SPATIAL RANDOM PERMUTATIONS AND DISTRIBUTION OF CYCLE LENGTHS

Background. This project suggests to investigate certain models of random permutations by means of numerical simulations (Monte Carlo methods). The model is at the crossroad between statistical mechanics and probability theory. It is motivated by the phenomenon of *Bose-Einstein condensation*, a phase transition that occurs in large systems of quantum bosonic particles. This project requires basic knowledge of Monte-Carlo simulations and nothing else. Here, only those features that are necessary to the project are described. But the topic is much broader, and there is ample room for a PhD thesis.

Spatial random permutations. Random permutations have been a hot topic in probability theory and applied mathematics because of their connections to random matrices and other systems. But our random permutations have an extra feature, namely that points are penalised if they travel far. More precisely, we consider permutations $\pi : \Lambda \to \Lambda$, where $\Lambda \subset \mathbb{Z}^d$ is a finite subset of the square lattice. The most interesting dimension is $d = 3$. Permutations receive the following weight:

$$\prod_{x \in \Lambda} \exp\Big\{ -\alpha \big| x - \pi(x) \big|^2 \Big\}.$$

Here, α is a parameter (it is proportional to the temperature of the system, in the statistical mechanics interpretation). The most probable permutation is the identity permutation — all particles keep their positions. There is an "energy-entropy competition" controlled by the parameter α: As α gets smaller, particles can travel farer away.

One is interested in the lengths of cycles. These fluctuate, and we would like to know their distribution. In particular, the presence of infinite, macroscopic cycles should signal the occurrence of Bose-Einstein condensation.

Literature. The model was somehow suggested by Feynman in 1953 [1], with subsequent contributions by Kikuchi et. al. [2, 3]. The link between infinite cycles and Bose-Einstein condensation was established by Sütő in the ideal Bose gas [4]. These notions were generalised in the mathematical articles [5, 6].

Of direct relevance here are the articles [7], [8], and [9]. Numerical experiments on this model have shown that infinite cycles occur at low temperature in dimension $d \geq 3$, and that the joint distribution of the lengths of long cycles converge to Poisson-Dirichlet. The article [9] is the outcome of fine numerical studies by Alexander Lovisolo, in the context of a complexity miniproject.

Research goals. An interesting research direction deals with the emergence of macroscopic cycles. Previous simulations suggest that the transition occurs at a deterministic time. We would like to obtain detailed data about it, and to identify the basic mechanisms.

Another research direction is to investigate the two-dimensional model. The situation is very different, with a possible Kosterlitz-Thouless phase transition, but existing studies have not achieved full clarification of this fact.

The mean to achieve this is Monte Carlo simulations, which is not too hard to implement here. There should be material for an interesting publication in an international journal.

References

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