Ramsey Spanning Trees and their Applications

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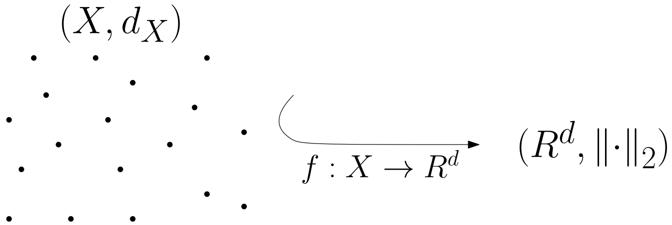
Michael Elkin, Ofer Neiman

Workshop on Data Summarization
University of Warwick

Metric Embeddings



Embedding $f : X \to \mathbb{R}^d$ has distortion α if for all $x, y \in X$

$$d_X(x,y) \le \|f(x) - f(y)\|_2 \le \alpha \cdot d_X(x,y)$$

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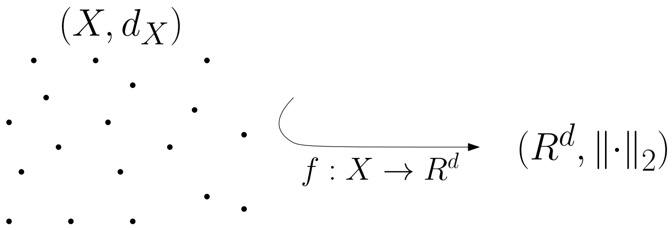

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Every n-point metric (X, d_X) is **embeddable** *into* *Euclidean* *space with* **distortion** $O(\log n)$.

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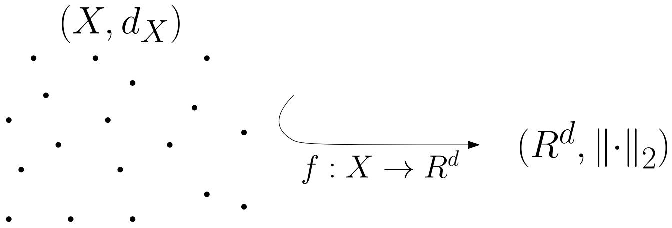

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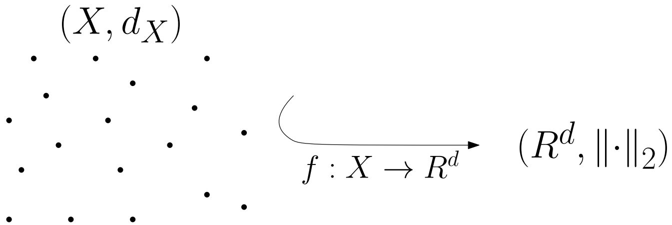
Asymptotically tight.

Metric Ramsey-Type Problem

For a fixed distortion $k > 1$, what is the largest subset $M \subset X$, s.t. (M, d_X) is embeddable into Euclidean space with **distortion** k?



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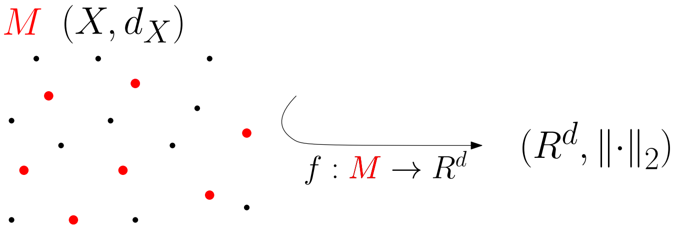

$$\forall x, y \in M, \qquad d_X(x,y) \le \|f(x) - f(y)\|_2 \le \boldsymbol{k} \cdot d_X(x,y)$$

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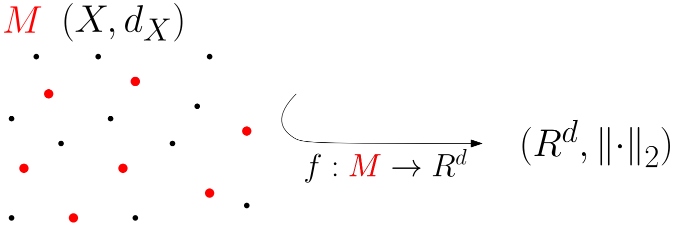

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Theorem (Mendel, Naor 07, following BFM86, BLMN05)

For every n-point metric space and $\boldsymbol{k} \ge 1$, there exists a subset M of size $n^{1-1/k}$ that can be embedded into Euclidean space with distortion $O(\boldsymbol{k})$.

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Ultrametric is a spacial kind of tree which is:

1. Very useful for divide an conquer algorithms.
2. Isometrically embeds into Euclidean space (i.e. distortion 1).

Our Second Result: Metric Ramsey-Type Problem

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Theorem (Our Secondary Result)

For every n-point metric space and $\boldsymbol{k} \geq 1$*, there is a* **deterministic algorithm** *that finds a subset* M *of size* $n^{1-1/k}$ *that can be embedded into* **ultrametric** *with distortion* $\mathbf{8} \cdot \boldsymbol{k}$*.*

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For every n-point metric space and $\boldsymbol{k} \geq 1$*, there is a* **deterministic algorithm** *that finds a subset* M *of size* $n^{1-1/k}$ *such that the hall metric can be embedded into* **ultrametric** *with distortion* $\mathbf{16} \cdot \boldsymbol{k}$ *w.r.t* $M \times X$.

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Corollary

For every n-point metric space and $\boldsymbol{k} \geq 1$,
there is a **set** $\mathcal{U}$ *of* $k \cdot n^{\frac{1}{k}}$ **ultrametrics** *and a mapping* **home** : $X \to \mathcal{U}$, *such that for every* $x, y \in U$,

$$d_{\mathbf{home}(x)}(x, y) \leq (\mathbf{16} \cdot \boldsymbol{k}) \cdot d_X(x, y)$$

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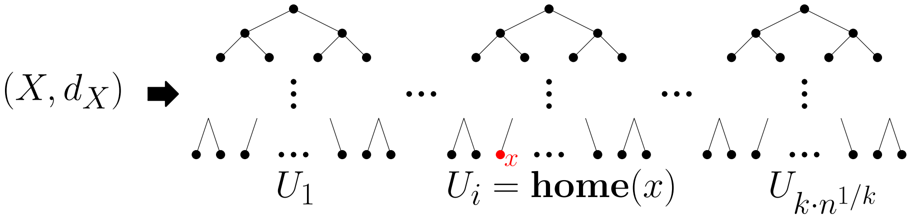

Distance Oracle

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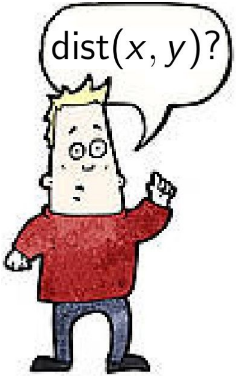
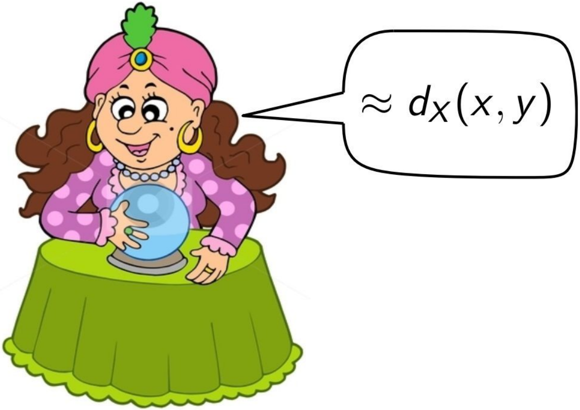
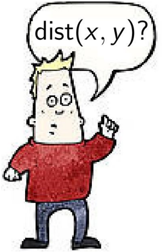


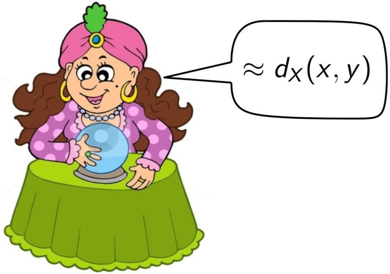


The properties of interest are size, distortion and query time.

