Colouring Office Blocks

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1 Introduction

We will be looking at the possible colourings of buildings with many rooms. This problem was motivated by the idea of office security with respect to wireless interent frequencies. It was originally posed as how many different frequencies must be used in an office building such that no offices that meet along a portion of their wall, floor or ceiling use the same frequency. We will be thinking of the frequencies as colours and attempting to colour the building in as few different coulours as possible. We will restrict ourselves to the case where the buiding is made up of cuboid offices. Reed & Allwright (2008) showed that for any positive integer k, we may construct buildings that require at least k colours. We will thus also restrict ourselves to buildings that are layed out in floors, that is we lay out a ground floor using offices of the same height and on top of this place a first floor with offices of the same height and so on. For such buildings we will show an upper and lower bound for the number of colours required.

2 Bounding the Number of Colours Required

Definition (Layered building). *A building is* layered *if we may fix a set of axes in $\mathbb{R}^3$ and a countable set $\zeta = \{z_k : k \in \mathbb{N}\} \subset \mathbb{R}$ such that $z_k < z_{k+1}$ for all $k \in \mathbb{N}$, and for each office there exists $k \in \mathbb{N}$ such that its floor lies in the plane $\mathbf{z} = z_k$ and its ceiling in the plane $\mathbf{z} = z_{k+1}$.*

Definition (n-coloured). *For a positive integer n we say a building can be n-coloured if there is an assignment of n colours to its offices such that no offices that meet along a portion of their wall, floor or ceiling are given the same colour.*

2.1 Proposition. *A layered building can be 8-coloured.*

Proof. Begin with the ground floor, that is the set of offices with their floor lying in the plane $\mathbf{z} = z_0$. Project this into the plane via $(x, y, z) \mapsto (x, y, z_0)$.

By the four-colour theorem this projection may be four coloured. Find such a colouring and colour each office with the colour given to its projection, we thus have a 4-colouring of the ground floor, label the colours used 1, 2, 3 and 4. Similarly find a 4-colouring of the first floor and here use colours 5, 6, 7 and 8. Proceed in the same manner using colours 1 to 4 for even numbered floors and 5 to 8 for odd numbered floors. By construction, no neighbouring offices from the same floor are given the same colour and any offices immediately above or below are coloured using colours from a different set of four. Thus we have an 8-colouring for the layered building. □

Note: This bound does not require the offices to be cuboids, it is sufficient for the offices to be prisms of arbitary two dimensional shapes (not necessairily the same shapes for each office) in the x-y plane, each with the same height.

2.2 A Planar Map With Rectangular Countries That Requires Four Colours

To show n is a lower bound it is sufficient to find a building that requires at least n colours to colour it. A logical first step when dealing with cuboids is to attempt to find a planar map which uses only rectangular countries but still requires the maximum four colours. Figure 1 is an example of such a map. The countries labelled A–C all share borders with each other so must be coloured with 3 different colours, wlog let these be 1 for A, 2 for B and 3 for C. Then consider D, it borders A and C so we must either colour it with a fourth colour, or use colour 2. If we do use colour 2, apply the same arguement to E and if this is not coloured with a fourth colour we must use 1. We are then left with F, if we haven't already used a fourth colour this borders C, D and E which have colours 3, 2 and 1 respectively. This forces F to be coloured using a fourth colour. Thus Figure 1 requires 4 colours.

