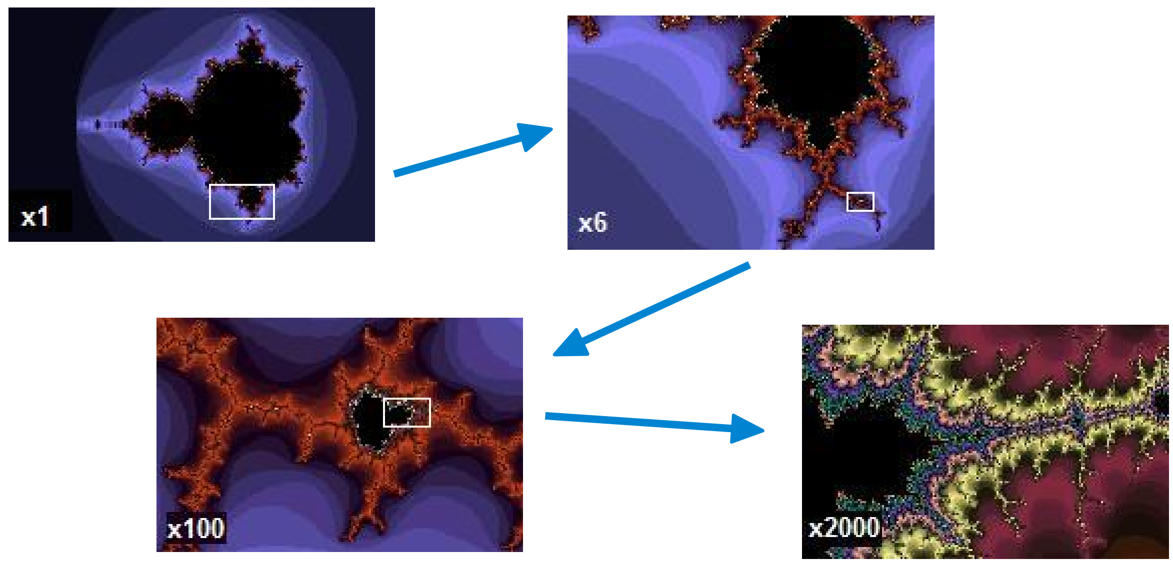
Self-Organised Criticality

Fractals

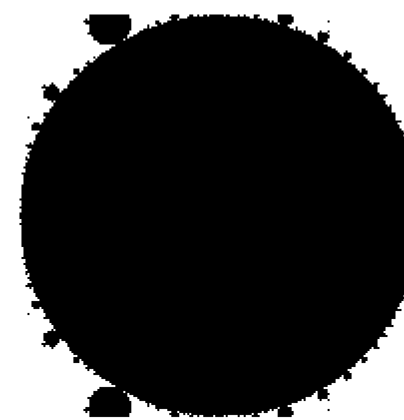
B. Mandelbrot



$$z_{n+1} = z_n^2 + c$$

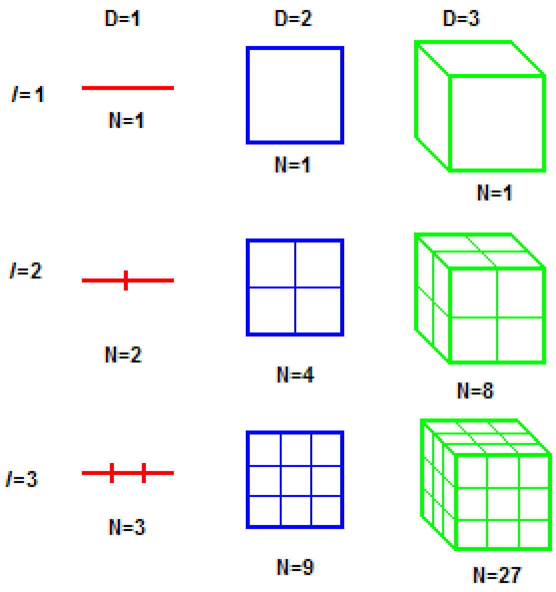
$$c \in M \iff \lim_{n\to\infty} |z_{n+1}| \leq 2$$

