Walking Randomly, Massively, and Efficiently

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Joint work with **Jakub Łącki** (Google), **Slobodan Mitrović** (MIT), and **Piotr Sankowski** (University of Warsaw)

(Idealized) Past

Single machine directly accessing the entire data set

The Cloud

Setting: Data sets distributed across several machines

Why: full access by a small number of machines not feasible

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Develop algorithms that leverage the platform's parallelism!

Massive data processing systems: MapReduce, Spark, Hadoop, Dryad, IBM Streams, Pregel, . . .

Talk Plan

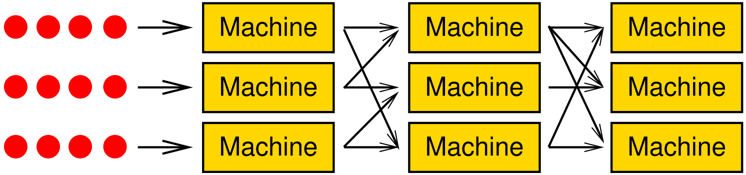
- The Massively Parallel Computation model
- Our Results
- Algorithms for Undirected Graphs
  - Lower Bounds
  - Applications to Property Testing
- Algorithms for Directed Graphs via Series of Transformations

Massively Parallel Computation

Model: Massively Parallel Computation (MPC)

M machines S space per machine

Input: m edges from a graph on n vertices



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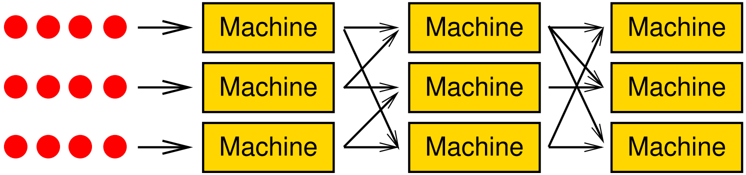

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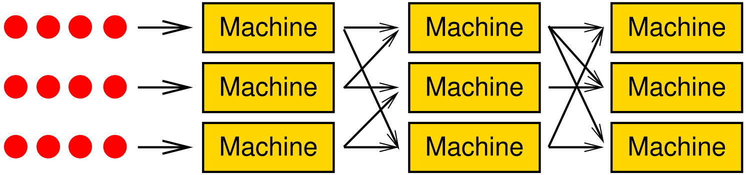

- Initially: each machine receives $\sim m/M$ edges
- Single round:
  1. Each machine performs computation
  2. Each machine sends and receives at most $O(S)$ data

Model: Massively Parallel Computation (MPC)

- Introduced by Karloff, Suri, Vassilvitskii (2010)
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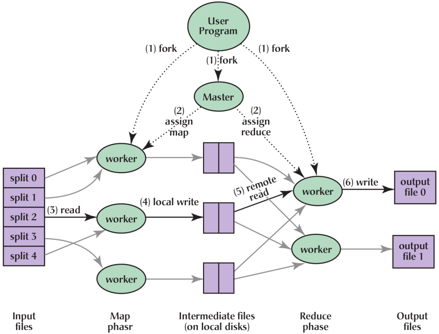

(from Dean, Ghemawat "MapReduce: Simplified Data Processing on Large Clusters")

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  - A refined version asks for near-linear total space: $M \times S = m^{1+o(1)}$

Three Main Memory Regimes

- Superlinear: $S = n^{1+\Omega(1)}$
  - Many early papers [Karloff, Suri, Vassilvitskii 2010] [Lattanzi, Moseley, Suri, Vassilvitskii 2011] ...
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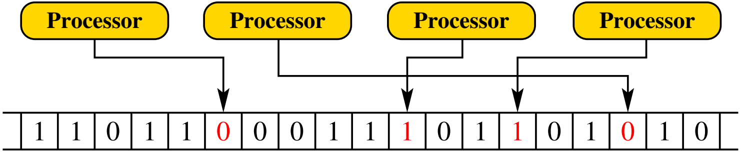
  - Very similar to the CONGESTED CLIQUE model
- Sublinear: $S = O(n^{\alpha})$ for $\alpha \in (0, 1)$
  - Most interesting for large sparse graphs
  - Results in this talk
  - Beating $O(\log n)$ becomes a challenge

