MINIPROJECT IN COMPLEXITY — FEBRUARY 2010 — DANIEL UELTSCHI

SPATIAL RANDOM PERMUTATIONS AND CYCLE DISTRIBUTION

Background. This project suggests to investigate certain models of random permutations by means of numerical simulations (Monte Carlo methods). The model is at the crossroad between statistical mechanics and probability theory. It is motivated by the phenomenon of *Bose-Einstein condensation*, a phase transition that occurs in large systems of quantum bosonic particles. This project requires basic knowledge of Monte-Carlo simulations and nothing else. Here, only those features that are necessary to the project are described. But the topic is much broader, and there is ample room for a PhD thesis.

Spatial random permutations. Random permutations have been a hot topic in probability theory and applied mathematics because of their connections to random matrices and other systems. But our random permutations have an extra feature, namely that points are penalised if they travel far. More precisely, we consider permutations $\pi : \Lambda \to \Lambda$, where $\Lambda \subset \mathbb{Z}^d$ is a finite subset of the square lattice. The most interesting dimension is $d = 3$. Permutations receive the following weight:

$$\prod_{x \in \Lambda} \exp\left\{-\alpha |x - \pi(x)|^2\right\}.$$

Here, α is a parameter (it is proportional to the temperature of the system, in the statistical mechanics interpretation). The most probable permutation is the identity permutation — all particles keep their positions. There is an "energy-entropy competition" controlled by the parameter α: As α gets smaller, particles can travel farer away.

One is interested in the lengths of cycles. These fluctuate, and we would like to know their distribution. In particular, the presence of infinite, macroscopic cycles should signal the occurrence of Bose-Einstein condensation.

Literature. The model was somehow suggested by Feynman in 1953 [1], with subsequent contributions by Kikuchi et. al. [2, 3]. The link between infinite cycles and Bose-Einstein condensation was established by Sütő in the ideal Bose gas [4]. These notions were generalised in the mathematical articles [7, 8].

Of direct relevance here are the articles [5] and [6]. Numerical experiments on this model have shown that infinite cycles occur at low temperature in dimension $d \geq 3$. A puzzling observation is that the average length of the longest cycle seems identical to that in random permutations with *uniform distribution*. This is very surprising, because the permutations here are very much influenced by the spatial structure — namely, each site is sent to a neighbourhood with high probability. One would like to understand more about this.

Research goals. The main goal is to obtain detailed statistics about the length of cycles that are present in these random permutations. The mean to achieve this is Monte Carlo simulations, which should not be hard to implement here. It will be especially interesting to compare the results with uniform random permutations. If these turn out to be identical, one can try to provide an explanation. In the other case, it will be interesting to comment about the difference. There should be material for an interesting publication in an international journal.

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