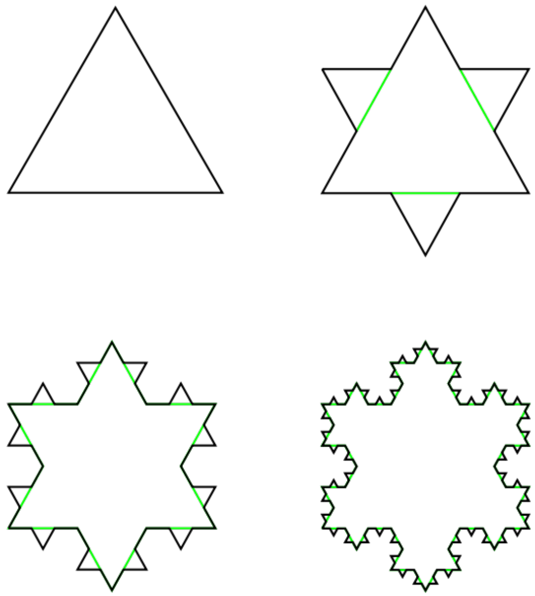
Self similarity



Self-Organised Criticality

Fractals





$$N \propto \epsilon^{-D}$$

Fractal dimension

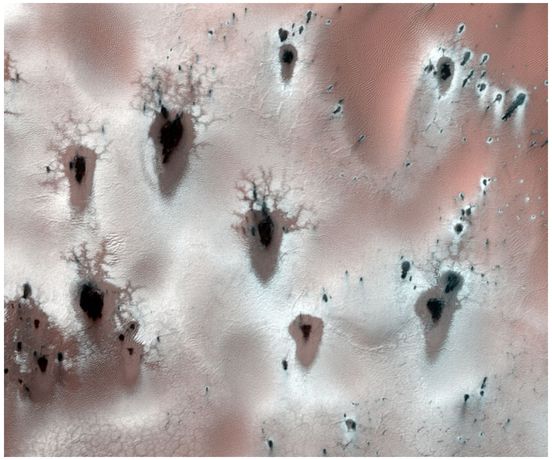
Koch snowflake

$$\log_\epsilon N = -D$$

$$D \simeq 1.2619$$

Self-Organised Criticality

Fractals

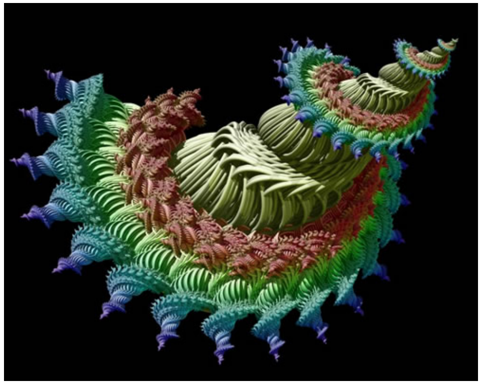


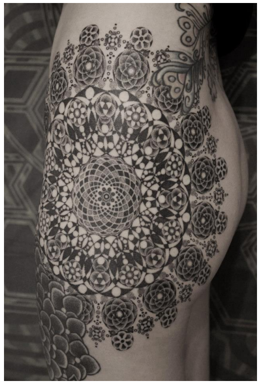
Barnsley Fern

$$f(x,y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

Self-Organised Criticality

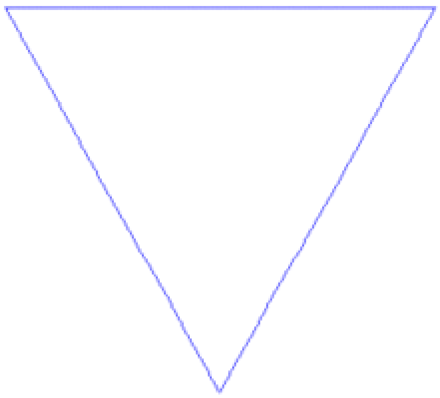
Fractals



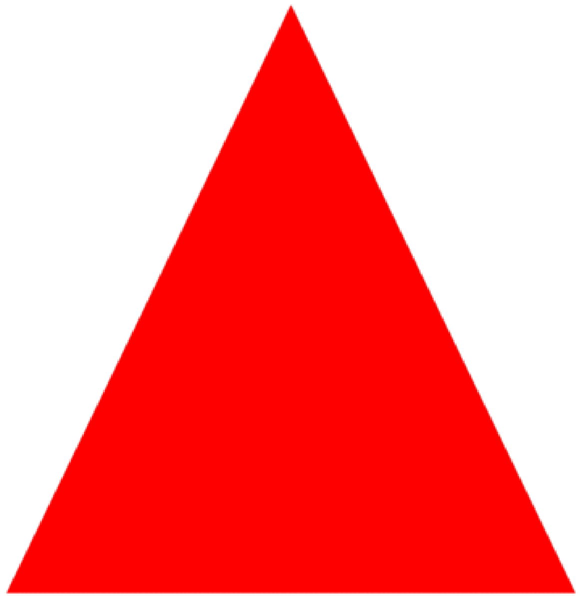


Self-Organised Criticality

Fractals



Koch snowflake

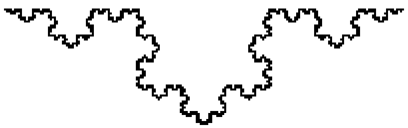


Sierpinski triangle

Self-Organised Criticality

Power laws

Self similarity /
Scale invariance



$$f(x)?$$

$$f(cx) \propto f(x)$$

$$f(x) = ax^{-k}$$

$$f(cx) = c^{-k} f(x)$$

$$N \propto \epsilon^{-D}$$

Fractal dimension

Self-Organised Criticality

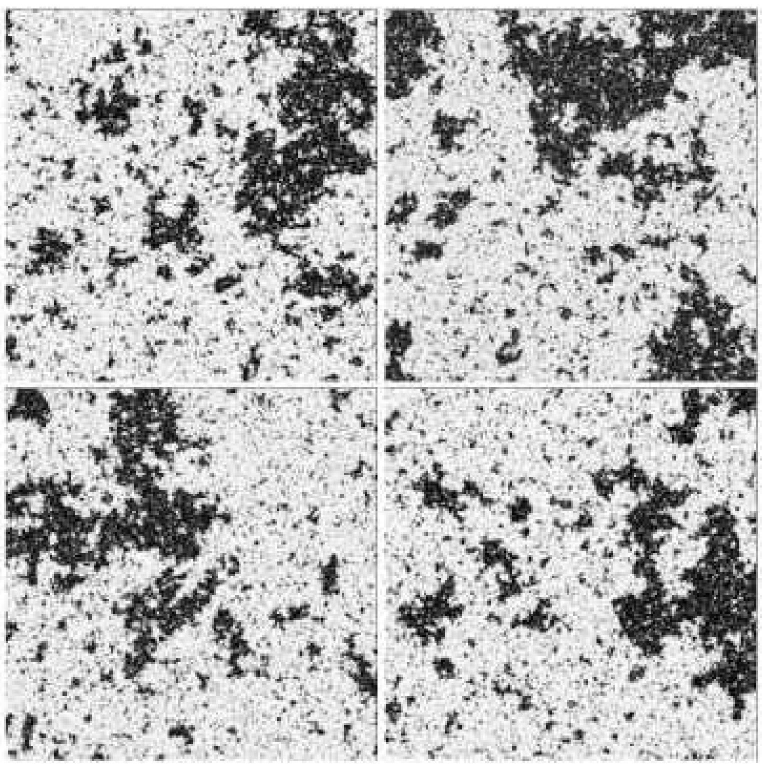
Power laws

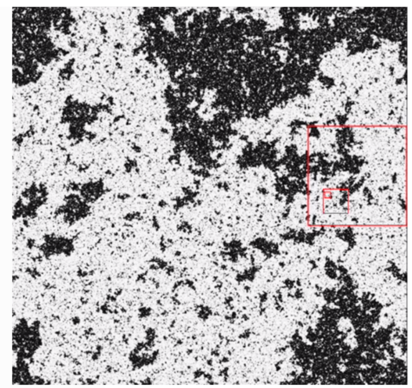
$$t = \frac{T - T_c}{T_c}$$

$$m \propto |t|^{\tilde{\beta}}$$

$$\chi \propto |t|^{-\gamma}$$

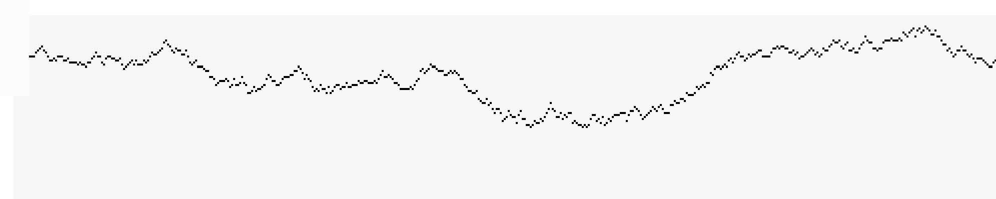
$$\xi \propto |t|^{-\nu}$$





Ising at criticality

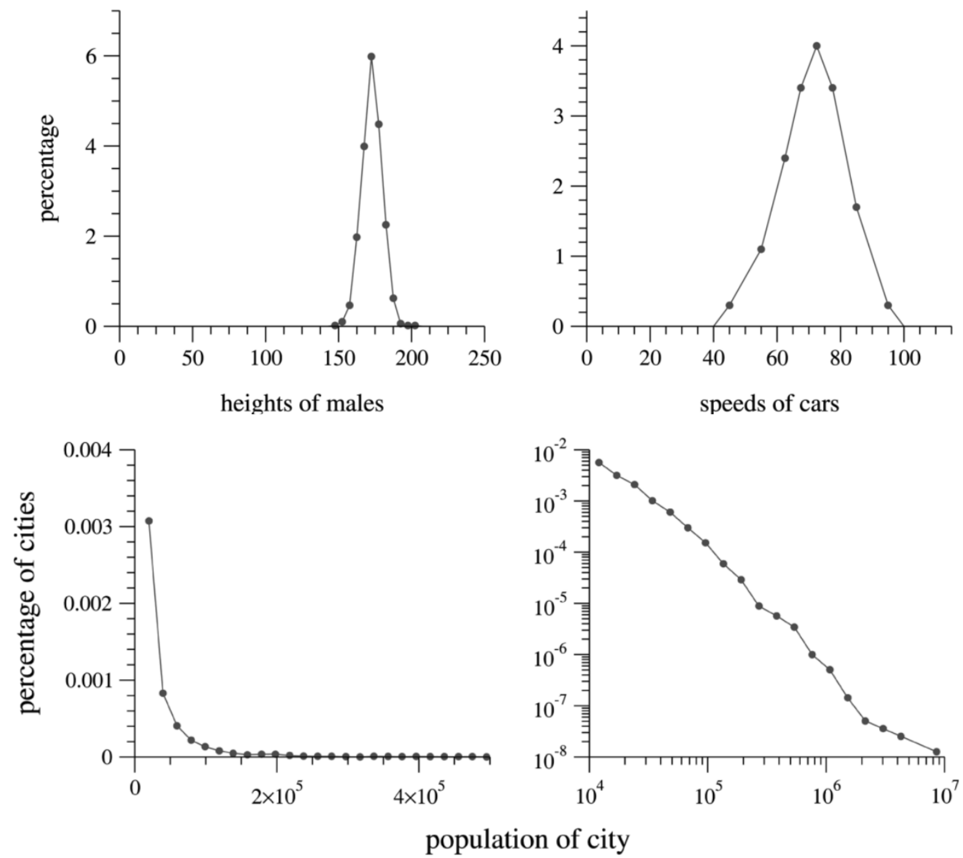
$$\text{Mean field: } \tilde{\beta} = 1/2, \gamma = 1, \nu = 1/2$$