Main goal: minimize the number of rounds

- PRAM:
  - can usually be simulated in the same number of rounds
  - often $\Omega(\log n)$ rounds



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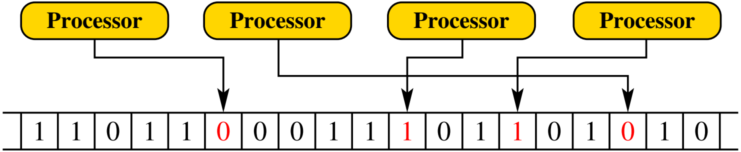


- Fewer parallel rounds than best PRAM algorithms?

$O(1)$ or $O(\mathrm{poly}(\log \log N))$ rounds of MPC?

Our Results

Random Walks

Random walk on a graph:

At every step select a random outgoing edge

In general, the set of options could be weighted

Why study random walks?

Useful primitive! Sample applications:

- PageRank and rating web pages
- optimal PRAM algorithms for connectivity
- partitioning graphs
- minimizing query complexity in property testing
- graph matchings in regular graphs
- generating random spanning trees
- volume estimation
- counting problems

Our Results

Setting: strongly sublinear space per machine, i.e., $O(n^{\alpha})$ for $\alpha \in (0, 1)$

Generate a small number of length-L random walks from every vertex

- undirected graphs: $O(\log L)$ rounds
- directed graphs: $O((\log \log n)^2 + \log^2 L)$ rounds

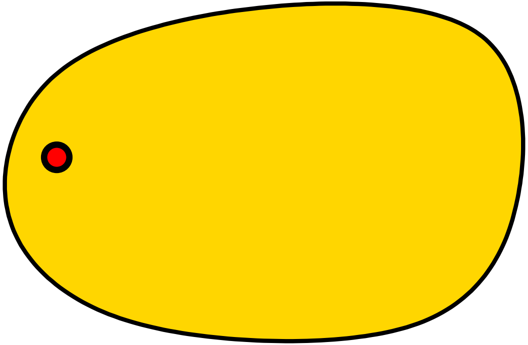
PageRank: $O((\log \log n)^2 + \log^2(1/\epsilon))$ rounds

- multiplicative approximation for all vertices
- $\epsilon =$ teleportation probability

Undirected Graphs

Basic Challenges

Trivial: compute random walk of length L in $O(L)$ rounds



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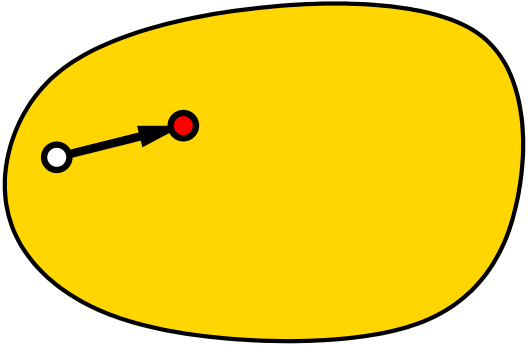

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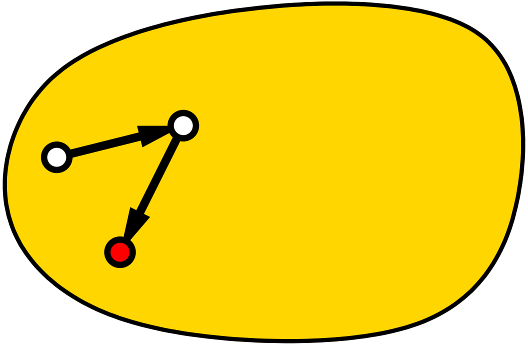