Distance Oracles: State of the Art

DO	Distortion	Size	Query	Deterministic?
TZ05	$2k-1$	$O(k \cdot n^{1+1/k})$	$O(k)$	no
MN07	$128k$	$O(n^{1+1/k})$	$O(1)$	no
W13	$(2+\epsilon)k$	$O(k \cdot n^{1+1/k})$	$O(1/\epsilon)$	no
C14	$2k-1$	$O(k \cdot n^{1+1/k})$	$O(1)$	no
C15	$2k-1$	$O(n^{1+1/k})$	$O(1)$	no
RTZ05	$2k-1$	$O(k \cdot n^{1+1/k})$	$O(k)$	yes
W13	$2k-1$	$O(k \cdot n^{1+1/k})$	$O(\log k)$	yes





Our contribution: Deterministic Distance Oracles

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This paper	$8(1+\epsilon)k$	$O(n^{1+1/k})$	$O(1/\epsilon)$
This paper+C14	$2k-1$	$O(k \cdot n^{1+1/k})$	$O(1)$

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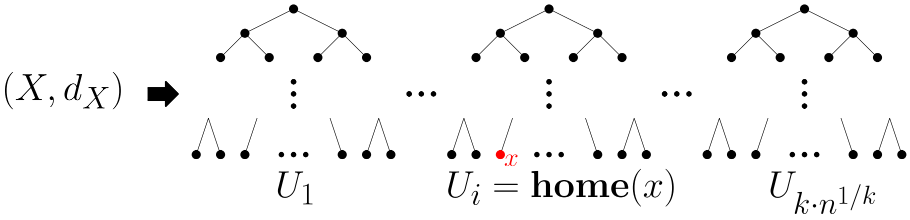

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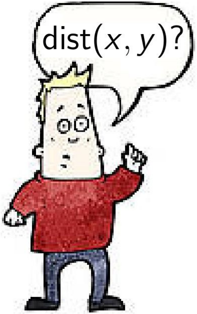

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Theorem (Tree Distance Oracle, HT84, BFC00)

For every **tree metric***, there is an exact distance oracle of* **linear size** *and* **constant** *query* **time***.*

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For any n-point metric space, there is a distance oracle with :

Distortion	**Size**	**Query time**
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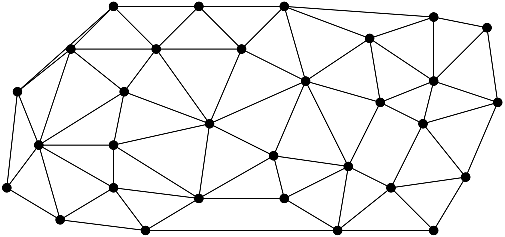
C15 *(Randomized)*	$2k-1$	$O(n^{1+1/k})$	$O(1)$

Ramsey Spanning Tree Question

Given a **weighted graph** $G = (V, E, w)$, and a fixed distortion $\boldsymbol{k} > 1$, what is the **largest subset** $M \subset V$, such that: there is a **<u>spanning tree</u>** T of G with distortion $\boldsymbol{k}$ w.r.t $M \times V$?



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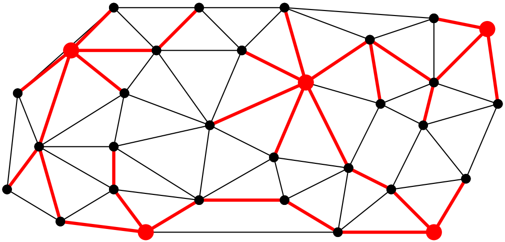

For all $v \in M$ and $u \in V$, $\quad d_T(v, u) \leq k \cdot d_G(v, u)$.

Main Result

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Theorem (Main Result)

For every n-vertex **weighted graph** $G = (V, E, w)$ *and* $\boldsymbol{k} \geq 1$, *there exists a* **subset** M *of size* $n^{1-1/k}$ *and* **spanning tree** T *of* G *with* **distortion** $O(\boldsymbol{k} \cdot \log\log n)$ *w.r.t* $M \times V$.

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Theorem (Mendel, Naor 07)

For every n-point **metric space** (X, d_X) *and* $\boldsymbol{k} \geq 1$, *there exists a subset* M *of size* $n^{1-1/k}$ *and an* **ultrametric** U *over* X *with distortion* $O(\boldsymbol{k})$ *w.r.t* $M \times X$.

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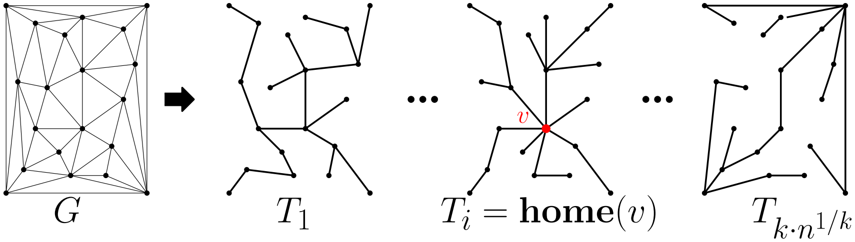

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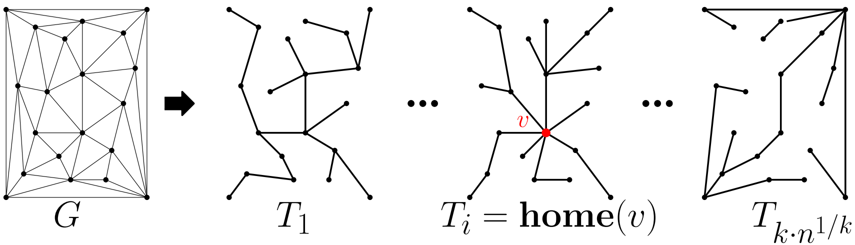
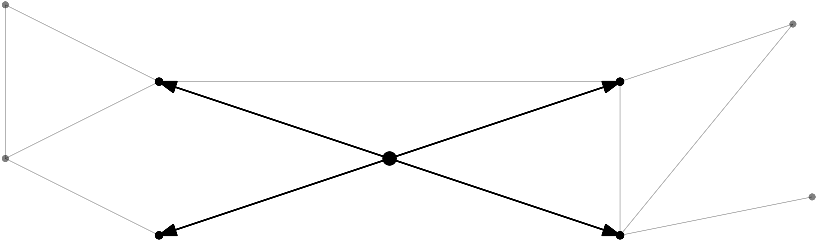


The union of all the trees in $\mathcal{T}$ creates an $O(\boldsymbol{k} \cdot \log\log n)$-**spanner** with $O(k \cdot n^{1+\frac{1}{k}})$ edges.