Brownian motion

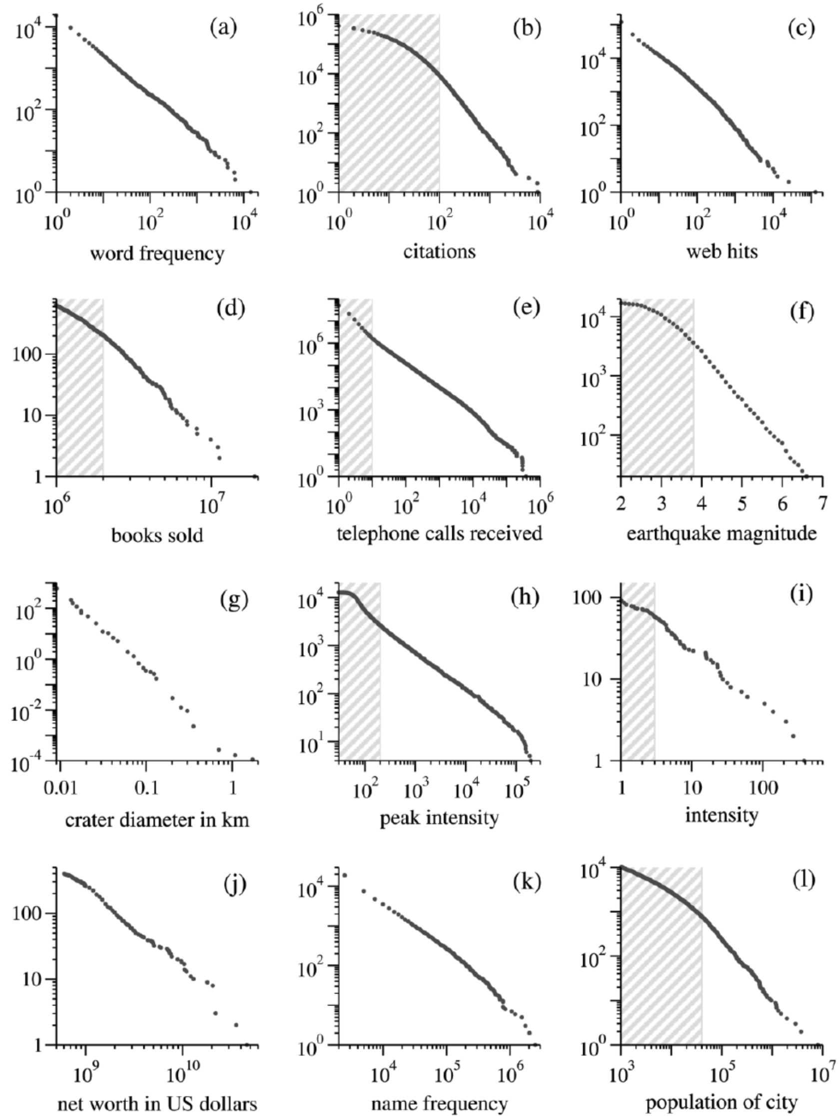
Self-Organised Criticality

Power laws

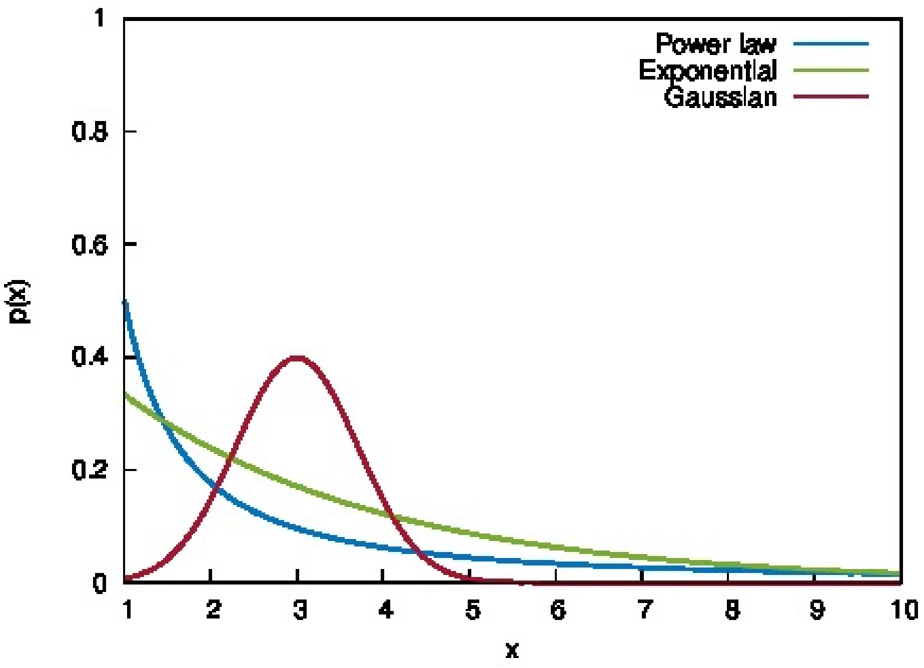


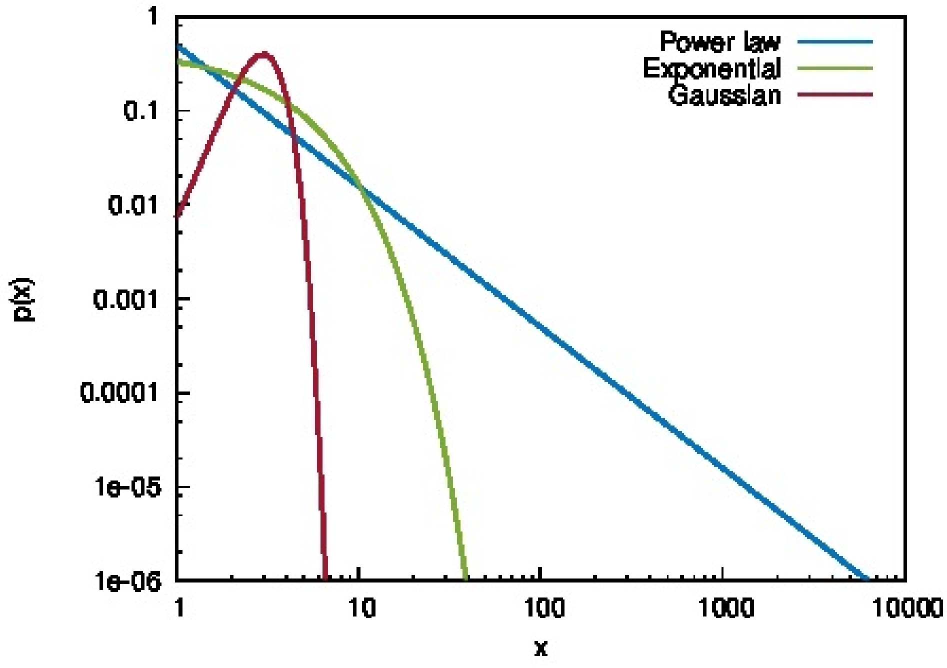
MEJ Newman (2005) “Power laws, Pareto distributions and Zipf's law”, *Contemp Phys*

