Media and Motion

Amal Alphonse, Simon Bignold, Yuchen Pei
Mathematics Institute

Aims

Although the motivation of our work is the biology of the heart, our aims are primarily to simulate the spiral waves that arise during arrhythmias on heart-like surfaces. Specifically, we aim to

- create spiral waves on a 2D plane using both the finite element (FE) and finite difference (FD) methods
- create spiral waves on a static unit sphere and static ellipsoid
- create spiral waves on a oscillating sphere with a range of different magnitudes of oscillation and compare to the static case
- simulate inhomogeneities (surface holes and areas of low conductivity) in the sphere

The Barkley Model

To simulate the formation of spiral waves we use a coupled system of partial differential equations (PDEs) that lie on a surface Γ:

$$\begin{aligned} \dot{u} + u\nabla_\Gamma \cdot \mathbf{v} - a\Delta_\Gamma u &= f(u,v) \qquad \text{in } (0,T)\times\Gamma \\ \dot{v} + v\nabla_\Gamma \cdot \mathbf{v} &= g(u,v) \qquad \text{in } (0,T)\times\Gamma, \end{aligned}$$

where

$$\begin{aligned} f(u,v) &= \frac{1}{\epsilon}u(1-u)\left(u - \frac{v+b}{c}\right) \\ g(u,v) &= u - v, \end{aligned}$$

with initial conditions $u(0,\cdot) = u_0(\cdot)$ and $v(0,\cdot) = v_0(\cdot)$, and model parameters a, b, c and ϵ all in $\mathbb{R}^+$. By $\mathbf{v}$ we denote the velocity of the surface.
When Γ is a surface with $\partial\Gamma$ non-empty (eg., a sphere with holes), we use Neumann boundary conditions

$$\nabla_\Gamma u(x) \cdot \nu_{\partial\Gamma}(x) = 0 \qquad \forall x \in \partial\Gamma.$$

This model is called the *Barkley model*. We use this model as it has been shown to produce spiral waves, is relatively simple and numerically efficient.
We look at three different kinds of surfaces Γ:

1. **Squares** Here $\Gamma \subset \mathbb{R}^2$, and $\dot{u}$ and $\dot{v}$ are the usual time derivatives and $\Delta_\Gamma = \Delta$ is the Laplacian.
2. **2D Fixed Surface** The walls of heart chambers are 2D surfaces that cannot be embedded in $\mathbb{R}^2$, unlike the square. So it is more realistic to model the equations on surfaces like spheres and ellipsoids. Here, $\dot{u}$ and $\dot{v}$ are again the normal time derivatives, however Δ_Γ is now the Laplace–Beltrami operator.
3. **Moving Surfaces** The pulsation of the heart requires us to simulate the spiral waves on moving surfaces, where $\dot{u}$ and $\dot{v}$ are material derivatives and Δ_Γ is the Laplace–Beltrami operator on Γ_t.

2D Simulations

As stated above we wish to simulate the generation of spiral waves on a square. In this case we simulate using both the finite element and finite difference methods. After comparing results we conclude that they produce similar results.

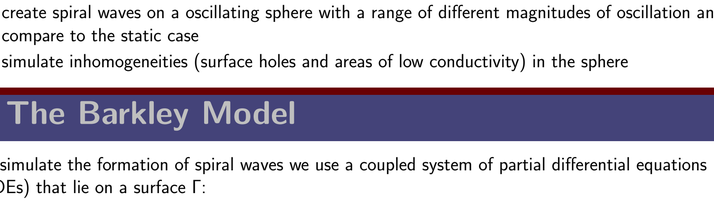

Figure: Spiral on a square. *Left*: using FD method with explicit RHS; *Right*: using FE method with explicit RHS.

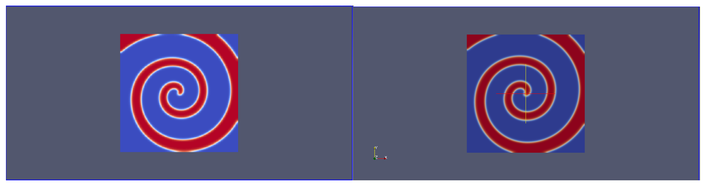

Figure: Spiral on a square. *Left*: using FD method with semi-implicit RHS; *Right*: using FE method with semi-implicit RHS.

Static Sphere

The unit sphere is relatively simple to implement and has similar embedding properties to the heart chambers and so is a good first example. We keep refining the grid resolution until the solution appears to converge.

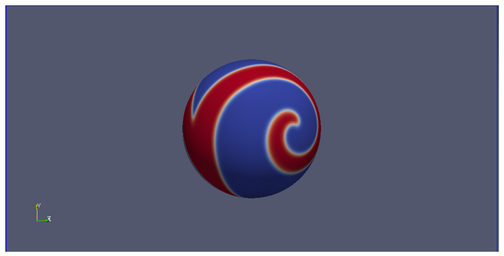

Figure: Spiral waves on a unit sphere at time 6.7

Static Ellipsoid

An ellipsoid resembles the shape of the chambers of the heart more accurately. For simplicity we choose to extend along the y-axis by a factor of 1.5. We see that there's no qualitative difference between the spiral waves we generate on a static ellipsoid and the spiral waves on the static sphere.

