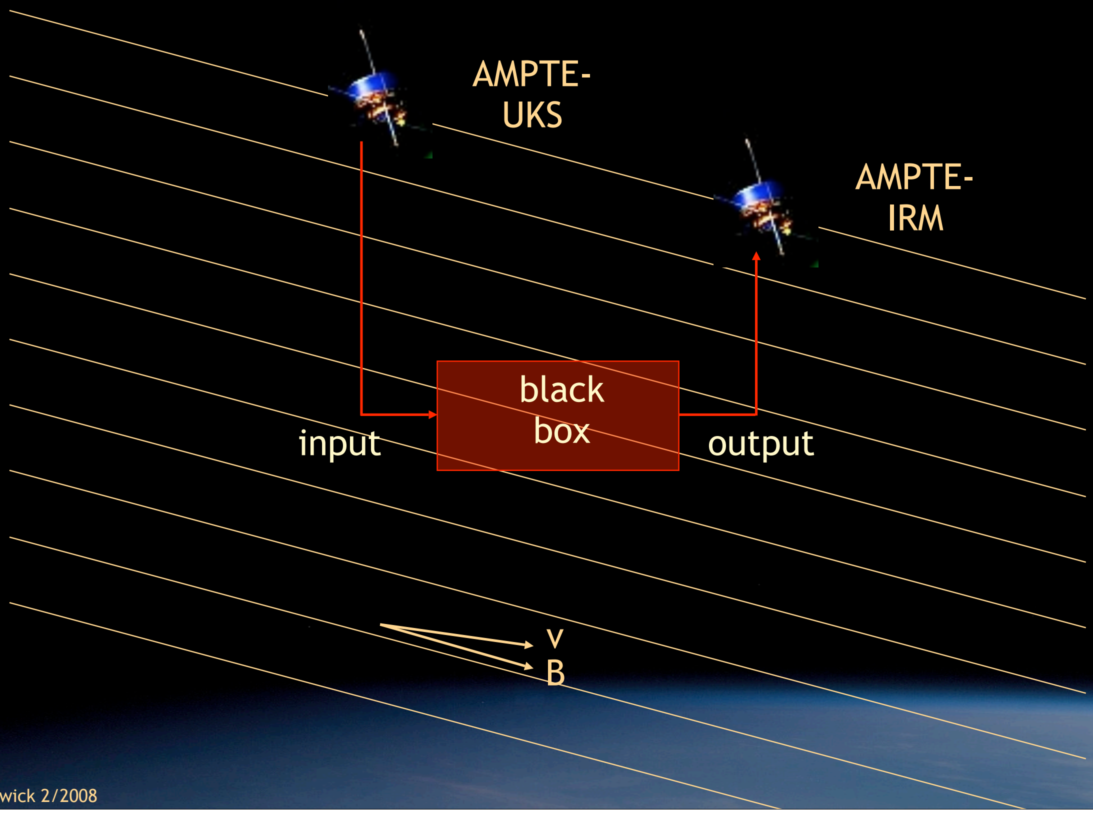
- → are the whistlers instabilities triggered by the SLAMS ?
- → were the whistlers generated farther upstream and are they now frozen into the wavefield ?
- → are the whistlers inherently part of the SLAMS (= solitary waves) ?
- → is all of this an instrumental effect ?

Physical picture

To answer this question unambiguously,
we need spatial resolution,
i.e. multipoint measurements

A transfer function approach

Nonlinear Transfer Function



Model of the black box

- First approximate the spatial derivative (Ritz & Powers, 1980)

$$\frac{\partial u}{\partial x} \leftrightarrow \frac{u_{output} - u_{input}}{\Delta x}$$

- Then assume the random phase approximation (mostly valid for broadband spectra)

$$\langle u_p u_q^* \rangle = \delta_{pq} P_p$$

- The Volterra model then naturally leads to a kinetic equation

The kinetic equation

- The kinetic equation tells us how the spatial variation of the spectral energy at a given frequency varies according to linear/ nonlinear processes (Monin & Yaglom, 1976)

$$\frac{\partial P_p}{\partial x} = \gamma_p P_p + \sum_{m+n=p} T_{m,n} + \sum_{m+n+k=p} T_{m,n,k} + \cdots$$

The spectral energy transfers **T** tell us how much energy is exchanged between Fourier modes : the are the key to the dynamical evolution of the wavefield

P_p : power spectral density at frequency ω_p

linear growth or damping

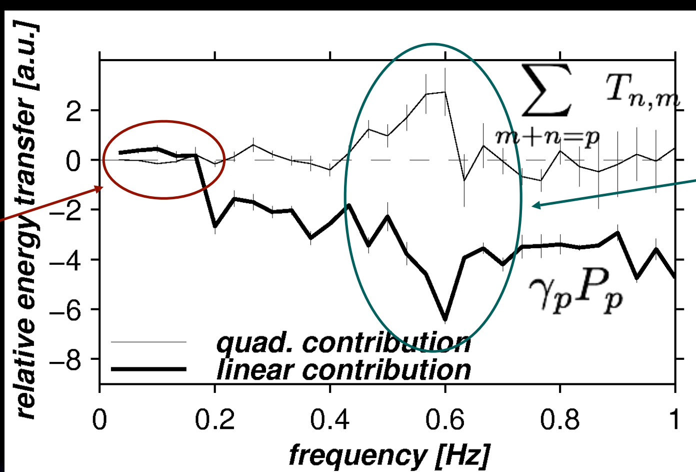
losses/gains due to quadratic effects

losses/gains due to cubic effects

The linear term

- We first consider the energy balance
- How much of the spatial variation of the spectral energy is due to linear / nonlinear effects ?