Self-Organised Criticality



Self-Organised Criticality



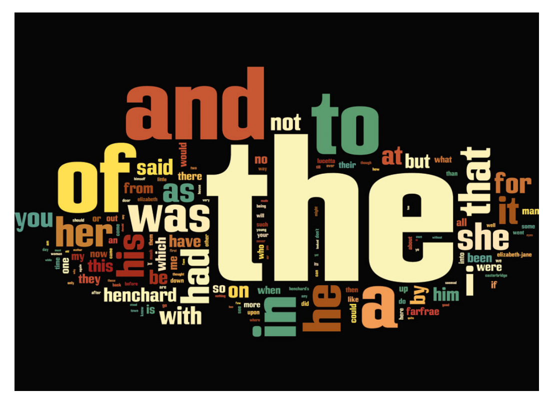


Self-Organised Criticality

Pareto principle

Gutenberg–Richter law

Richardson's law



Zipf's law

Self-Organised Criticality

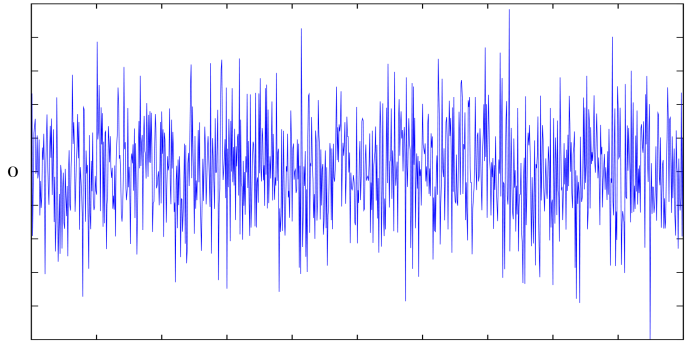
Examples of power-law functions

More than a hundred power-law distributions have been identified in physics (e.g. sandpile avalanches and earthquakes), biology (e.g. species extinction and body mass), and the social sciences (e.g. city sizes and income).[13] Among them are:

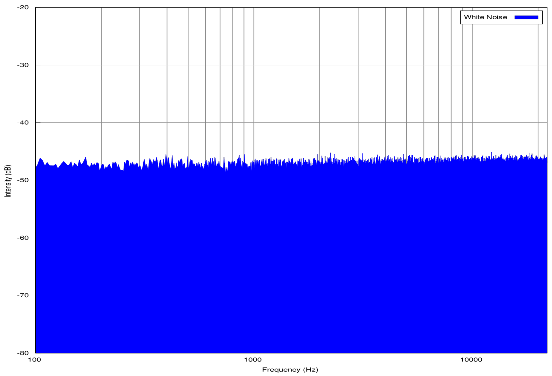
- The frequency-dependency of acoustic attenuation in complex media
- The Stevens' power law of psychophysics
- The Stefan–Boltzmann law
- The input-voltage–output-current curves of field-effect transistors and vacuum tubes approximate a square-law relationship, a factor in "tube sound".
- Square-cube law (ratio of surface area to volume)
- Kleiber's law relating animal metabolism to size, and allometric laws in general
- A 3/2-power law can be found in the plate characteristic curves of triodes.
- The inverse-square laws of Newtonian gravity and electrostatics, as evidenced by the gravitational potential and Electrostatic potential, respectively.
- Self-organized criticality with a critical point as an attractor
- Exponential growth and random observation (or killing)[14]
- Progress through exponential growth and exponential diffusion of innovations[15]
- Highly optimized tolerance
- Model of van der Waals force
- Force and potential in simple harmonic motion
- Kepler's third law
- The initial mass function of stars
- The M-sigma relation
- Gamma correction relating light intensity with voltage
- The two-thirds power law, relating speed to curvature in the human motor system.
- The Taylor's law relating mean population size and variance of populations sizes in ecology
- Behaviour near second-order phase transitions involving critical exponents
- Proposed form of experience curve effects
- The differential energy spectrum of cosmic-ray nuclei
- Fractals
- Pareto distribution and the Pareto principle also called the "80–20 rule"
- Zipf's law in corpus analysis and population distributions amongst others, where frequency of an item or event is inversely proportional to its frequency rank (i.e. the second most frequent item/event occurs half as often the most frequent item, the third most frequent item/event occurs one third as often as the most frequent item, and so on).
- The safe operating area relating to maximum simultaneous current and voltage in power semiconductors.
- Supercritical state of matter and supercritical fluids, such as supercritical exponents of heat capacity and viscosity.[16]
- Zeta distribution (discrete)
- Yule–Simon distribution (discrete)
- Student's t-distribution (continuous), of which the Cauchy distribution is a special case
- Lotka's law
- The scale-free network model
- Pink noise
- Neuronal avalanches[4]
- The law of stream numbers, and the law of stream lengths (Horton's laws describing river systems) [*citation needed*]
- Populations of cities (Gibrat's law)[*citation needed*]
- Bibliograms, and frequencies of words in a text (Zipf's law)[*citation needed*]
- 90–9–1 principle on wikis (also referred to as the 1% Rule)[*citation needed*]
- Richardson's Law for the severity of violent conflicts (wars and terrorism){Lewis Fry Richardson, The Statistics of Deadly Quarrels, 1950}
- Gutenberg–Richter law of earthquake magnitudes

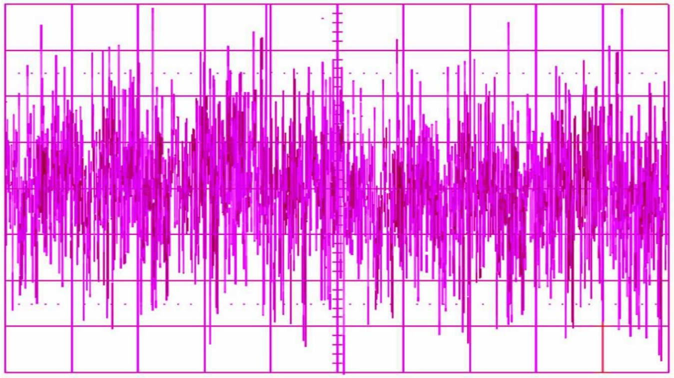
Self-Organised Criticality

Pink noise



$$S(f) \sim \text{Uniform}$$

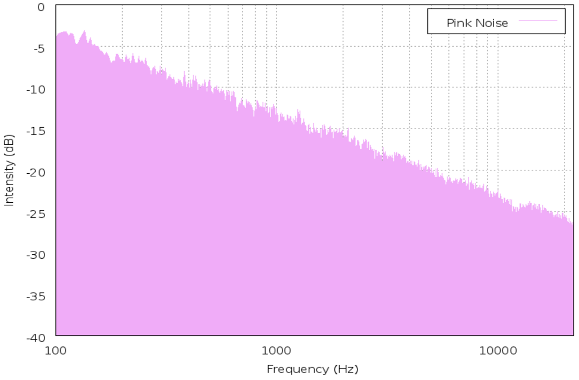




$$S(f) \propto \frac{1}{f}$$

$$S(f) \propto \frac{1}{f^{\alpha}}$$

$$(\alpha \simeq 1)$$

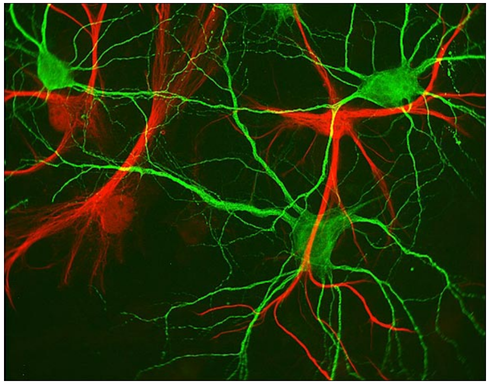


Self-Organised Criticality

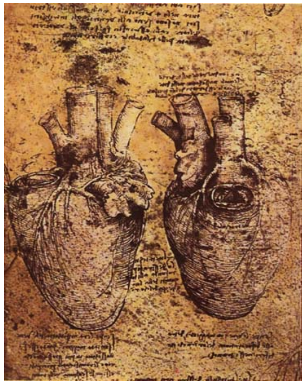
Pink noise

Tides and river heights

Quasar light emission



Firings of single neurons



Heart beat

Resistivity in solid state devices

Music

Self-Organised Criticality

PHYSICAL REVIEW LETTERS

VOLUME 59 | 27 JULY 1987 | NUMBER 4

Self-Organized Criticality: An Explanation of $1/f$ Noise

Per Bak, Chao Tang, and Kurt Wiesenfeld

Physics Department, Brookhaven National Laboratory, Upton, New York 11973

(Received 13 March 1987)

We show that dynamical systems with spatial degrees of freedom naturally evolve into a self-organized critical point. Flicker noise, or $1/f$ noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.