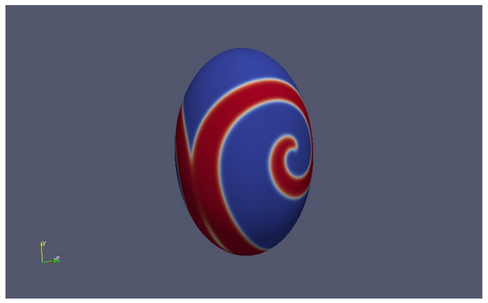

Figure: Spiral waves on a ellipsoid at time 6.7

Oscillating Sphere

The oscillating sphere is a way of simulating the pumping action of the heart. To simulate this, we apply a factor of

$$1 + \alpha\sin(2\pi\beta t)$$

along the y-axis, where $|\alpha| < 1$ defines the deformation and β defines the period. We try a range of values of α going from 0.1 to 0.5 in steps of 0.1 and set $\beta = 0.1$ to give a period of 10 time units.

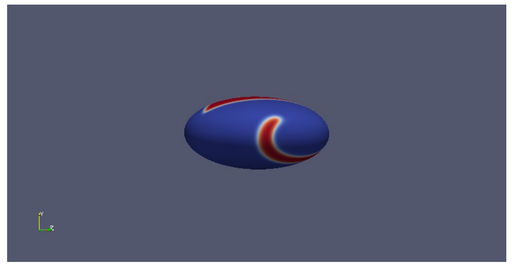

Figure: Spiral wave on an oscillating sphere at the point of minimum deformation

All our simulations produce spiral waves however the solution dramatically reduces twice on the way from the initial conditions to the spiral wave which is not seen in the static case. Also for large deformations we saw the wave-form become deformed at the points of largest oscillation and the center of the spiral meandering away from its original location.

Inhomogeneities

Up to now we have considered only surfaces of uniform conductivity. In real life, this is unrealistic as the heart has blood vessels running through it and may have areas of damaged tissue where conductivity is reduced. These leads to problems with heart rhythm and may be the source of spiral waves. To simulate the effect this has on the waves we consider two approaches:

- a physical hole in the surface, and
- reducing the diffusion coefficient (equivalent to reducing conductivity) in a region by
  - setting it zero
  - reducing it by a constant factor
  - reducing it continuously to zero in the centre

A Physical Hole

This is the simplest type of hole. We filter elements from the sphere and add zero Neumann boundary conditions at the boundary of the new hole. We show two different sizes of holes.

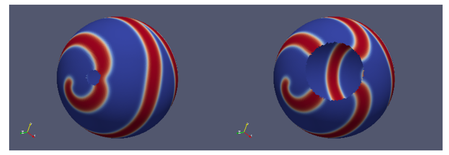

Figure: Physical holes. *Left*: hole with radius 0.1, *Right*: hole with radius 0.4

We can see that the spiral wave is intercepted by the hole and reforms after it. Also, the centre of the spiral wave also moves towards the hole as time increases. This is called spiral *locking*.

Regions of Reduced Conductivity

As mentioned above, we considered three cases of reduced conductivity:

- **Zero conductivity** In this case we set the diffusion coefficient to zero in a region. Unfortunately this creates a jump in the coefficient which leads to numerical instabilities as seen in the overshoot of the left figure (the solution should be between 0 and 1).
- **Reduced conductivity** Instead of setting the diffusion constant to zero we reduce it by a factor. This approximates the diffusion constant being set to zero and the approximation becomes better as the divisor becomes larger. This method also has the advantage that it allows us to simulate a range of conductivities. Compared to the above case, it has similar behaviour but no noticeable overshoot and therefore is numerically better.
- **Continuous conductivity** To address the instabilities caused by the jump in the diffusion coefficient, we let it vary *continuously* to zero. Therefore for a hole centred at $\mathbf{x_0}$ with radius R we multiply the diffusion coefficient by a factor:

$$1 + \exp\left(\frac{-R}{10}\right) \times \exp\left(1000\left(|\mathbf{x}-\mathbf{x_0}|^2 - R^2\right)\right) - \exp\left(\frac{-|\mathbf{x}-\mathbf{x_0}|^2}{10R}\right)$$

where the first and third factor combine to give a function that decays like a Gaussian (with variance $5R$) to zero in the center of the whole. The second term is chosen to make the factor equal to 1 at $\{|\mathbf{x}-\mathbf{x_0}| = R\}$ (i.e., at the edges of the hole) and to decay very fast as we enter the hole.

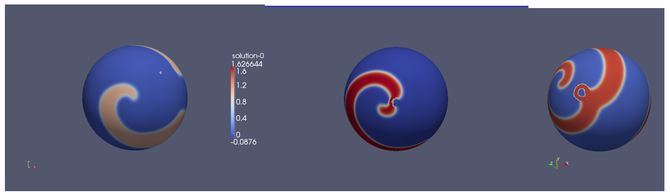

Figure: Sphere with inhomogeneities. *Left*: zero conductivity, *Centre*: reduced conductivity, *Right*: continuously reduced conductivity.

Acknowledgements

Thanks to:

- Dwight Barkley and Andreas Dedner
- CSC and EPSRC