Crash consolidation notes on Fourier Theory.

Read in conjunction with your notes from Prof Chapman, and other refs (eg. BOAS, books found by Quentin, many others).

They undoubtedly contain

- items illegible
- ~~minor (? major - hope not)~~
- minor slips (I hope no major ones)
- mismatch with notation of others.

SO DISCUSS THEM WITH FELLOW STUDENTS.

RCB 22 Feb 09

For CO907:

Some Guidance on Fourier Methods (in one dimension)

These notes are an informal supplement, not course material.

Fourier transform

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt \, e^{-i\omega t} f(t)$$

what this achieves is that

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{i\omega t} \, \tilde{f}(\omega)$$

– a reconstruction of the original from a superposition of pure oscillations. This is what $\tilde{f}(\omega)$ "means".

[aside: $e^{i\omega t} = \cos \omega t + i \sin \omega t$, but I think all are happy about that?]

- The factor of 2π is annoying, and by adapting the definitions it can be moved around – but not got rid of!

- Conventions vary for the 'frequency' variable.

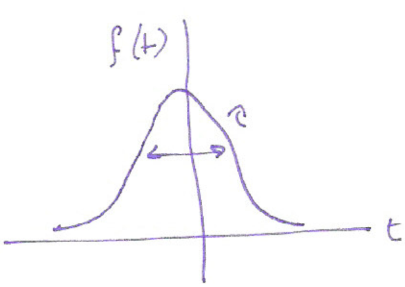
$$\omega = 2\pi f$$

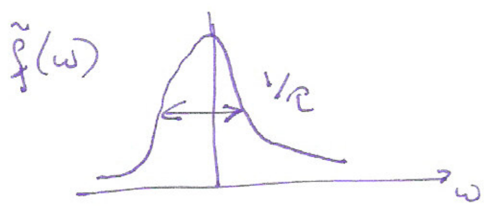
ω: angular frequency (radians/time); f: frequency (cycles/time)

Notation clash! not the same f as further above. – Sorry.

- Not infinite range of time required.

- Simple examples – IMPORTANT INSIGHT "UNCERTAINTY PRINCIPLE"





Rich detail in shape, but generic behaviour as to width!

$$\Delta t \, \Delta \omega \sim 1$$

Fourier Series

We never have data over an infinite range of time!

To make sense of data over just $0 \leq t \leq T$ boundary conditions have to be imposed at $t=0$ and $t=T$

eg. one of

$$\begin{cases} f(0) = f(T) = 0 & \rightarrow \text{(Fourier) Sine } \cancel{\text{transform}}\ \text{Series} \\ f'(0) = f'(T) = 0 & \rightarrow \text{Cosine } \cancel{\text{Transform}}\ \text{Series} \\ f(0) = f(T) & \rightarrow \text{(complex) Fourier Series} \end{cases}$$

Using the last choice $f(0) = f(T)$ we have

$$\tilde{f}_n = \int_0^T f(t)\, e^{-i\omega_n t}\, dt \qquad \text{but} \quad \omega_n = n \cdot 2\pi/T \quad \text{ONLY}$$

because we need $e^{+i\omega_n T} = e^{i\omega_n 0}$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\, \tilde{f}_n\, e^{i\omega_n t}$$

can be thought of as $\frac{\Delta\omega}{2\pi}$

Discrete Fourier Series

We also rarely have anything like continuous time data.

Assuming (originally or after some processing) equally spaced time points $t_m = m\vartheta$ $\qquad \vartheta =$ timestep

The above becomes

$$\tilde{f}_m = \sum_m \underbrace{f(t_m)}_{f_m}\, e^{-i\omega_n t_m} \cdot \vartheta$$

whilst reconstructing gives $f_m = \sum_{n=0}^{N-1} \frac{1}{T}\, \tilde{f}_n\, e^{i\omega_n t_m}$

Not all frequencies are required, as clearly multiple cycles of oscillation between successive time points are meaningless.

Introduce $N = T/\vartheta$ number of sample points

and it should not surprise you that we have to use only N frequencies ω_n, $n = 0$ to $N-1$ (or an equivalent shifted range).

~~Note that $\omega_n t_m =$~~

Note that $\omega_n t_m = n \cdot \frac{2\pi}{T} \, m \vartheta = n m \frac{2\pi}{N}$

and it is obvious that ϑ and T appear naturally as their ratio (dimensional analysis!).