PACS numbers: 05.40.+j, 02.90.+p

Self-Organised Criticality

RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 40, NUMBER 10 1 OCTOBER 1989

$1/f$ noise, distribution of lifetimes, and a pile of sand

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Kim Christensen and Hans C. Fogedby

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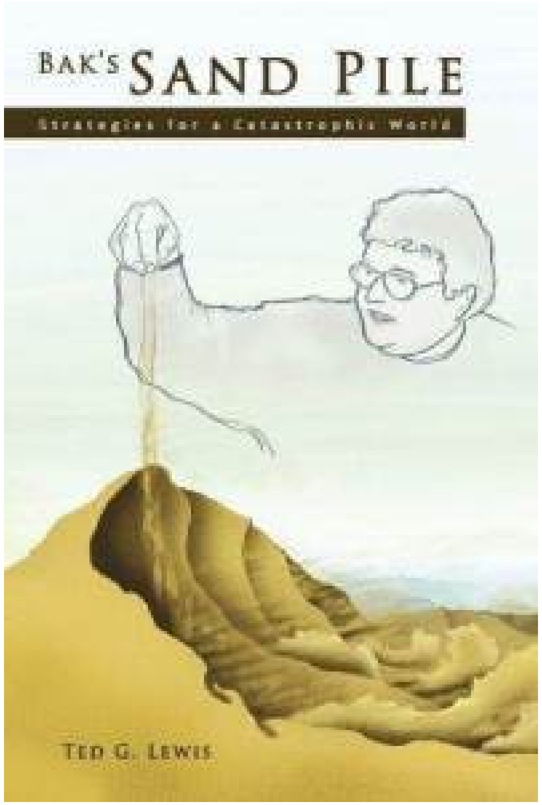
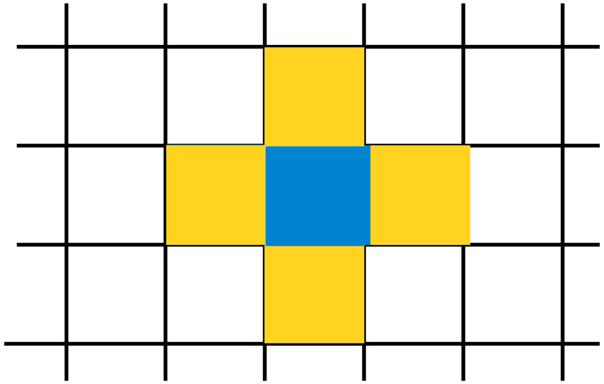
(Received 26 June 1989)

A connection between the distribution of lifetimes and the power spectrum is derived. It is shown that the flow of sand down the slope in the cellular automaton model, considered recently by Bak, Tang, and Wiesenfeld [Phys. Rev. Lett. **59**, 381 (1987)], has a $1/f^2$ power spectrum in one and two dimensions. The flow over the rim of the system behaves similar to the transport in a real sand pile as measured by Jaeger, Liu, and Nagel [Phys. Rev. Lett. **62**, 40 (1989)].

Self-Organised Criticality





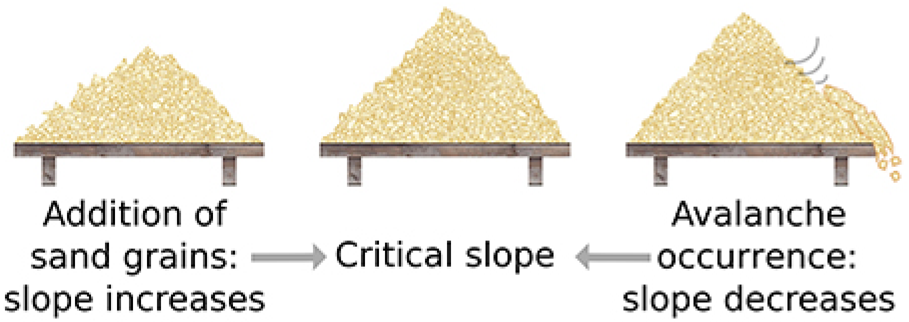
If $z(x,y) \geq 4$:

$$z(x,y) \rightarrow z(x,y) - 4$$

$$z(x \pm 1, y) \rightarrow z(x \pm 1, y) + 1$$

$$z(x, y \pm 1) \rightarrow z(x, y \pm 1) + 1$$

Self-Organised Criticality



Self-Organised Criticality

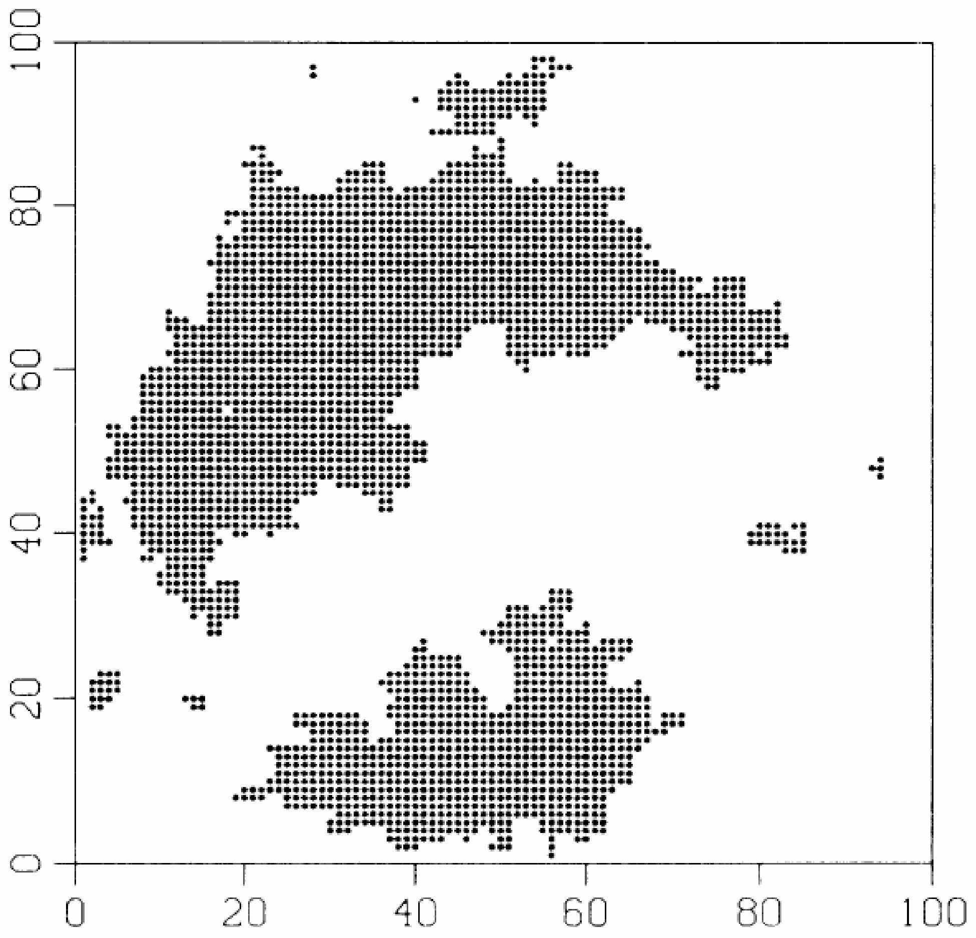


FIG. 1. Self-organized critical state of minimally stable clusters, for a 100×100 array.

Self-Organised Criticality

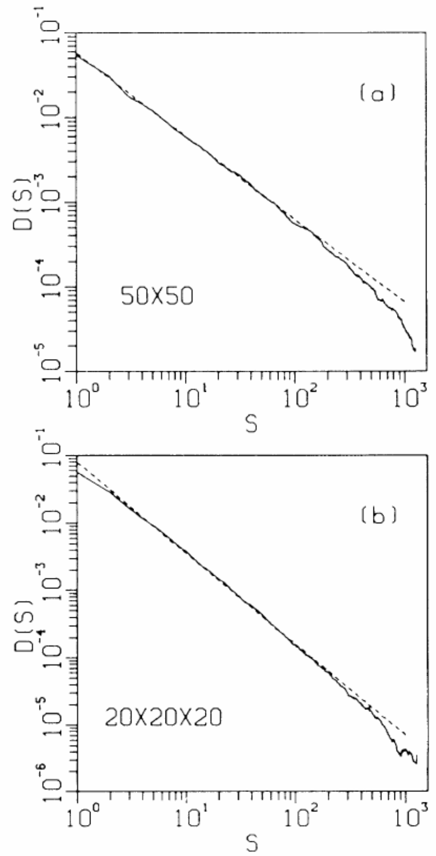


FIG. 2. Distribution of cluster sizes at criticality in two and three dimensions, computed dynamically as described in the text. (a) 50×50 array, averaged over 200 samples; (b) 20×20×20 array, averaged over 200 samples. The data have been coarse grained.