Application: Compact Routing Scheme



- **Huge** network

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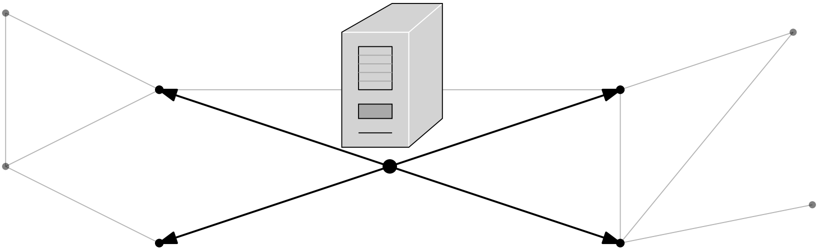

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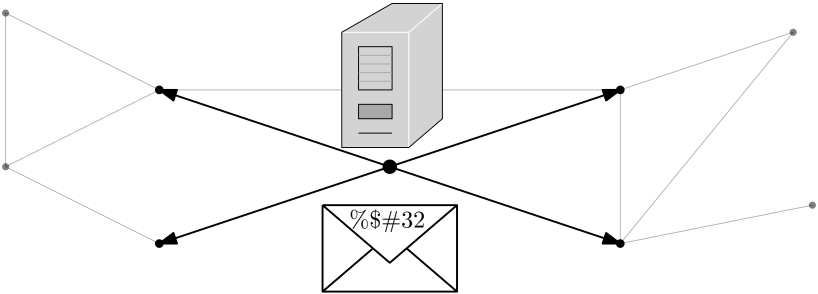

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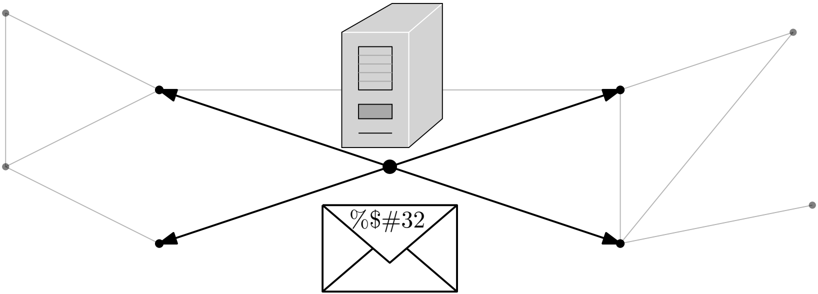

- **Huge** network
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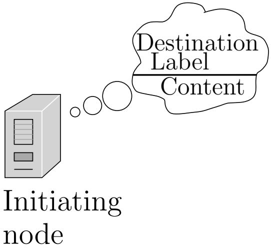
- Task: **route** packages throughout the network.
- **Store** the whole network in each node is **unfeasible**.

Compact Routing Scheme



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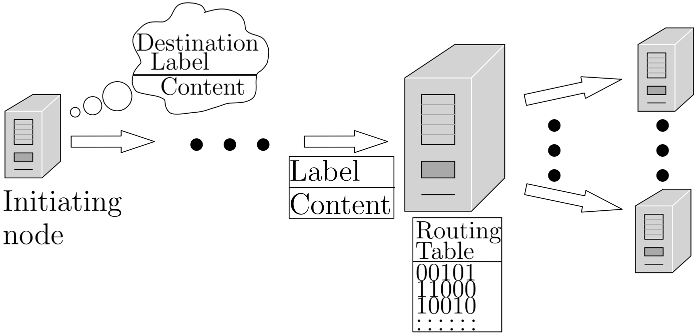

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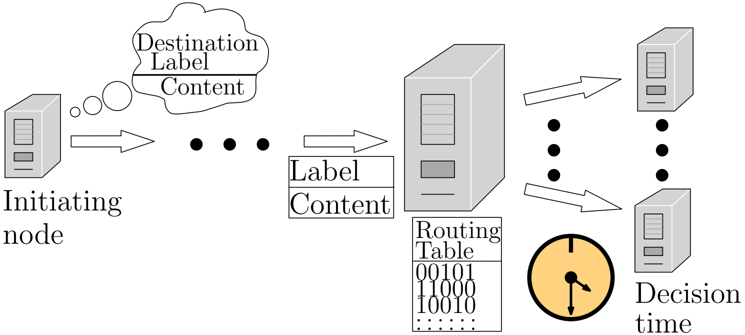

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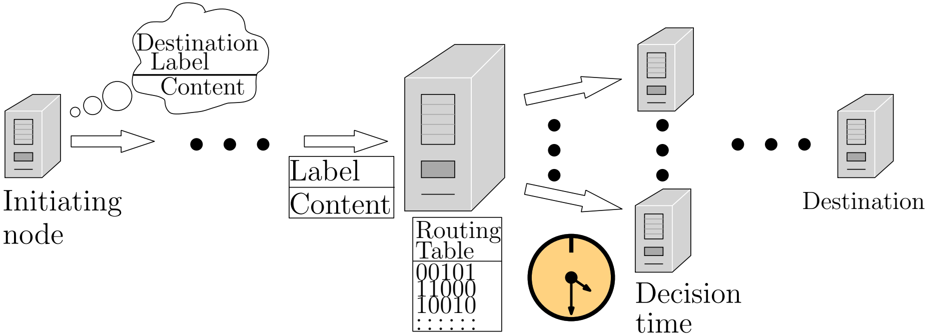

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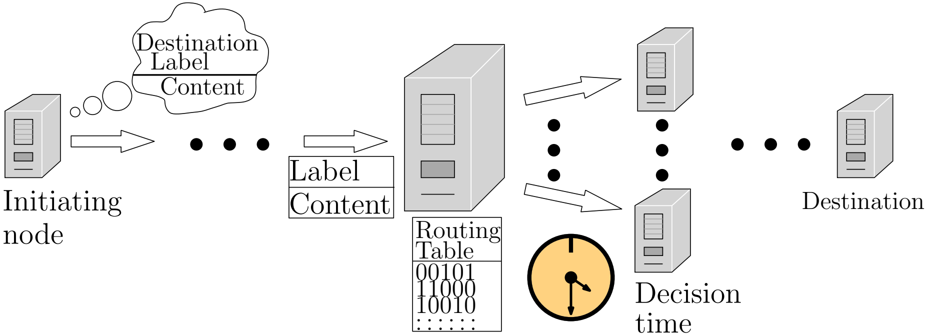
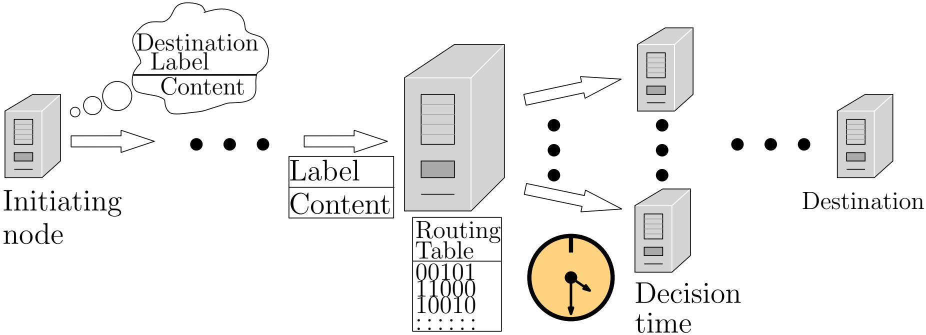


In order to keep other **parameters small**, we will allow **stretch**.

