Correlation for Permutations

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Sets and Permutations

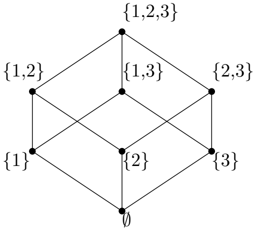
Extremal Set Theory

- $X = [n] = \{1, 2, \ldots, n\}$ (finite ground set)
- Power set $\mathcal{P}(X)$ (set of all subsets of X)
- $\mathcal{F} \subseteq \mathcal{P}(X)$ (a family of sets)
- Results involving relations between properties of $\mathcal{F}$

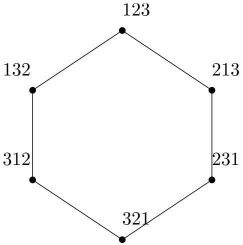
Permutations

- S_n set of all permutations (ordered n-tuples) of X.
- $\mathcal{F} \subseteq S_n$ (a family of permutations)
- Aim: Results inspired by extremal set results

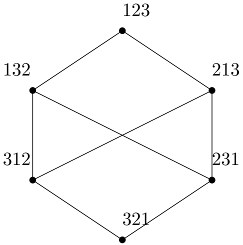
Preview



The subset lattice for $n = 3$



The weak order for $n = 3$



The strong order for $n = 3$

Aim

We seek results about permutation orders (righthand two figures) inspired by results on the hypercube (lefthand figure).

The Harris-Kleitman Inequality

Up-sets

$\mathcal{F} \subseteq \mathcal{P}(X)$ is an up-set if: $F \in \mathcal{F}, \quad x \in X \implies F \cup \{x\} \in \mathcal{F}$

Theorem (Harris 1960, Kleitman 1966)

If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ are up-sets then

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geqslant \mathbb{P}(\mathcal{A}) \times \mathbb{P}(\mathcal{B}) \qquad \text{(where } \mathbb{P}(\mathcal{F}) = |\mathcal{F}|/2^n\text{)}$$

Application

In a random graph $G \sim G(N, 1/2)$, the events “ G contains a triangle” and “G is Hamiltonian” are positively correlated.

- Let $X = E(K_N)$.
- A graph corresponds to a subset of X.
- A monotone property corresponds to an up-set.

The Permutation Setting

Permutation set-up

- $X = \{1, 2, \ldots, n\}$ (finite ground set)
- S_n set of all permutations (ordered n-tuples) of X.
- $\mathcal{F} \subseteq S_n$ (a family of permutations)
- For now, random means uniform so

$$\mathbb{P}(\mathcal{F}) = \frac{|\mathcal{F}|}{n!}$$

- How should we define up-set?
  In $\mathcal{P}(X)$ these came from the containment partial order.

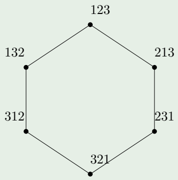
Permutation Orders

Weak Order $<_w$

If $1 \leqslant x < y \leqslant n$ then $p <_w q$ when

$$p = (\dots yx \dots)$$
$$q = (\dots xy \dots)$$

(Swap x, y in adjacent places into correct order)



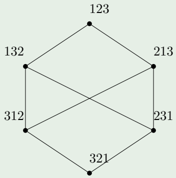
The weak order for $n = 3$

Strong order $<_s$

If $1 \leqslant x < y \leqslant n$ then $p <_s q$ when

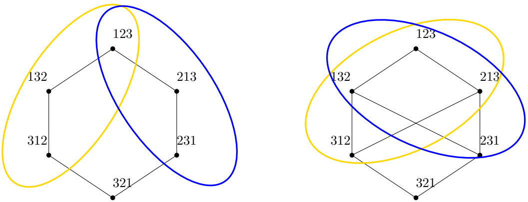
$$p = (\dots y \dots x \dots)$$
$$q = (\dots x \dots y \dots)$$

(Swap any x, y into correct order)



The strong order for $n = 3$

Permutation Up-Sets (Weak and Strong)



Weak up-set examples

The set of all $p \in S_n$ with "i before j" (where $1 \leqslant i < j \leqslant n$).

Strong up-set examples

- All $p \in S_n$ with element 1 in one of first k positions.
- All $p \in S_n$ which have $\leqslant k$ inversions (ie can be written as the product of $\leqslant k$ adjacent transpositions).
- All $p \in S_n$ which move no element by more than k places.

Positive Correlation: Weak up-sets

Counterexample to Positive Correlation in $<_w$

$$\mathcal{A} = \{p \in S_n : \text{ 1 appears before 2}\}, \qquad \mathbb{P}(\mathcal{A}) = 1/2$$
$$\mathcal{B} = \{p \in S_n : \text{ 2 appears before 3}\}, \qquad \mathbb{P}(\mathcal{B}) = 1/2$$
$$\mathcal{A} \cap \mathcal{B} = \{p \in S_n : \text{ 1 before 2 before 3}\}, \qquad \mathbb{P}(\mathcal{A} \cap \mathcal{B}) = 1/6$$

Question

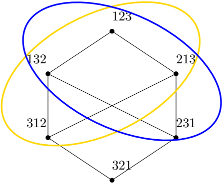
Is this the least correlated that weak up-sets can be?

Theorem (JLL, 2020)

No! There are weak up-sets with $\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{B}) = 1/2$ *and*

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = o(1).$$

Positive Correlation: Strong up-sets



Question

Do we have positive correlation for strong up-sets?