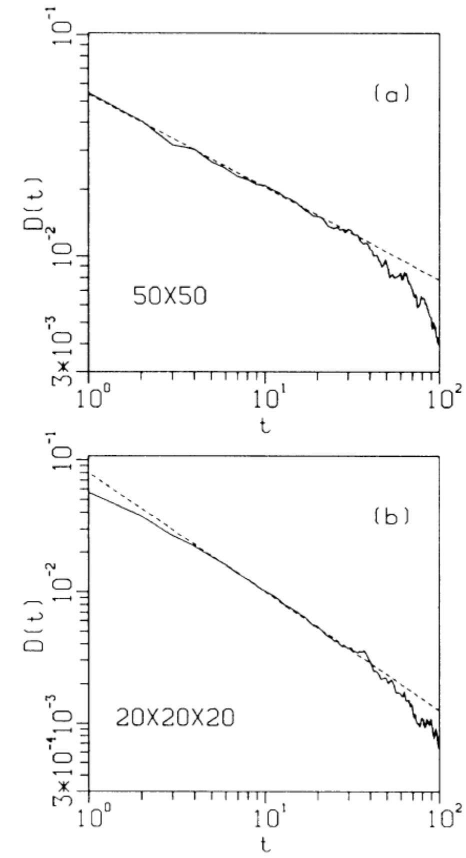


FIG. 3. Distribution of lifetimes corresponding to Fig. 2. (a) For the 50×50 array, the slope $\alpha \approx 0.42$, yielding a "$1/f$" noise spectrum $f^{-1.58}$; (b) 20×20×20 array, $\alpha \approx 0.90$, yielding an $f^{-1.1}$ spectrum

Self-Organised Criticality

Avalanche dynamics in a pile of rice

Vidar Frette*, **Kim Christensen, Anders Malthe-Sørenssen, Jens Feder, Torstein Jøssang & Paul Meakin**

Department of Physics, University of Oslo, PO Box 1048, Blindern, N-0316 Oslo, Norway

THE idea of self-organized criticality[1] (SOC) is commonly illustrated conceptually with avalanches in a pile of sand grains. The grains are dropped onto a pile one by one, and the pile ultimately reaches a stationary 'critical' state in which its slope fluctuates about a constant angle of repose, with each new grain being capable of inducing an avalanche on any of the relevant size scales. Some numerical models of sand-pile dynamics do show SOC[1–8], but the behaviour of real sand piles remains ambiguous[9–18]. Here we describe experiments on a granular system—a pile of rice—in which the dynamics exhibit self-organized critical behaviour in one case (for grains with a large aspect ratio) but not in another (for less elongated grains). These results show that SOC is not as 'universal' and insensitive to the details of a system

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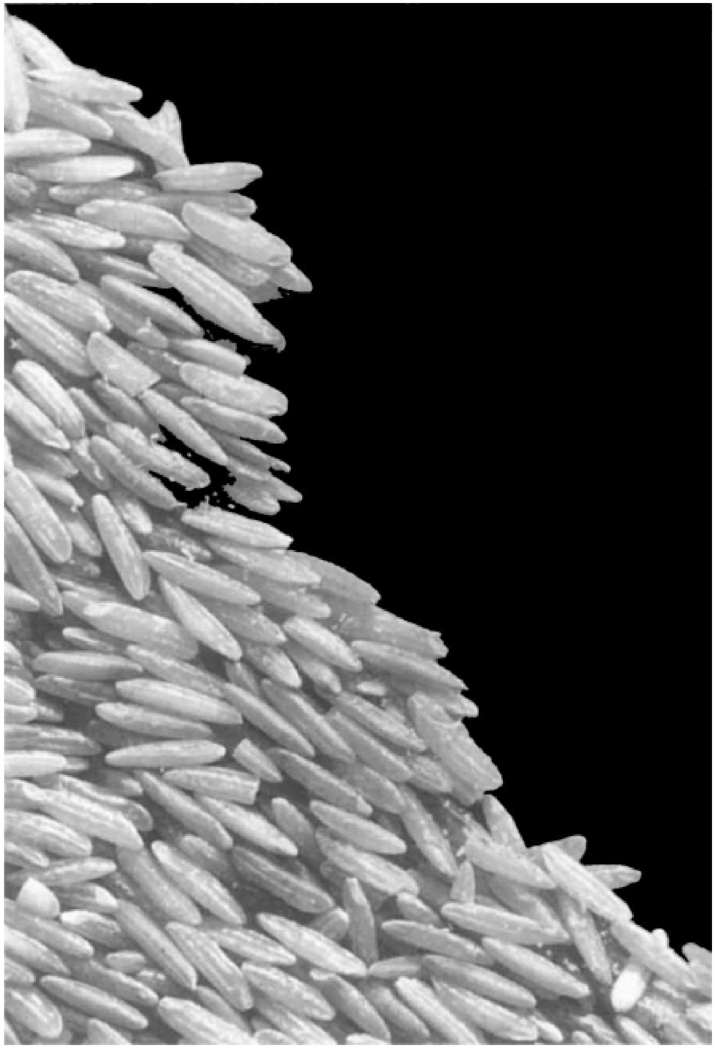
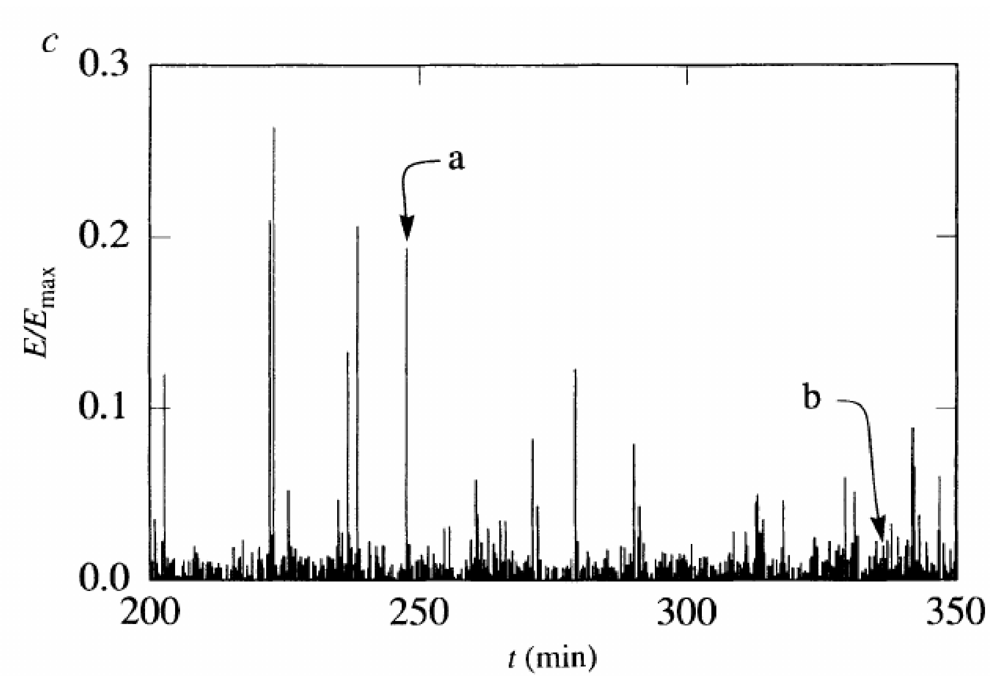
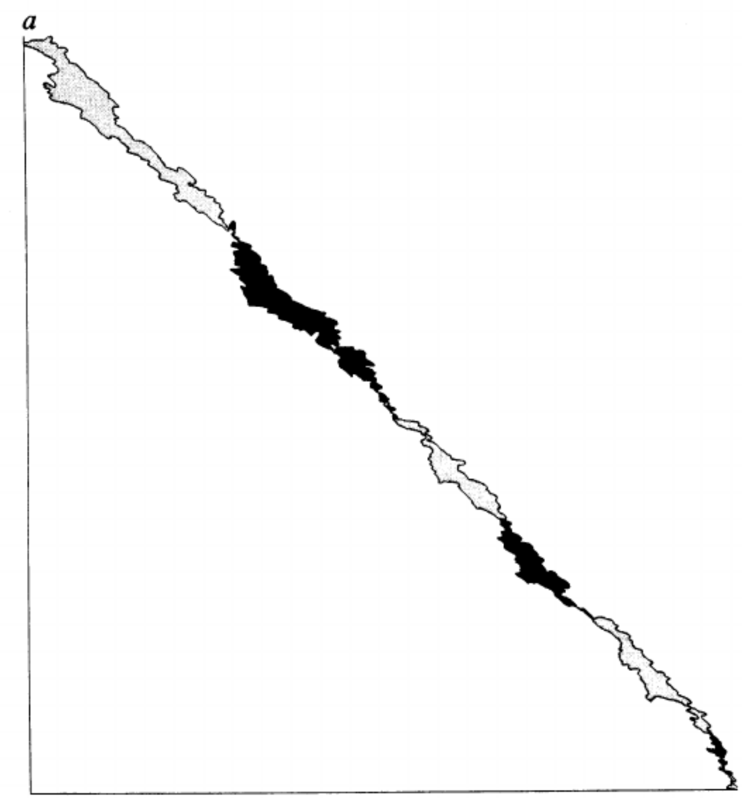
NATURE · VOL 379 · 4 JANUARY 1996