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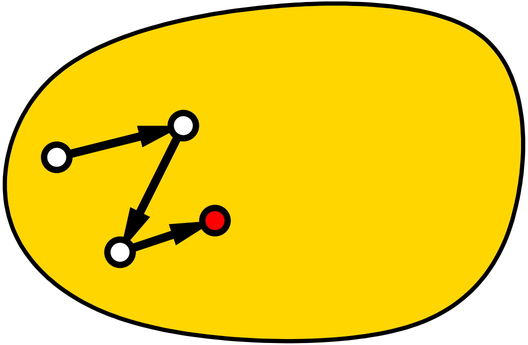

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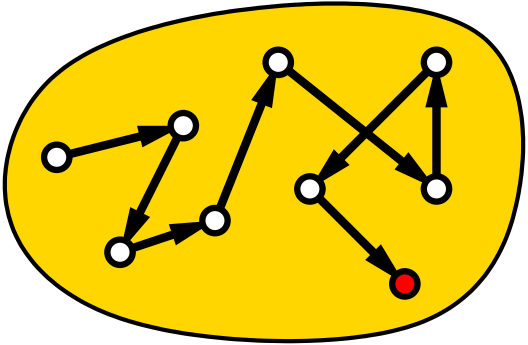

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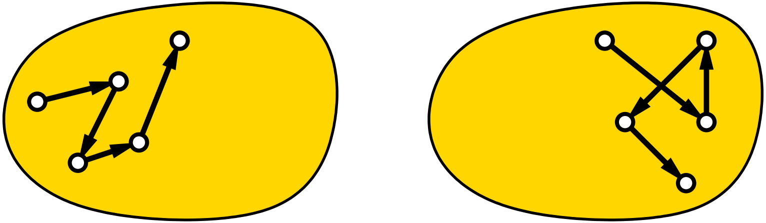
- Random walks are memoryless
- Compute different sections and stitch them together?
- For L-step random walk, compute independently the first and second half of length $L/2$ via recursion?



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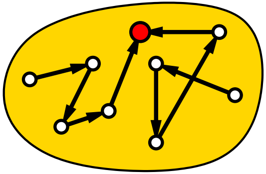
- With many random walks, they could collide



Undirected Graphs

What we achieve: many random walks from each vertex

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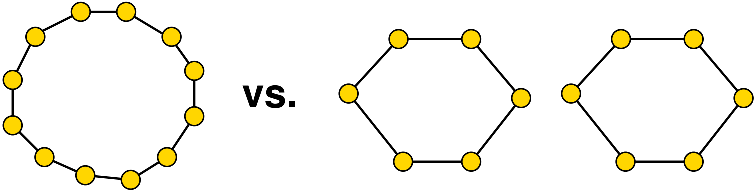
How:

- Start from the stationary distribution: $\frac{\deg(v)}{2m}$ for vertex v
  - After any number of steps, the distribution will be the same
- Sample slightly more edges for consecutive steps to ensure that number of continuations is sufficient
- Roughly $O(\deg(v) \cdot \log n)$ random walks from vertex v
- Use $O(\log L)$ rounds to combine edges into random walks

Is $O(\log L)$ Rounds Optimal?

- Space per machine: $S = n^{\alpha}$ for $\alpha \in (0, 1)$
- Problem: One or two cycles?



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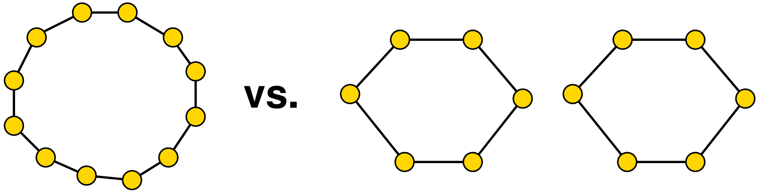