Because of this it is natural to write

~MATLAB fft $\quad \tilde{F}_n = \sum_{m=0}^{N-1} F_m \, e^{-i \frac{mn}{N} 2\pi}$ $\qquad F_m = f_m$

but $\tilde{F}_n = \tilde{f}_n / \vartheta$

~MATLAB ifft $\quad$ and $F_m = \sum_{n=0}^{N-1} \frac{1}{N} \tilde{F}_n \, e^{+i \frac{mn}{N} 2\pi}$

Now N = length of data (T in units of ϑ)

m = time in units of ϑ

n = frequency in units of $1/T$

[n/N = frequency in units of $1/\vartheta$]

There is a final catch about the frequency variable n: it is only meaningful <u>modulo N</u> as $\frac{m(n+N)}{N} = \frac{mn}{N} + m$ and $e^{2\pi i m} = 1$.

Thus as $(N-k)_{\mathrm{mod}\,N} = (-k)_{\mathrm{mod}\,N}$ the high frequencies are equivalent to low negative ...

We should also recap ~~that~~ the uncertainty principle in the discrete coordinates.

We had $\Delta t \; \Delta\omega \sim 1$

which translates directly to $\Delta m \; \Delta\left(\frac{n}{N}\right) \sim 1$

or $\Delta m \; \Delta n \sim N$

Spread in discrete time (units of ϑ)

Spread in discrete frequency (units of $1/T$)

Footnote

On the face of it, calculating

$$\tilde{F}_n = \sum_{m=0}^{N-1} F_m \, e^{-imn\frac{2\pi}{N}}$$

involves N elementary operations for each of N Fourier cpts, total cost $\sim N^2$.

The "Fast Fourier Transform" algorithm achieves the result in only $\sim N \log N$ operations, by ingenious tricks.

All previous analysis related to things LINEAR in original signal (e.g. $\tilde{f}$ is linear in f). Now we look at some bi-linear and important quantities.

1. Auto correlation Function – picks up correlation across time

$$A(\tau) = \frac{1}{T}\int dt \; f(t)\, f(t+\tau)$$

natural prefactor for a time stationary process

2. Power Spectrum

$$S(\omega) = \frac{1}{T} \left| \tilde{f}(\omega) \right|^2 \qquad \text{ie abs}^2 \text{ of F.T.}$$

– a positive definite measure of how much different frequencies contribute to the signal (~~pow~~ and specifically its "power")

~~Magic Result~~ Key Result and insight:

$$\boxed{S(\omega) \text{ is the Fourier transform of } A(\tau)}$$

ie $S(\omega) = \tilde{A}(\omega) = \int_{-\infty}^{\infty} d\tau \, e^{-i\omega\tau} A(\tau)$

$= 2\int_0^\infty \cos\omega\tau \; A(\tau)\, d\tau$ also,

as $A(\tau) = A(-\tau)$.

A closely related result is the

Convolution Theorem.

Let $$h(t) = \int dt' \, f(t-t') \, g(t')$$
$$\equiv f * g$$
which you can think of as "f smoothed by g"

Then $$\tilde{h}(\omega) = \tilde{f}(\omega) \, \tilde{g}(\omega)$$

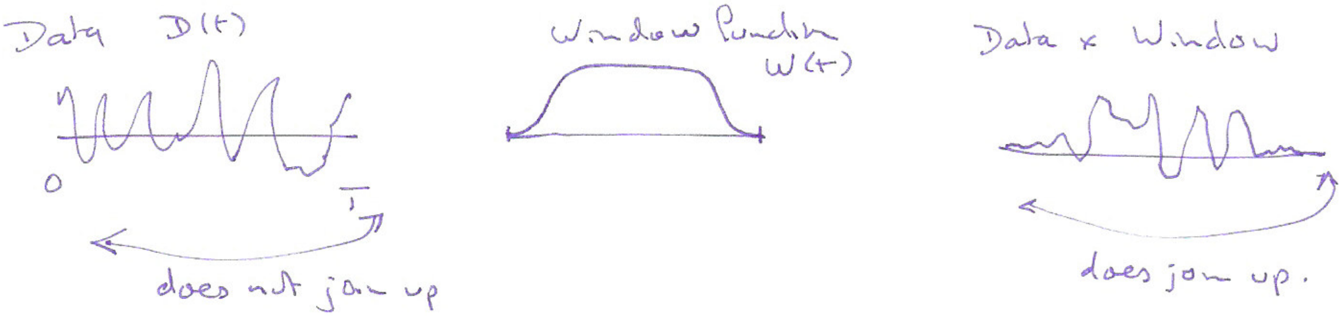
That is if we denote $\tilde{f} = \mathcal{F}[f]$
↑ operation of taking FT

then $$\mathcal{F}[f*g] = \mathcal{F}[f] \, \mathcal{F}[g]$$
FT of ~~a~~ convolution = product of FT's.