Compact Routing Scheme



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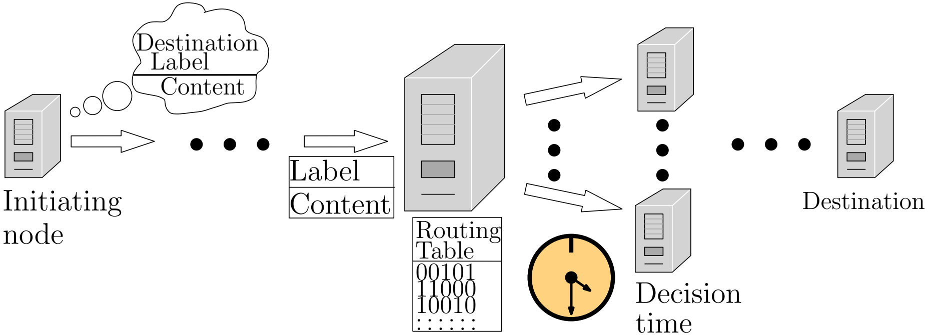

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Theorem (Thorup, Zwick, 01)

For any n-vertex **tree** $T = (V, E)$, *there is a routing scheme with :*

Stretch	**Label**	**Table**	**Decision time**
1	$O(\log n)$	$O(1)$	$O(1)$

Routing using Ramsey Spanning Trees

For every n-vertex weighted graph $G = (V, E, w)$ and $\boldsymbol{k} \geq 1$, there is a **set** $\mathcal{T}$ of $k \cdot n^{\frac{1}{k}}$ **spanning trees** and a mapping **home** : $V \to \mathcal{T}$, such that for every $u, v \in V$,

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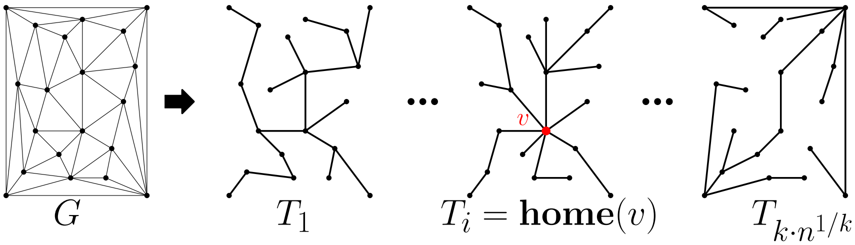

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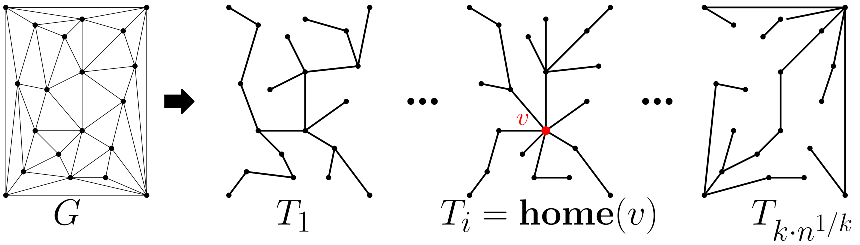

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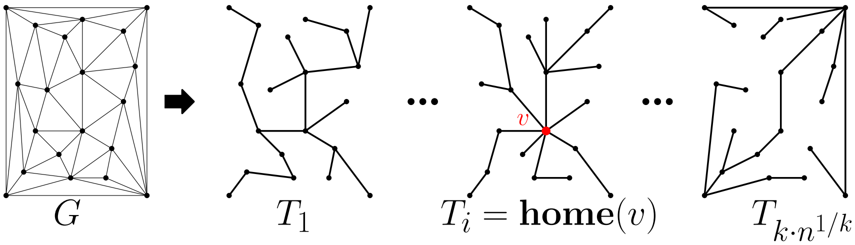

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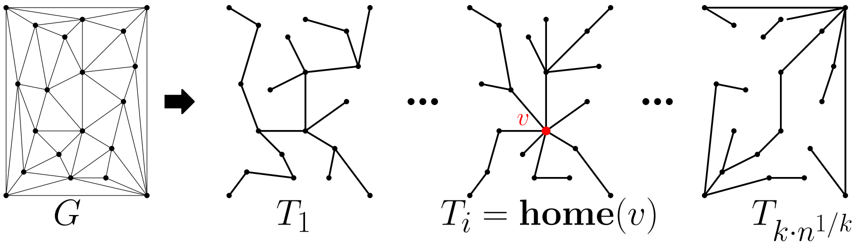

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Theorem (Ramsey based Compact Routing Scheme)

For any n-vertex graph, there is a routing scheme with :

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By choosing $k = \log n$, we get:

	Stretch	Label	Table	D. time
Here	$O(\log n \cdot \log\log n)$	$O(\log n)$	$O(\log n)$	$O(1)$
[TZ01]	$O(\log n)$	$O(\log^2 n)$	$O(\log n)$	$O(1)$ ($O(\log n)$)

Technical Ideas

Theorem (Main Result)

For every n-vertex **weighted graph** $G = (V, E, w)$ *and* $\boldsymbol{k} \geq 1$, *there exists a* **subset** M *of size* $n^{1-1/k}$ *and* **spanning tree** T *of* G *with* **distortion** $O(\boldsymbol{k} \cdot \log\log n)$ *w.r.t* $M \times V$.

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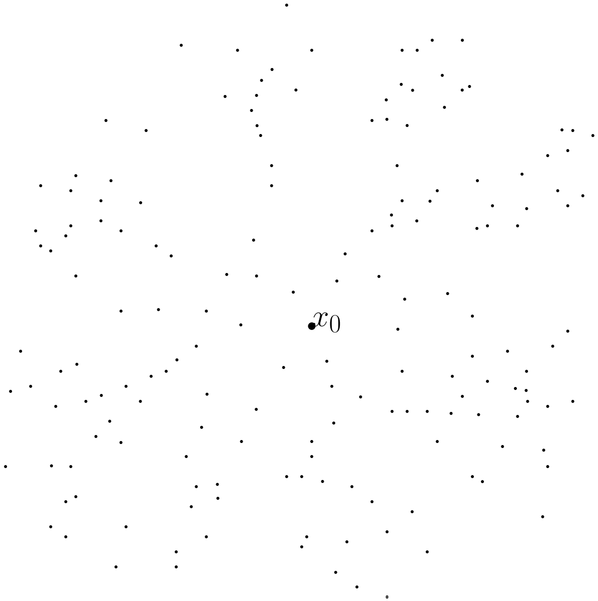
- Framework: Petal decomposition.
- Hierarchically padded decompositions.
- Region growing.

