BPTree: improved space for insertion-only ℓ_2 heavy hitters

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joint work with Vladimir Braverman (Johns Hopkins), Stephen Chestnut (G-Research), Nikita Ivkin (Johns Hopkins), Zhengyu Wang (Harvard), and David Woodruff (CMU)

Finding frequent items

A (fake) search engine query log from Nov 7th:

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18:59:12	`mlb playoffs`
19:07:40	`wiki trump`
19:07:42	`cream of wheat wiki`
19:07:58	`p vs np`
19:09:37	`aa flight status 1597`
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- for word $i \in \mathcal{U}$, "heavy" means $x_i > \varepsilon \|x\|_2$, where $x \in \mathbb{R}^{|\mathcal{U}|}$ has x_i equal to # occurrences of word i in stream

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- Henceforth: $k := 1/\varepsilon^2$, want to find (ℓ_2-approximate) "top-k"
- Could define in terms of $\|x\|_p$ for other p, but known $f(k) \cdot n^{o(1)}$ space possible iff $p \le 2$ [BarYossef-Jayram-Kumar-Sivakumar'04], and up to slight change in problem defn can black-box solve ℓ_p HH optimally using optimal ℓ_q algo. if $p < q$ [Jowhari-Sağlam-Tardos'11].

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`query()`: Must output $L \subseteq [n]$ s.t.

(1) $|L| = O(k)$, and

(2) L contains every k-HH

Works on heavy hitters

- sampling (folklore)
- Frequent [Misra-Gries'82]
- LossyCounting [Singh-Motwani'02]
- SpaceSaving [Metwally-Agrawal-ElAbbadi'05]
- SampleAndHold [Estan-Varghese'03]
- Multi-stage bloom filters [Chabchoub-Fricker-Mohamed'09]
- Sketch-guided sampling [Kumar-Xu'06]
- CountMin sketch [Cormode-Muthukrishnan'05]
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- HSS (Hierarchical CountSketch) [Cormode-Hadjieleftheriou'08]
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Bounds attained for ℓ_2-heavy hitters

(k denotes $1/\varepsilon^2$)

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reference	data structure	space (words)
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OPEN: $O(k)$ words?

Insertion-only ℓ_2 heavy hitters: the BPTree

[Braverman-Chestnut-Ivkin-Nelson-Wang-Woodruff'17]

BPTree

Plan of attack

- **Defn.** $H \in [n]$ is **super-heavy** if $x_H^2 > 1000 \sum_{j \neq H} x_j^2$
- We will reduce finding $L \subset [n]$ containing all heavy hitters, $|L| = O(k)$, to the following problem:

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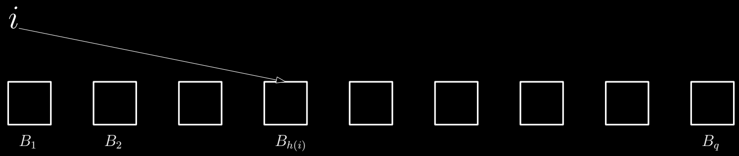
- If can solve "super-heavy" in space S, our final algorithm will have space $O(S \cdot k \log k) \implies$ want $S = O(1)$

Reduction to finding super-heavy items

The reduction: $h : [n] \to [q]$ from 2-wise indep. family, $q = 5000k$



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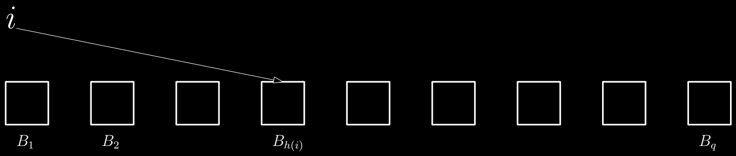

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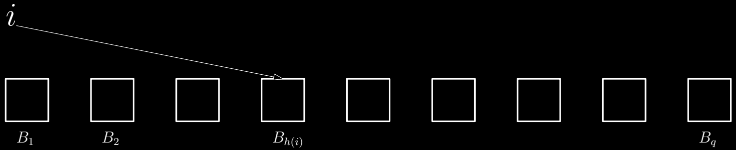

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$$\mathbb{P}_h(\exists j \in HH \backslash \{i\},\ h(j) = h(i)) \le \frac{1}{5000} \quad \text{(i isolated from rest of HH)}$$

$$\mathbb{P}_h\Big(\sum_{\substack{j \notin HH \\ h(j)=h(i)}} x_j^2 \ge \frac{1}{1000k}\|x\|_2^2\Big) \le \frac{1}{5} \quad \text{(very little non-HH mass in $B_{h(i)}$)}$$