Since the inverse is almost the same, we get also that
$$\mathcal{F}[fg] = \mathcal{F}[f] * \mathcal{F}[g] \times \frac{1}{2\pi}$$
FT of product = convolution of FT's.

Applications – windowing and smoothing.

(1) Windowing addresses the need to analyse just a section of data, plus the problem that the data will not match boundary conditions (eg not periodic).



(1) Transforming the windowed data means taking

$$\mathcal{F}\left[D(t)\, W(t)\right] = \tilde{D}(\omega) * \tilde{W}(\omega)$$

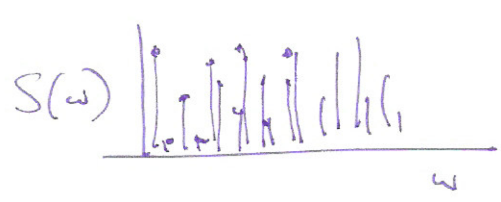
Now if $W(t)$ is smooth and wide $\Delta t \sim T$

then $\tilde{W}(\omega)$ is smooth and narrow $\Delta\omega \sim 1/T$

and all we end up doing is locally smoothing the resulting Fourier transform.

(2) Smoothing Power Spectra.

Each Fourier cpt is linearly independent (indeed orthogonal) of all the others, so they contain no ~~real~~ ~~averaging~~ statistical averaging. The result is that raw Power Spectra are always a mess



You know you want to smooth $S(\omega)$ but what does that mean in terms of the data?

Smoothing gives

$$\tilde{g}(\omega) * S(\omega)$$

$\uparrow$

smoothing function

which is clearly the Fourier transform of a product,

$$\tilde{g}(\omega) * S(\omega) = \mathcal{F}\left(A(\tau)\, g(\tau) \right)$$

Thus smoothing S is equivalent to multiplying $A(\tau)$ by an envelope function. If $\tilde{g}(\omega)$ is ~~broad~~ narrow, $g(\tau)$ is broad and if it is broader than the range over which $A(\tau)$ falls off then we have done little damage to the data! In other words, smoothing S by $\tilde{g}(\omega)$ ~~(narrow)~~ cuts out long time features (? noise) in $A(\tau)$.

Wavelet Spectra

Many wavelet transforms can be thought of as Fourier transform of data under a sliding window.

Centre of wavelet

e.g. $\cancel{W_{t,\omega}} = \int dt' \, e^{-i\omega t'} f(t') \, W\left(\frac{t'-t}{\tau}\right)$

W

width.

~~Now this is just a convolution~~

~~Whilst this can be thought of as a convolution, it is best first to consider t=0 and think of it as the~~

Now put $t=0$ so this is the FT of a simple product.

Clearly ~~frequency~~ the original FT then gets convolved by the FT of the window, spreading frequencies by of order $\Delta\omega \simeq 1/\tau$.

Therefore: no point considering frequency more accurately than $1/\tau$

for frequency $\gg 1/\tau$, we have surplus resolution in frequency and could gain by using smaller τ and hence finer resolution in time.

Hence in practice ~~many~~ one often hard wires a relationship

$\omega\tau =$ a few cycles

and combines $e^{-i\omega t'} W =$ 

Bispectra.

FT was linear in original data. $\propto f$

Autocorrelation $\leftrightarrow$ P. Spectrum bilinear $\propto f f$

At degree 3 the natural quantity to measure for a time stationary process is either

$$\frac{1}{T} \int f(t)\, f(t+\tau_1)\, f(t+\tau_2)\, dt = A(\tau_1, \tau_2)$$

generalizes autocorrel^n

or equivalently "Bispectrum"

$$S(\omega_1, \omega_2) = \tilde{f}(\omega_1 + \omega_2)\, \tilde{f}(-\omega_1)\, \tilde{f}(-\omega_2)$$

generalises power spectrum

It should not surprise you that $S(\omega_1, \omega_2)$ is a double F.T. of $A(\tau_1\ \tau_2)$.