Theorem (JLL, 2020)

Yes! If $\mathcal{A}, \mathcal{B} \subseteq S_n$ are strong up-sets then

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geqslant \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$$

Main Ingredient of Proof (Strong Case)

For $\mathcal{A} \subseteq S_n$, partition $\mathcal{A}$ as $\mathcal{A}'_1 \cup \mathcal{A}'_2 \cup \cdots \cup \mathcal{A}'_n$ where

$\mathcal{A}'_k = \{(p_1 \dots p_n) \in \mathcal{A} : p_k = n\}$ (element n in position k)

let $\mathcal{A}_k$ be the corresponding subset of S_{n-1} (delete n from each)

Let $\mathcal{A} \subseteq S_n$ be a strong up-set and $1 \leqslant x < y \leqslant n-1$

If $(\dots y \dots x \dots) \in \mathcal{A}_k$ then $(\dots y \dots n \dots x \dots) \in \mathcal{A}$
so $(\dots x \dots n \dots y \dots) \in \mathcal{A}$
so $(\dots x \dots y \dots) \in \mathcal{A}_k$
So each $\mathcal{A}_k$ is a strong up-set. This allows induction on n.

A little more work gives:

Theorem (JLL, 2020)

If $\mathcal{A}, \mathcal{B} \subseteq S_n$ are strong up-sets then $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geqslant \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$.

Non-Correlation: Weak up-sets

Take k large and $n = 2k - 1$.

$$\mathcal{A} = \{p \in S_n : k \text{ is in last half of elements } 1, 2, \ldots, k\}$$
$$\mathcal{B} = \{p \in S_n : k \text{ is in first half of elements } k, k+1, \ldots, n\}$$

Each is a weak up-set of size $n!/2$.
Let p be a random permutation

- If position of k in p is $< \left(\frac{1}{2} - \epsilon\right) n$ then whp $p \notin \mathcal{A}$
- If position of k in p is $> \left(\frac{1}{2} + \epsilon\right) n$ then whp $p \notin \mathcal{B}$

So $|\mathcal{A} \cap \mathcal{B}| \leqslant 2\epsilon n!$ as required.

Theorem (JLL, 2020)

For all $0 < \alpha, \beta < 1$, there exist weak up-sets $\mathcal{A}, \mathcal{B}$ with $\mathbb{P}(\mathcal{A}) = \alpha + o(1)$, $\mathbb{P}(\mathcal{B}) = \beta + o(1)$ and

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \begin{cases} o(1) & \text{if } \alpha + \beta \leqslant 1 \\ \alpha + \beta - 1 + o(1) & \text{if } \alpha + \beta \geqslant 1. \end{cases}$$

Back to Proof of Correlation in the Strong Order

If $(\ldots y \ldots x \ldots) \in \mathcal{A}_k$ then $(\ldots y \ldots n \ldots x \ldots) \in \mathcal{A}$
so $(\ldots x \ldots n \ldots y \ldots) \in \mathcal{A}$ $(*)$
so $(\ldots x \ldots y \ldots) \in \mathcal{A}_k$
So each $\mathcal{A}_k$ is a strong up-set.

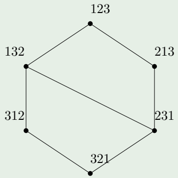
For weak up-sets this doesn't work – $(*)$ fails.
But could it work for some intermediate order?

Grid Order $<_g$

If $x < y < a_1, \ldots, a_m$ then $p <_g q$ when

$$p = (\ldots y a_1 \ldots a_m x \ldots)$$
$$q = (\ldots x a_1 \ldots a_m y \ldots)$$

(Swap x, y into correct order if all intermediate elements are larger)



The grid order for $n = 3$

Grid order for S_n is product order on $[n] \times [n-1] \times \cdots \times [2]$.

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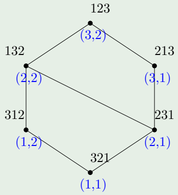

The grid order as $[3] \times [2]$

Grid order for S_n is product order on $[n] \times [n-1] \times \cdots \times [2]$.

Grid Orders

Grid Order $<_g$ again

For a permutation p define a vector $f(p) \in [n] \times [n-1] \times \cdots \times [2]$ by

$$f(p)_{n+1-k} = \text{position of } k \text{ among } \{1, 2, \ldots, k\}$$

If $p = 32514$ then

$$f(p) = (\underbrace{3}_{\text{5 is third in 32514}}, \underbrace{4}_{\text{4 is fourth in 3214}}, \underbrace{1}_{\text{3 is first in 321}}, \underbrace{1}_{\text{2 is first in 21}})$$

we have $p \leqslant_g q$ (grid order) if $f(p)_k \leqslant f(q)_k$ for all k.

Extensions

Working in the grid order environment gives:

- A second proof of main result using FKG inequality in grids.
- Positive correlation for up-sets in the grid order.
- Some non-uniform measures including …

Independently Generated Measures

For each k, let X_k be a rv taking values in $\{1, \ldots, n+1-k\}$. Pick $f(p)$ using X_k for coordinate k with each coordinate independent.

Have positive correlation for up-sets for these measures.

Mallows Measures (special case of above)

Fix $0 < q \leqslant 1$.

$$\mathbb{P}(p) = cq^{\mathrm{inv}(p)}$$

where $\mathrm{inv}(p)$ is the number of inversions in p.

Questions

- What other measures show positive correlation for strong up-sets? In particular what if

$$\mathbb{P}(p_1 p_2 \dots p_n) \propto q^{\sum_{i=1}^{n} |p_i - i|}$$

  for some $0 < q < 1$.
  (Special case of 1-dimensional Boltzmann distribution.)
- How does the correlation between up-sets behave in orders which interpolate between weak and strong orders?
- More applications?