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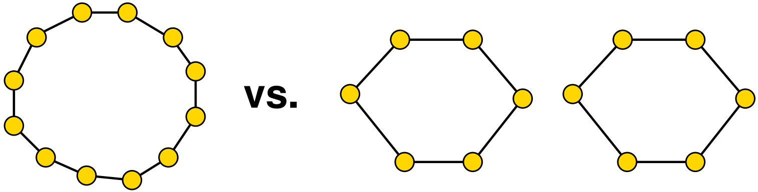

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- $\Omega(\log n)$ conditional lower bound for exact bipartiteness

Applications to Property Testing

Bipartiteness testing (similar to [Censor-Hillel, Fischer, Schwartzman, Vasudev 2016]):

- [Goldreich Ron 1999]: sampling $O(\sqrt{n})$ random walks from a random vertex is likely to detect an odd length cycle
- Can as well sample $O(1)$ random walks from all vertices

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Testing if a graph is an expander:

- Classic property testing: [Goldrech, Ron 2000] [Czumaj, Sohler 2007] ...
- Expander: two random walks collide with probability close to $1/n$
- Far from expander: higher probability for a random starting vertex
- Tweak the proof of Czumaj and Sohler (2007) to distribute starting points of random walks over all vertices

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Open question: Can this be done for testing clusterability?

How About Directed Graphs?

Difficulties:

- No explicit stationary distribution
- Values can be as low as $n^{-\Omega(n)}$

PageRank for Undirected Graphs

Definition of PageRank

PageRank: measure of importance of nodes in a graph

- Stationary distribution
- Random walk:
  - with probability $1 - \epsilon$, follow a random outgoing edge
  - with probability ϵ, teleport to uniformly selected vertex in the entire graph
- $\epsilon =$ teleportation probability

Alternate Definition

This process gives the same distribution [Breyer 2002]

- Select a vertex v uniformly at random
- Walk on the Markov chain until teleportation from some vertex u
- u distributed according to PageRank

Algorithm for Undirected PageRank

Algorithm:

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- Need at most $O(\epsilon^{-1} \log n)$ random walks from every vertex
- All random walks will teleport whp. after $O(\epsilon^{-1} \log n)$ steps

Important note: This works for directed graphs as long as someone gives us a collection of random walks with uniformly selected starting points

PageRank for Balanced Directed Graphs

c-Balanced Directed Graph

- Constant $c \in (0, 1)$
- For every vertex vertex v,

$$\mathrm{outdeg}(v) \geq c \cdot \mathrm{indeg}(b)$$

- Random incident edge is directed in the correct direction with non-trivial probability

Transformation

G is c-balanced graph

$P_G =$ PageRank transition probability matrix for G

$\bar{G} =$ undirected version of G

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Sequence: $0 = \delta_0 < \delta_1 < \ldots < \delta_{k-1} < \delta_k = 1$.

Intermediate PageRank transition probability matrices:

$$P_i = \delta_i P_G + (1 - \delta_i) P_{\bar{G}}$$

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Intermediate PageRank transition probability matrices:

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How:

- Know how to compute stationary distribution for P_0
- Want to compute stationary distribution for P_k
- We show how to move from P_i to P_{i+1} for $\delta_{i+1} - \delta_i \approx \frac{1}{\log \log n}$

Transition from P_i to P_{i+1}

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3. Use the surviving random walks to compute PageRank for P_{i+1}

PageRank
for General Directed Graphs

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- Replace vertices v with directed $O(\log n)$-paths
  - i-th edge: $\max n/2^i, \mathrm{indeg}(v)$ copies
- Correspondingly lower the teleportation probability
- Transition from $\epsilon = 1/2$ to $\epsilon/\log n$
  (again via series of transitions)

Generating Random Walks in Directed Graphs

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- $L =$ length of desired random walks
- Leverage the fact that we know the associated PageRank
- Set the teleportation probability to $1/L$
- Generate random walks from the PageRank Markov Chain
- Throw away those that teleported at least once
- A random walk "survives" with probability $\Omega(1)$

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