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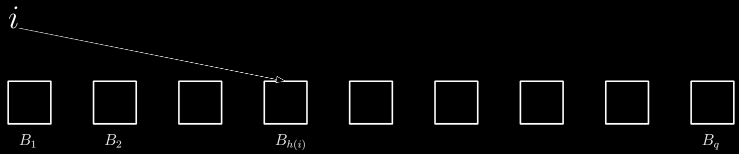

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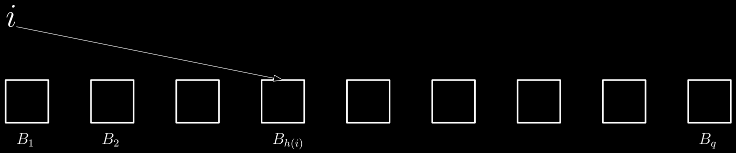

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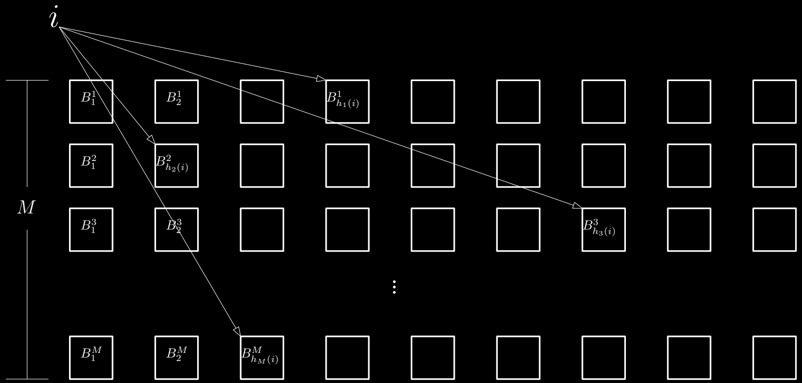
- Each B_r stores a data structure implementing a solution to the "super-heavy" problem w/ success prob. $\geq 9/10$, so we find i w.p. $\geq \frac{9}{10} \cdot (1 - \frac{1}{5} - \frac{1}{5000}) > \frac{7}{10}$

Final reduction

The reduction: $h_1, \ldots, h_M : [n] \to [q]$ from 2-wise indep. family, $q = 5000k$, $M = \Theta(\log k)$



Output
$L = \{i : i \text{ reported as super-heavy in at least half the rows}\}$

Analysis: Use last slide + Chernoff and union bound

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Will make use of ...

Core lemma: If $0 = y^{(0)}, \ldots, y^{(T)}$ is the evolution of a vector updated in an insertion-only stream and $\sigma \in \{-1, 1\}^n$ has 4-wise independent entries, then

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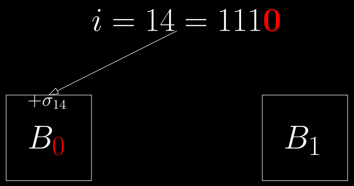
  - when we see $i \in [n]$ in stream, increment $B_{i[0]}$ by σ_i



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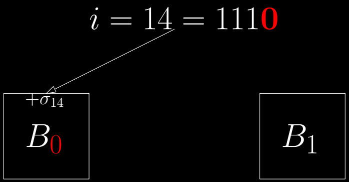

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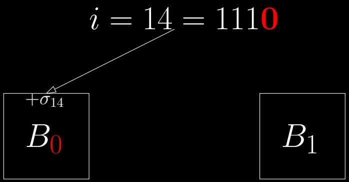

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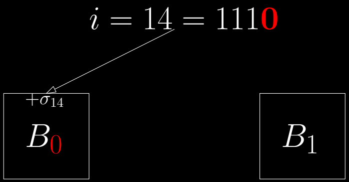

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- Remember *we know* $\|x^{(m)}\|_2$. Wait until some $|B_j| > .1\|x^{(m)}\|_2$, then we learn $H[0] = j$.

  "Core Lemma" applied twice (once to each bucket) implies two $\sum$'s above probably never exceed $.01\|x^{(m)}\|_2$

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- Final fix: Pick 2-wise permutation $\pi : [n^3] \to [n^3]$ and for each stream update i, feed $\pi(i)$ to algorithm. Then indices **are** random, and we can learn $H' = \pi(H)$. Then return $\pi^{-1}(H')$.