SLAMS are linearly unstable (growth)

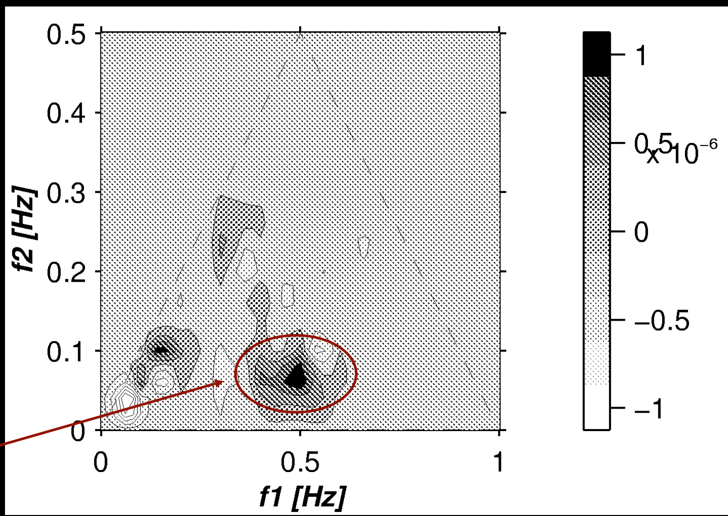


whistlers are linearly damped but receive energy through nonlinear couplings

The spectral energy transfers

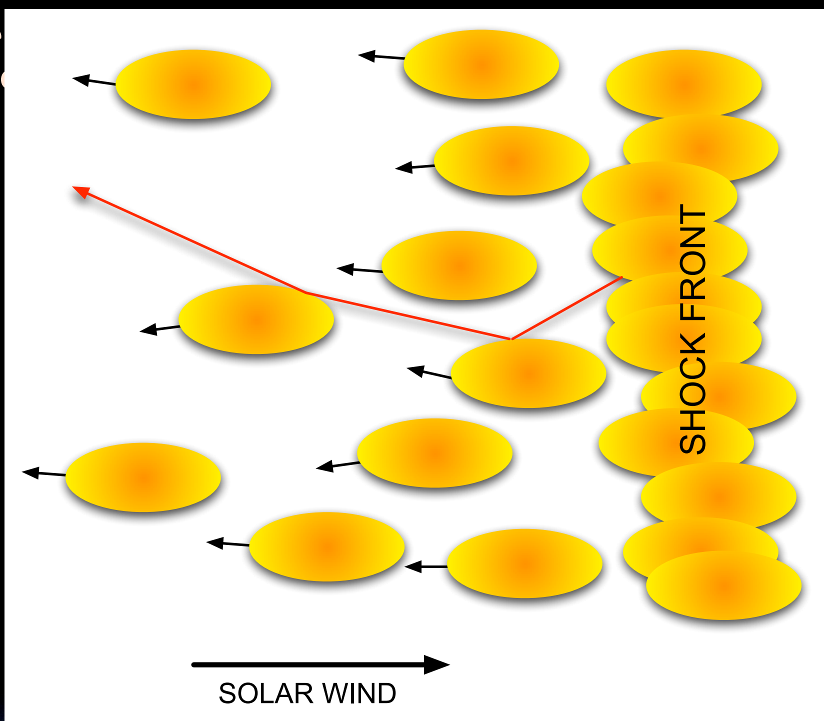
- The quadratic enery transfer $T_{m,n}$ quantifies the amount of spectral energy that flows from $\omega_m + \omega_p \rightarrow \omega_{m+p}$

waves at ~0.65 Hz receive energy through nonlinear coupling between 0.15 Hz and 0.5 Hz



Physical picture

- Whistle ... esult
  from a ...



Miscellaneous

- Volterra kernels, like HOS are sensitive to noise and finite sample effects. Careful ***validation*** is crucial.
- Nonlinear transfer function models have traditionally been estimated in the Fourier domain (Ritz & Powers, 1980)
- Kernels estimation by nonlinear ***parametric models*** (NARMAX = Nonlinear AutoRegressive Moving Average with eXogeneous inputs) today is a powerful alternative (Aguirre & Billings, 1995)

$$\begin{aligned} y_i \;=\; & -0.7610\, u_{i-8} + 0.0032\, u_{i-8} u_{i-1}^2 \\ & -0.0105\, u_{i-24}^2 u_{i-3} + 0.1022\, u_i \\ & +0.0017\, u_{i-23} u_{i-20} u_{i-8} \\ & -0.0115\, u_{i-14} u_{i-3} u_{i-1} \\ & +0.0103\, u_{i-23} u_{i-11} u_i \; + \; \epsilon_i \end{aligned}$$

Conclusions

- HOS have been there for a long time - but they're are still as relevant
- They are the right tools for describing weakly nonlinear wave interactions (weak turbulence, ...)
- But as for all higher order techniques, careful validation is compulsory to avoid pitfalls