Petal Decomposition



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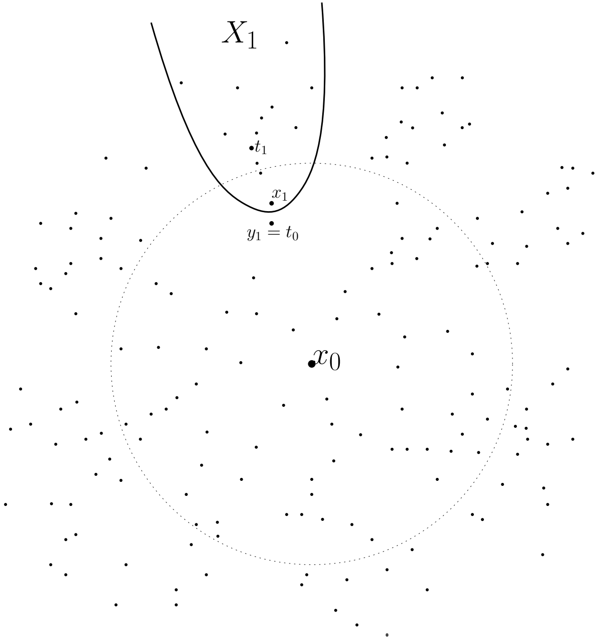

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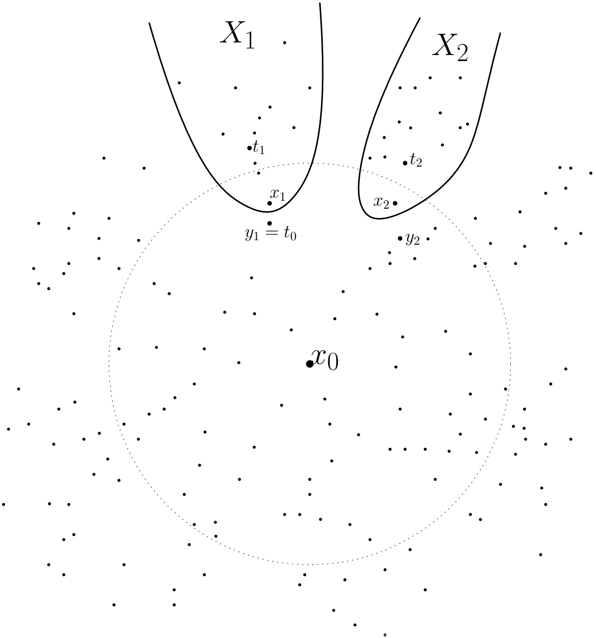

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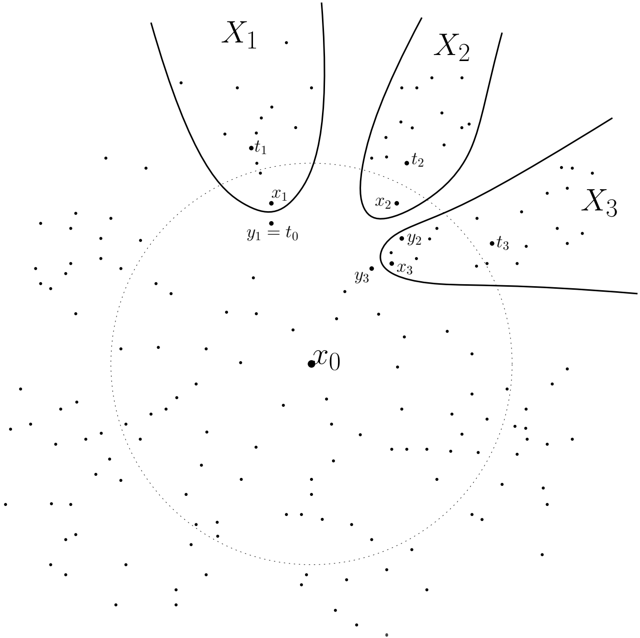

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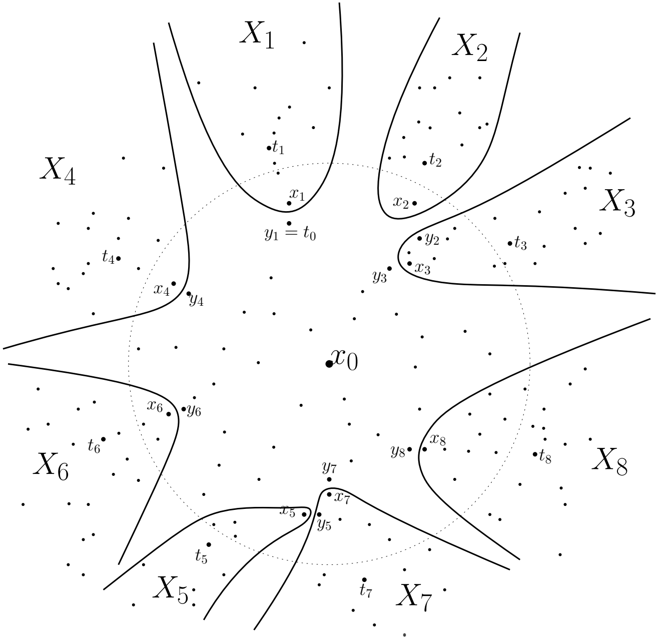

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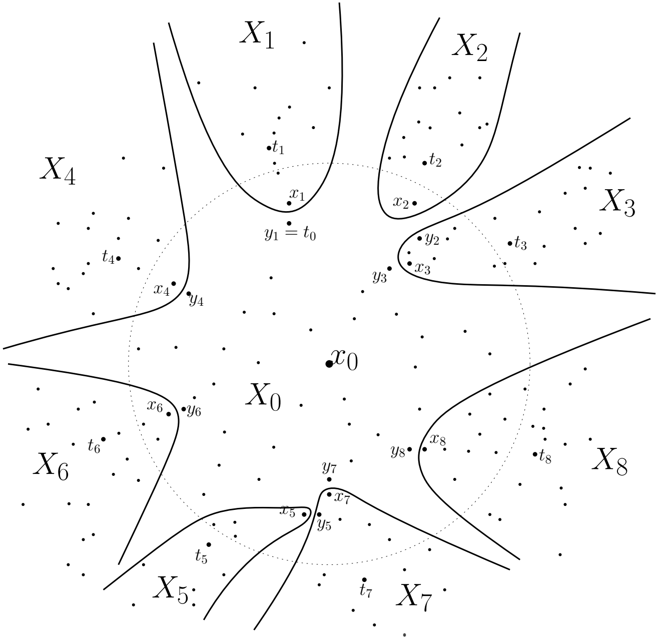