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  - The newly booted process missed out on some prefix of the stream, but if $\|x^{(m)}\|_2$ actually ends up $\approx 2^{R+j}$, we only missed out on mass leading up to $\|x\|_2 \approx 2^j$, so only missed $\approx 2^{-R}$ fraction of the final x_H occurrences. **QED**.

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Warmup

Simple random walk on a line.

$$y^{(t)} = (\overbrace{1, \ldots, 1}^{t}, \overbrace{0, 0, 0, 0, \ldots, 0}^{n-t})$$

- $\sigma \in \{-1, 1\}^n$, row of Π, has 4-wise independent entries

Warmup

Simple random walk on a line.

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- Will now show a proof (outline) of above standard result that can be adapted to handle 4-wise independent σ_i

Suprema of stochastic processes

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We have $V \subset B_{\ell_2^n}$ and want to upper bound

$$\alpha(V) := \mathbb{E} \sup_{v \in V} |\langle \sigma, v \rangle|$$

(in our case $V = \{\frac{y^{(t)}}{\sqrt{T}}\}_{t=0}^{T}$ and want to show $\alpha(V) \lesssim 1$)

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$$\begin{aligned}
\alpha(V) &= \int_0^\infty \mathbb{P}(\sup_{v \in V} | \langle \sigma, v \rangle | > \lambda) d\lambda \\
&= \int_0^\tau \overbrace{\mathbb{P}(\sup_{v \in V} | \langle \sigma, v \rangle | > \lambda)}^{\le 1} d\lambda + \int_\tau^\infty \overbrace{\mathbb{P}(\sup_{v \in V} | \langle \sigma, v \rangle | > \lambda)}^{\le \sum_{v \in V} \mathbb{P}(|\langle \sigma, v \rangle| > \lambda)} d\lambda \\
&\le \tau + |V| \cdot 2e^{-\tau^2/2} \\
&\lesssim \sqrt{\lg |V|} \ (\text{set } \tau = C\sqrt{\lg |V|})
\end{aligned}$$

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$$\implies \alpha(V) \lesssim \inf_{\varepsilon > 0} \left\{ \lg^{1/2} \mathcal{N}(V, \ell_2, \varepsilon) + \varepsilon\sqrt{n} \right\}$$

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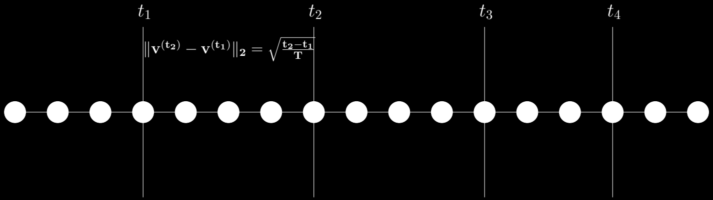
For us: will show $\mathcal{N}(V, \ell_2, \varepsilon) \simeq 1/\varepsilon^2$, so $\lg^{1/2}(1/\varepsilon) + \varepsilon\sqrt{n}$

Net size for random walk on line

Recall for us: $V = \{\frac{y^{(t)}}{\sqrt{T}}\}_{t=0}^{T}$, $v^{(t)} = \frac{1}{\sqrt{T}} \cdot y^{(t)}$.



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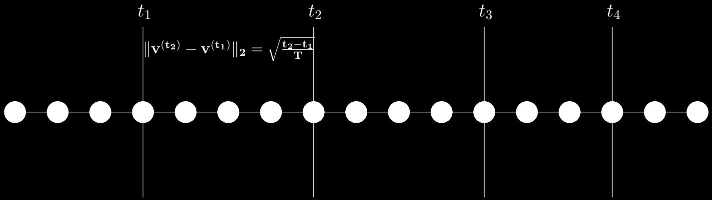

optimal ε-net is: $\{v^{(s\varepsilon^2 T)}\}$ for $s = 1, 2, \ldots, 1/\varepsilon^2$,
so $\mathcal{N}(V, \ell_2, \varepsilon) = 1/\varepsilon^2$

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&\qquad \times \lg^{1/2}(\mathcal{N}(V, \ell_2, \frac{1}{2^k}) \cdot \mathcal{N}(V, \ell_2, \frac{1}{2^{k-1}})) \\
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&\lesssim \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \lg^{1/2} \mathcal{N}\left(V, \ell_2, \frac{1}{2^k}\right) \quad \left(\le \sum_k \frac{\sqrt{k}}{2^k} = O(1)\right)
\end{aligned}$$

What about the 4-wise independence?