Self-Organised Criticality





Self-Organised Criticality

A forest-fire model and some thoughts on turbulence

Per Bak, Kan Chen
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and

Chao Tang
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Received 9 March 1990; revised manuscript received 1 April 1990; accepted for publication 7 April 1990
Communicated by A.R. Bishop

In the context of a forest-fire model we demonstrate critical scaling behavior in a "turbulent" non-equilibrium system. Energy is injected uniformly, and dissipated on a fractal. Critical exponents are estimated by means of a Monte Carlo renormalization-group calculation.

Self-Organised Criticality

VOLUME 69, NUMBER 11 PHYSICAL REVIEW LETTERS 14 SEPTEMBER 1992

Self-Organized Critical Forest-Fire Model

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(Received 30 June 1992)

A forest-fire model is introduced which contains a lightning probability f. This leads to a self-organized critical state in the limit $f \rightarrow 0$ provided that the time scales of tree growth and burning down of forest clusters are separated. We derive scaling laws and calculate all critical exponents. The values of the critical exponents are confirmed by computer simulations. For a two-dimensional system, we show that the forest density in the critical state assumes its minimum possible value, i.e., that energy dissipation is maximum.

Self-Organised Criticality

- A burning cell turns into an empty cell

- A tree will burn if at least one neighbor is burning

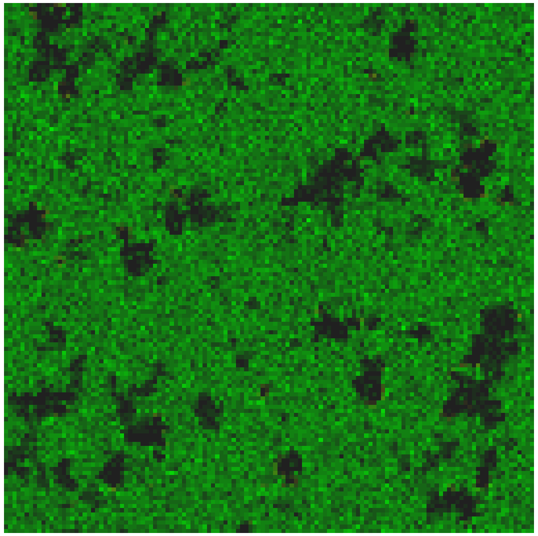
- A tree ignites with probability *f* even if no neighbor is burning

- An empty space fills with a tree with probability *p*

$$f \ll p \ll T_{\mathrm{smax}}$$

Longest fire

Control parameter: p/f



Self-Organised Criticality

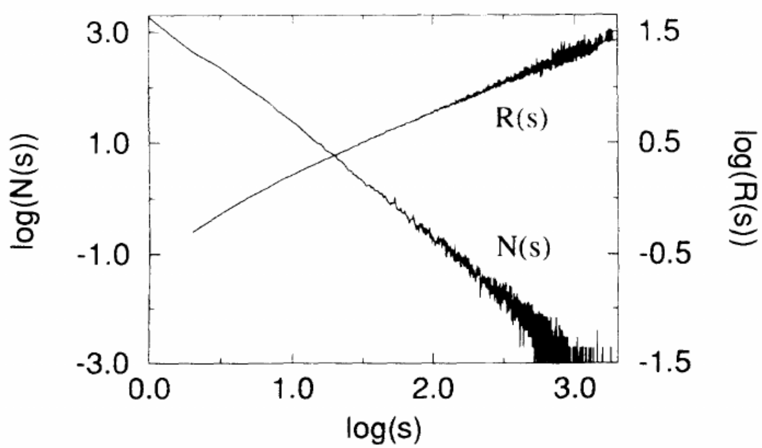


FIG. 2. Mean number of clusters and mean cluster radius as functions of the cluster size for $f/p = 1/70$ and $d = 2$.

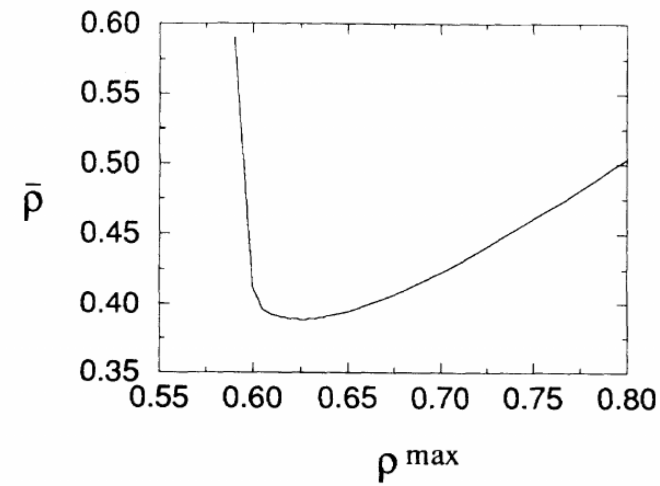


FIG. 3. Mean forest density as a function of maximum forest density for $d = 2$.