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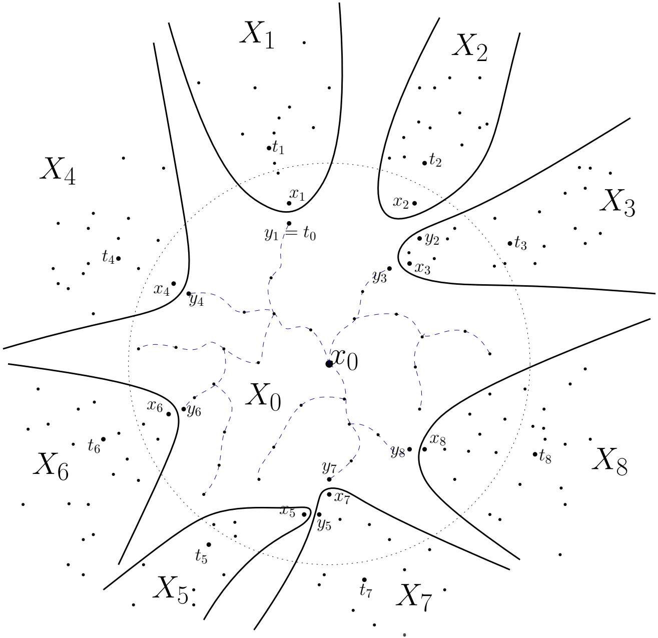

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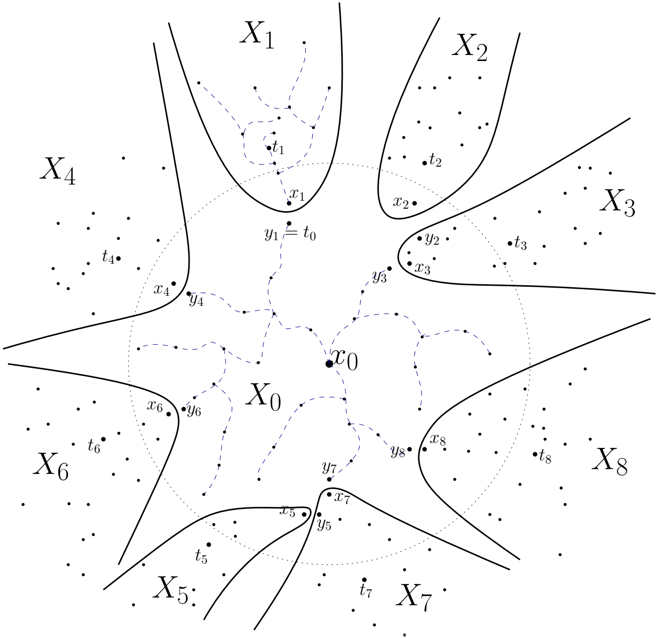

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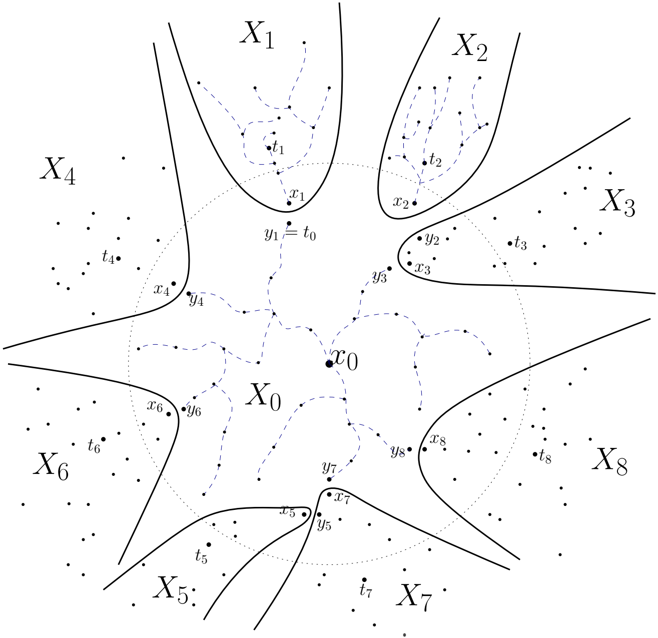

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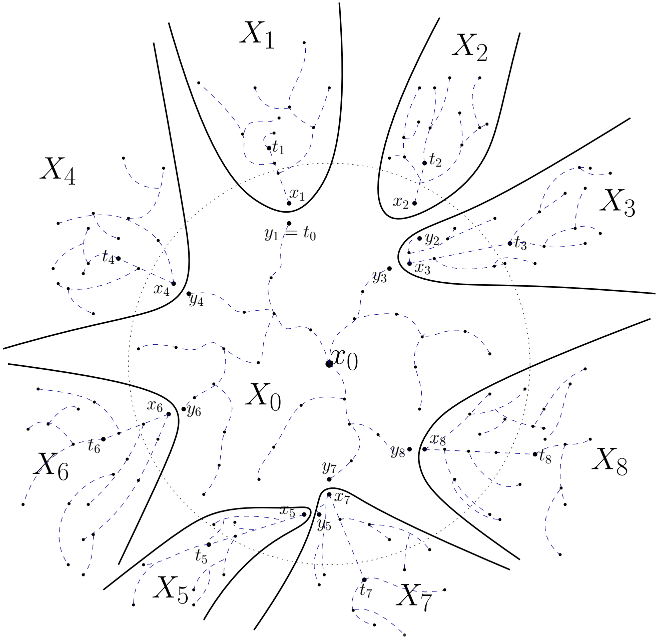

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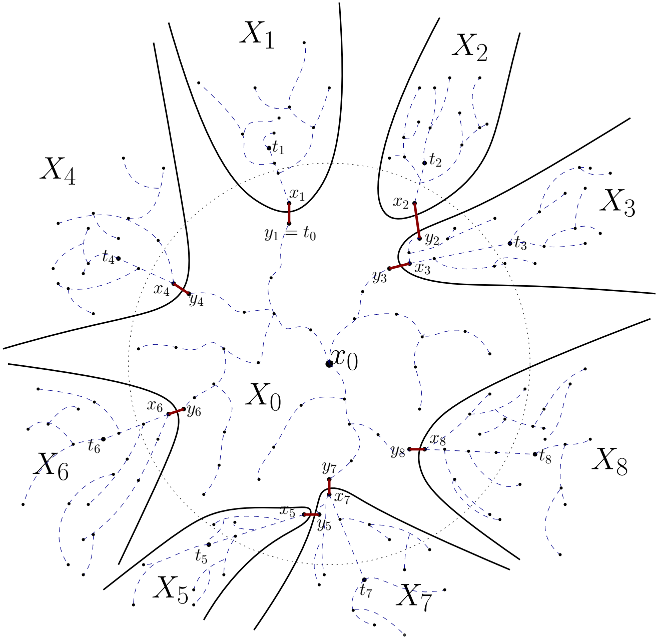
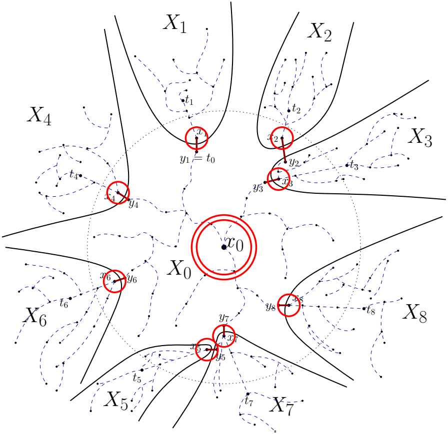


Petal Decomposition

- Each cluster X (petal) has a center vertex x.