Dudley chaining with p-wise independence

Where it all started: Khintchine inequality says $\mathbb{P}_\sigma(|\langle \sigma, v\rangle| > \lambda) \le 2e^{-\lambda^2/(2\|v\|_2^2)}$.

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Khintchine says $\mathbb{E}|\langle\sigma, v\rangle|^p \le (\sqrt{p}\cdot\|v\|_2)^p$ for all $p \ge 1$

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If use above new tail bound in Method 1 and push through the Dudley argument, and note $|\{v(k) - v(k-1) : v \in V\}| \le 2|V_k|$, obtain a new "Dudley-esque" bound for our V:

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$$\begin{aligned}\alpha(V) &\lesssim \sum_{k=1}^{\infty} \frac{1}{2^k}\cdot\sqrt{p}\cdot(\mathcal{N}(V, \ell_2, \frac{1}{2^k}))^{1/p}\\ &\le \sum_{k=1}^{\infty}\sqrt{p}\cdot\frac{2^{2k/p}}{2^k}\\ &\lesssim 1 \text{ (for } p \ge 3)\end{aligned}$$

Yay – done with the warmup!

Recap: what we showed (and what's left)

Core lemma: If $0 = y^{(0)}, \dots, y^{(T)}$ is the evolution of a vector updated in an insertion-only stream and $\sigma \in \{-1, 1\}^n$ has 4-wise independent entries, then

$$\mathop{\mathbb{E}}_{\sigma} \sup_{t \in [T]} \left| \left\langle \sigma, v^{(t)} \right\rangle \right| \lesssim \|v^{(T)}\|_2 \quad (\text{where } v^{(t)} := \frac{y^{(t)}}{\|y^{(T)}\|_2})$$

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We showed: we proved core lemma in special case

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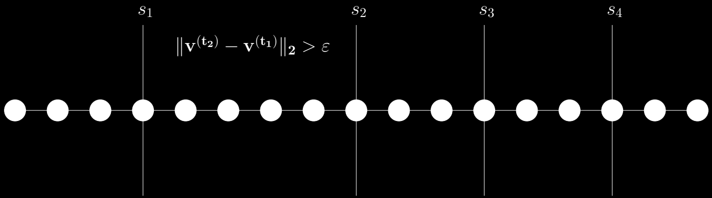
Same proof works!



- Our ε-net will be $V' = \{v^{(0)} := v^{(t_0)}, v^{(t_1)}, \ldots, v^{(t_R)}\}$
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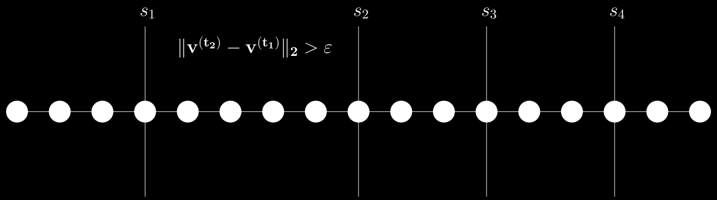

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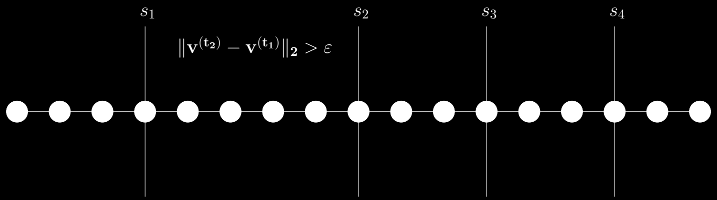

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$$
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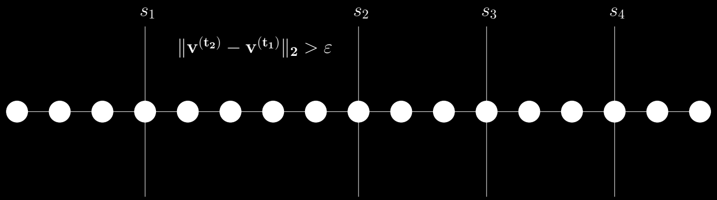

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&> R \cdot \varepsilon^2 \quad (\Longrightarrow R < 1/\varepsilon^2)
\end{aligned}
$$

Open Problems

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- $O(k)$ words of memory for insertion-only ℓ_2 heavy hitters?
- Does core lemma hold with 2-wise independence?