Self-Organised Criticality

VOLUME 71, NUMBER 24 PHYSICAL REVIEW LETTERS 13 DECEMBER 1993

Punctuated Equilibrium and Criticality in a Simple Model of Evolution

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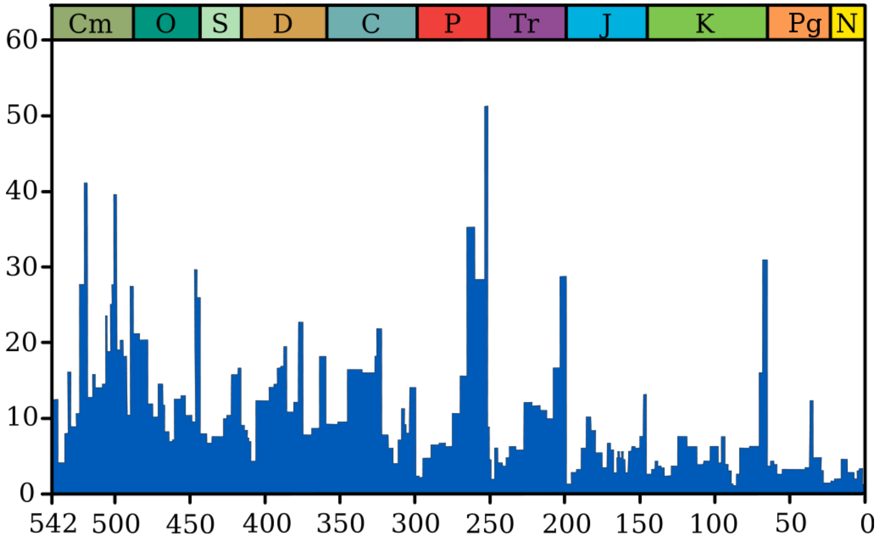
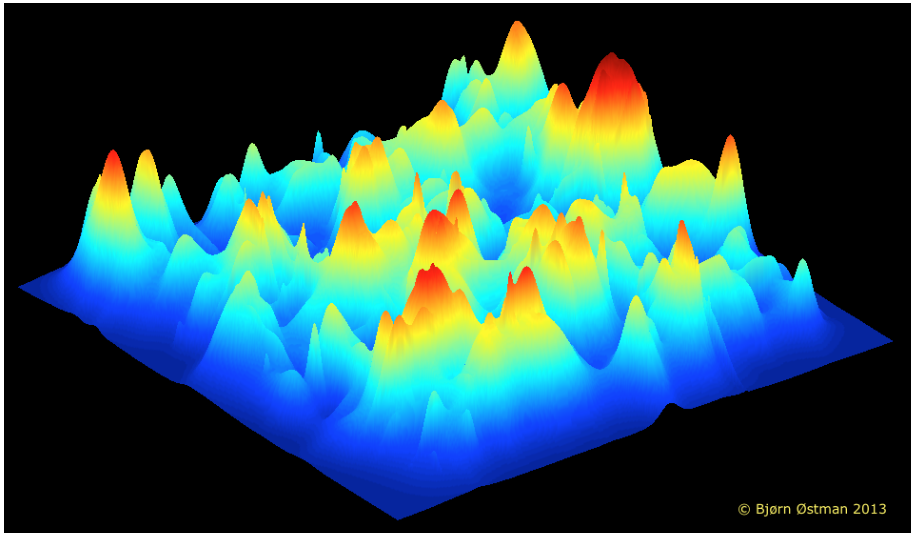
(Received 7 July 1993)

A simple and robust model of biological evolution of an ecology of interacting species is introduced. The model self-organizes into a critical steady state with intermittent coevolutionary avalanches of all sizes; i.e., it exhibits "punctuated equilibrium" behavior. This collaborative evolution is much faster than noncooperative scenarios since no large and coordinated, and hence prohibitively unlikely, mutations are involved.

Self-Organised Criticality



Self-Organised Criticality

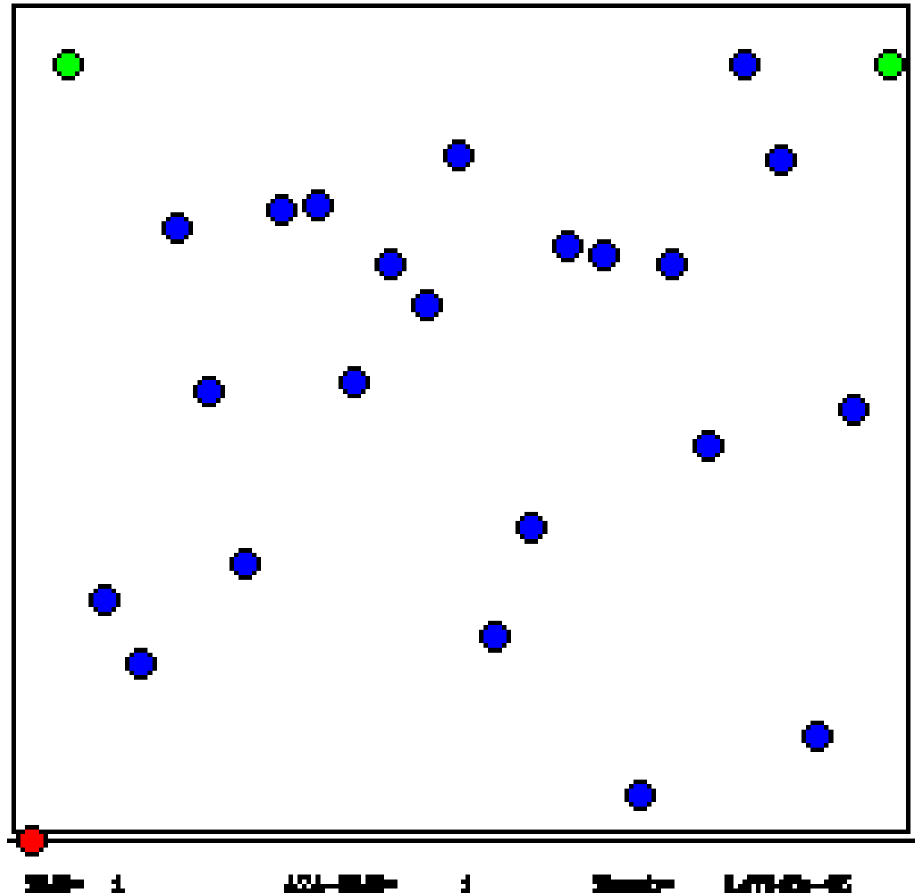


Self-Organised Criticality

Our model, intended to represent the main features of all of this, is defined and simulated as follows: (i) N species are arranged on a one-dimensional line with periodic boundary conditions. (ii) A random barrier, B_i, equally distributed between 0 and 1, is assigned to each species. At each time step, the ecology is updated by (iii) locating the site with the lowest barrier and mutating it by assigning a new random number to that site, and (iv) changing the landscapes of the two neighbors to the right and left, respectively, by assigning new random numbers to those sites, too.

Bak & Sneppen (1993) *Phys Rev Lett*

Self-Organised Criticality



Self-Organised Criticality

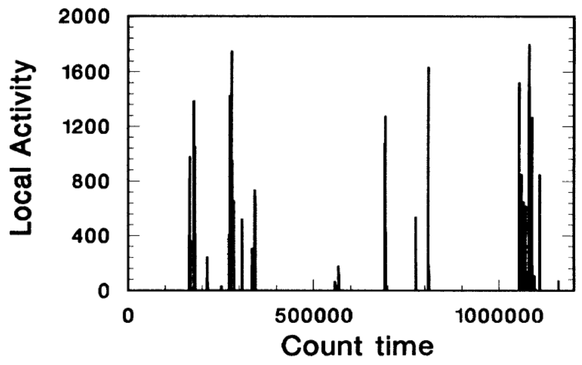


FIG. 4. Punctuated equilibrium behavior. Activity vs time in a local segment of ten consecutive sites is shown for a system of size $N = 512$. Time is measured in units of the number of mutations. In real time, the intermittency is further enhanced by the exponential enlargement of the periods of stasis.

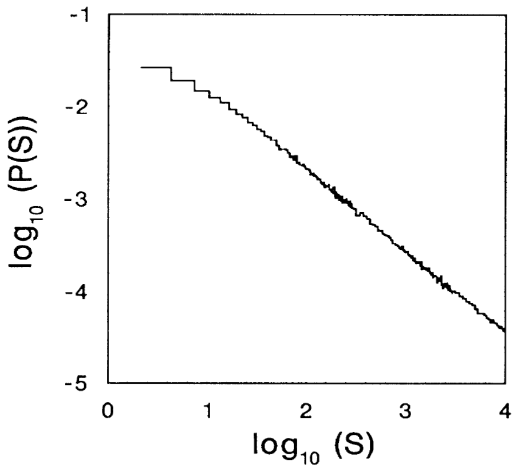


FIG. 5. Distribution of avalanche sizes in the critical state. Here an avalanche is defined by subsequent sequential activity below punctuation of the barrier $B = 0.65$.