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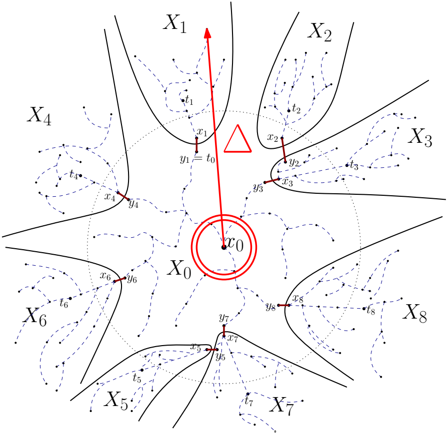

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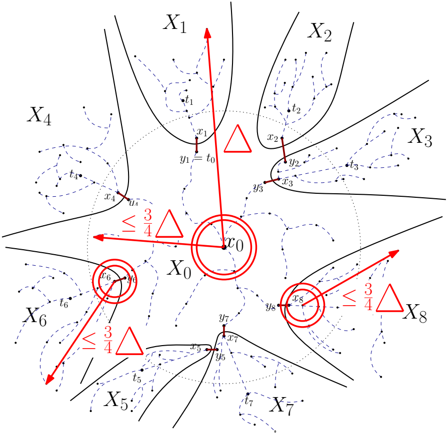
- The radius decrease by $\frac{3}{4}$ factor in each hierarchi. step.



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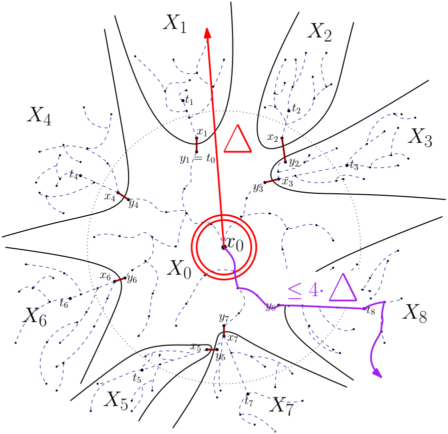

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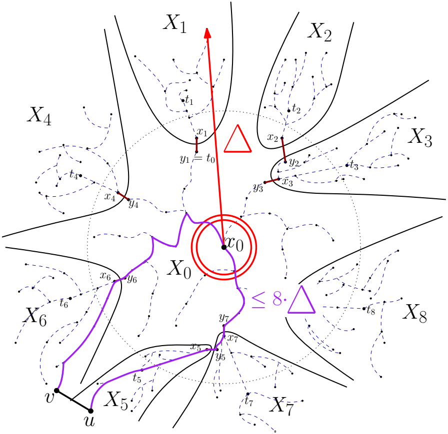
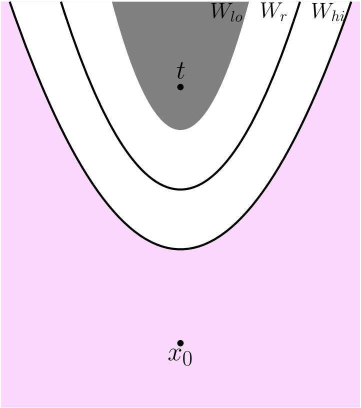


Corollary

Suppose v, u were separated while being in cluster of radius Δ. Then $d_T(v,u) \leq 8 \cdot \Delta$.

Petal Growth

Degree of freedom:
parameter $R \in [\text{lo}, \text{hi}]$ ($\text{hi} - \text{lo} = \frac{\Delta}{8}$).



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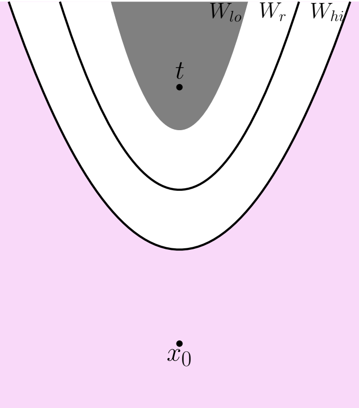
Monotonicity: $r' \leq r \Rightarrow W_{r'} \subseteq W_r$.



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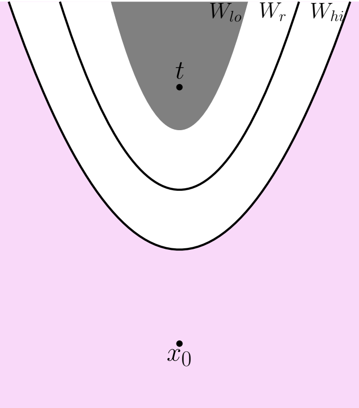

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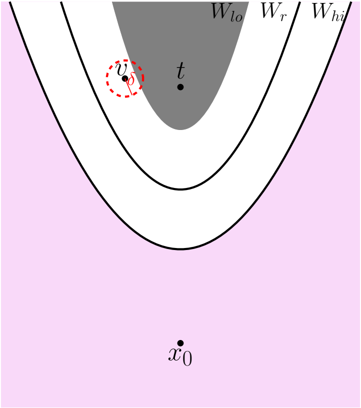
Vertex v s.t. $B(v, \delta) \subseteq W_r$ is **padded**.



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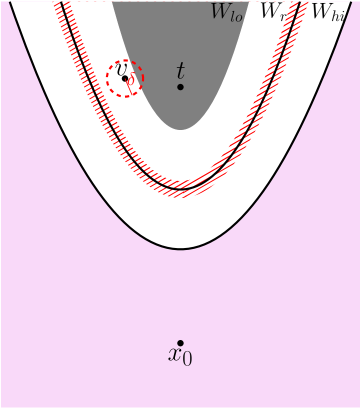

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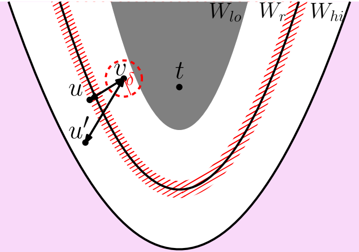

$d_G(u,v) = \Omega(\delta)$

$d_T(u,v) = O(\Delta)$

$\frac{d_T(u,v)}{d_G(u,v)} = O(\frac{\Delta}{\delta}) = O(k \cdot \log\log n)$

Padded vertices suffer distortion **at most** $\boldsymbol{\Delta/\delta} = O(k \cdot \log\log n)$!

