Metastability and extinction in birth-death systems

Metastability is ubiquitous in nature. Given that we observe a species today means that it has not been extinct so far. However, the possibility of ultimate extinction is a sad reality for any species and has to be taken into account in mathematical models for population dynamics. Since for realistic descriptions the state space should also be finite, the only true stationary state of probabilistic models often corresponds to extinction. This leads to the fundamental question: How do we describe the long-term behaviour of a surviving species?
More generally, it is clear that the paradigm of studying the stationary state has to be adjusted for many complex systems. Birth-death systems are simple prototype models for such situations with a vast area of applications, from the kinetics of chemical reactions, biological populations and epidemics to the dynamics of financial markets and ecology (see [1] and references therein).

Details.

So-called 'quasi-stationary distributions (QSDs)' are a well established concept in the mathematics literature for general birth-death systems with absorbing states. These are limiting distributions conditioned on non-absporption of the process, a short introduction can be found e.g. in the first two sections of [3] in a biological context. For large but finite systems QSDs are metastable distributions in the sense that they have a very long lifetime before the system eventually gets absorbed. Mathematically, QSDs can be characterized by an eigenvalue problem very similar to stationary distributions. For Markov chains with generator G and state space $S \cup \{0\}$ where 0 is an absorbing state, we have

$$\boldsymbol{\pi}\, G = -\theta\, \boldsymbol{\pi}\ , \quad \text{with the eigenvalue} \quad \theta = \sum_{i\in S} \pi_i g_{i0} \tag{1}$$

corresponding to the quasi-stationary probability current into the absorbing state.
More recently and fairly independently of the mathematical literature, metastable behaviour has been studied within the statistical mechanics of complex systems [1,2]. The authors analyse simple birth-death systems, where a first idea of the dynamic properties is obtained from the mean-field rate equations. The full probabilistic description is provided by the master equation, which is analyzed using generating functions and perturbative methods from Physics.

Aim of the project.

The aim of this project is to combine both approaches mentioned above to get a better understanding of the metastable behaviour in complex systems. This should then be used to devise an efficient agent-based simulation algorithm to sample from metastable states in a controlled way. It can be further developed to include spatial information and study the dynamics of so-called metapopulations [4], consisting of several colonies which are only weakly interacting.
Besides the computational aspect, the project provides an introduction to the theory of metastability, which is an important aspect of complex systems. It is also relevant in understanding the finite-size behaviour of critical phenomena which is vital when trying to apply the theory of phase transitions to the real world. The project can be extended to PhD in the direction of particular applications such as the study of biodiversity [2], or towards more theoretical aspects of metastability, which is a current topic of major research in applied probability theory.

References.

[1] M. Assaf, B. Meerson: *Spectral Theory of Metastability and Extinction in Birth-Death Systems*, PRL **97**, 200602 (2006) and *Spectral Theory of Metastability and Extinction in Branching-Annihilation Reaction*, PRE **75**, 031122 (2007)

[2] T. Reichenbach, M. Mobilia, E. Frey, *Mobility promotes and jeopardizes biodiversity in rock-paper-scissors games*. Nature **448**, 1046-1049 (2007)

[3] I. Nasell, *On the quasi-stationary distribution of the stochastic logistic epidemic*. Math. Biosci. **156**, 21-40 (1999)

[4] P.K. Pollett, *Modelling quasi-stationary behaviour in metapopulations*. Mathematics and Computer Simulation **48**(4-6), 393-405 (1999)

Contact details:

Stefan Grosskinsky
Mathematics and Complexity
University of Warwick

office: D1.10
phone: +44 2476522673
S.W.Grosskinsky@warwick.ac.uk