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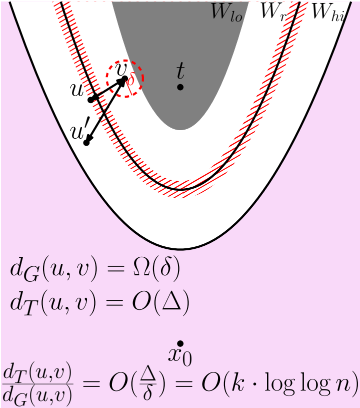
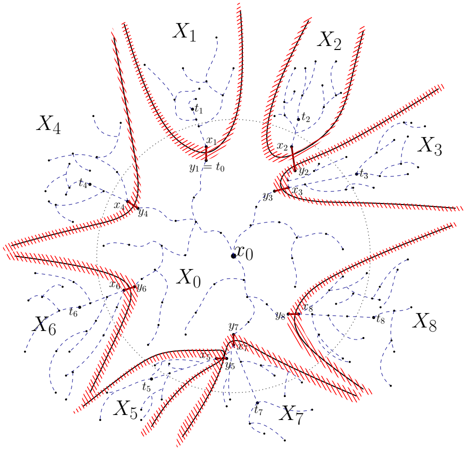


Padded vertices suffer distortion **at most** $\boldsymbol{\Delta/\delta}$ $= O(k \cdot \log\log n)$!

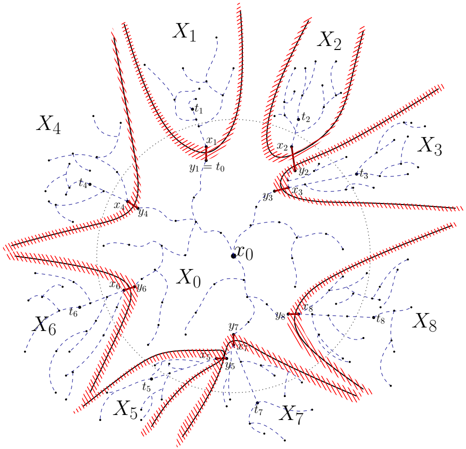
Goal: find r, with **many padded vertices**! (sparse restricted area).

Petal Decomposition



Petal Decomposition

A vertex which is **padded in all the levels** will have small distortion w.r.t all other vertices.



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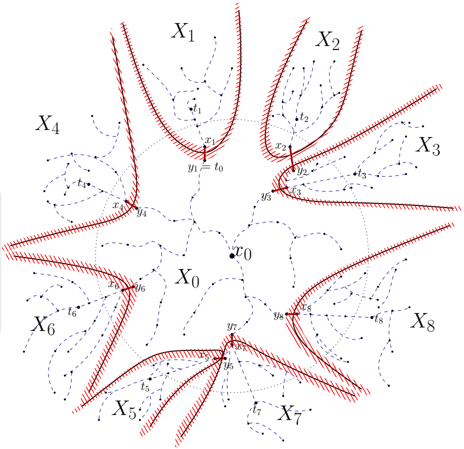
Goal: choose parameters ($r \in [lo, hi]$) s.t. at least $n^{1-\frac{1}{k}}$ vertices will be **padded in all levels**.



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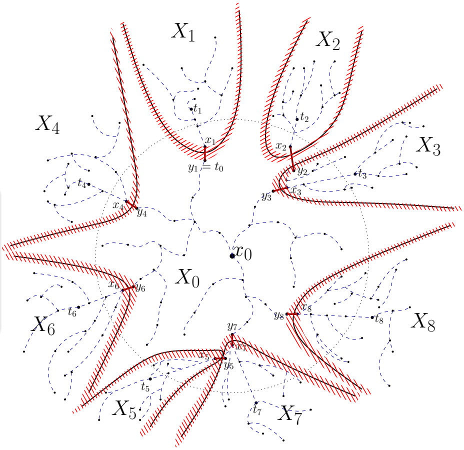
A vertex is called **active** if it is **padded** in all levels **up till now**.

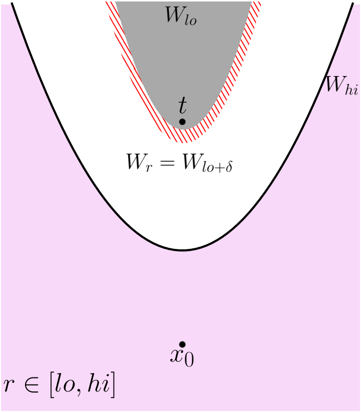


Region Growing

For petal W_r:

Active $x \in W_{r-\delta}$ **remains** active.

Active $x \in W_{r+\delta} \setminus W_{r-\delta}$
ceases to be active.



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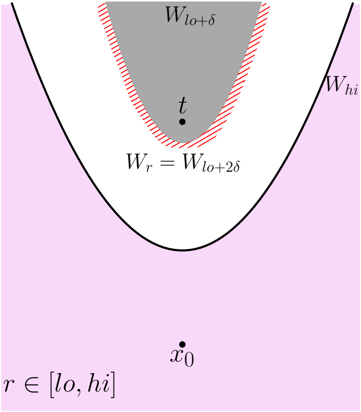

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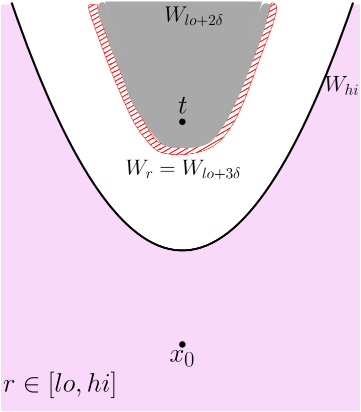

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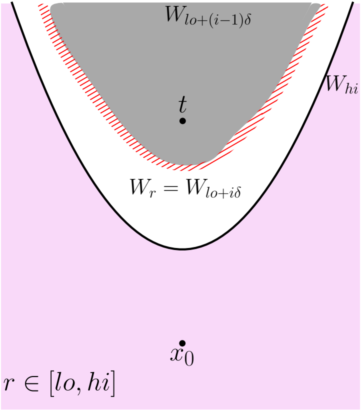

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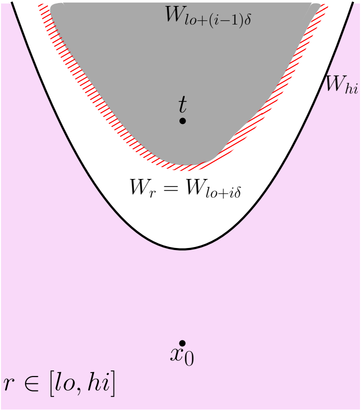

Intuition

There is $r \in [lo, hi]$ such that $W_{r-\delta}$ is **large enough** *compared to $W_{r+\delta}$.*

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Corollary

At least $n^{1-1/k}$ *vertices* **remain active** *at the* **end** *of the process.*

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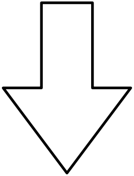

Theorem (Main Result)

For every n-vertex **weighted graph** $G = (V, E, w)$ *and* $\boldsymbol{k} \geq 1$, *there exists a* **subset** M *of size* $n^{1-1/k}$ *and* **spanning tree** T *of* G *with* **distortion** $O(\boldsymbol{k} \cdot \log\log n)$ *w.r.t* $M \times V$.

Open Questions

1. **Remove** the $\log\log n$ factor.

Conjecture

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2. Improve construction for deterministic distance oracle.

Distance Oracle	**Distortion**	**Size**	**Query**
This paper+C14	$2k-1$	$O(k \cdot n^{1+1/k})$	$O(1)$
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3. Find more **applications** to Ramsey spanning trees!