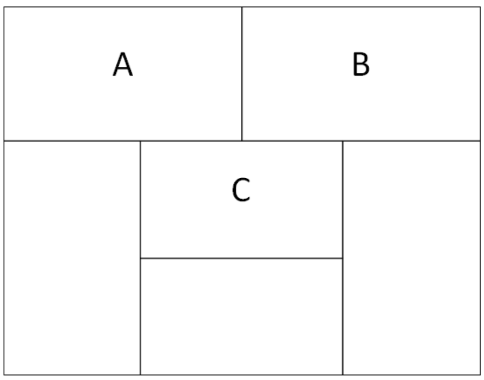
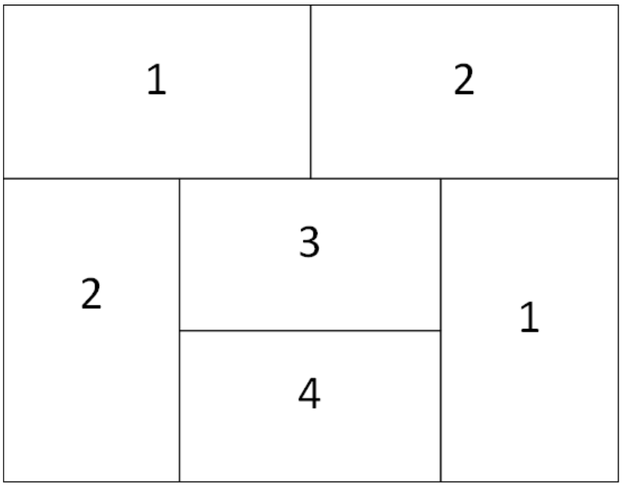


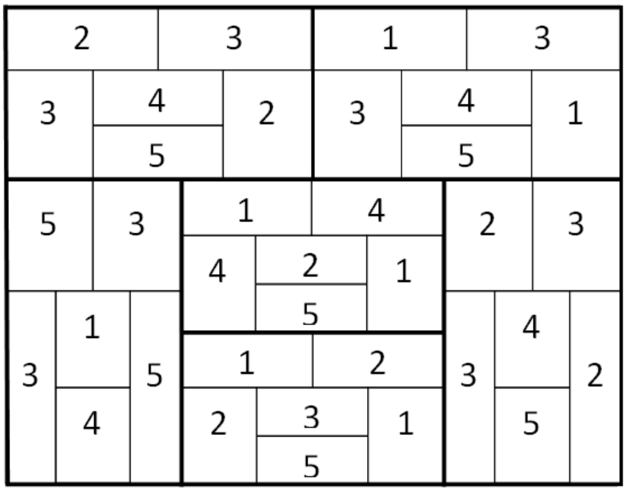
Figure 1: An arrangement of rectangles requiring four colours.

2.3 Construction of a Building Requiring 6 Colours

We can use the pattern in Figure 1 to construct a building which requires 6 colours. The floor plan for this building is shown in Figure 2. To construct the building we use the following steps. The ground floor is one copy of the pattern from Figure 1. For the first floor we repeat the pattern above each room from the ground floor, giving six repeated units, each of the pattern from Figure 1. This is shown in Figure 2, with the ground floor plan superimposed using the bold lines. The second floor is then simply a large room covering the entire floor.



(a) Ground floor



(b) First floor

Figure 2: The ground and first floor with a suggested colouring.

A colouring is suggested in the figure, this is completed by colouring the second floor room with a sixth colour, giving a 6-colouring of this building. We will now show 6 colours are required.

2.4 Proposition. *Any colouring of the building constructed in § 2.3 requires at least 6 colours.*

Proof. Begin by looking at the possible colourings of the first floor. First note that if we use 6 or more colours then we are done. We also know at least 4 colours are needed so we are left with two cases:

Case 1: First floor is coloured with four colours.

In this case, as each of the repeated units requires 4 colours then the 4 colours must all be used in each repeating unit. Thus every office on the ground floor has a room of each of the four colours above it. The ground floor must then be coloured using an entirely new set of colours and also requires four, so such a colouring requires 8 colours.

Case 2: First floor is coloured with five colours.

In this case the room on the second floor has a room of each of the five colours below it. This room then must be coloured with a sixth colour. □

3 Conclusions

We were able to prove that 8 colours are always sufficient for a layered building and when considering a layered building with cuboid offices at least 6 colours are required. We could make no further progress from this however. From attempting to better these bounds we discovered a subtelty of the problem, the optimal colouring of any floor or subsection of a building is rarely part of the optimal colouring of the whole. For example, in the case of the building from Figure 2, the first floor can be 4-coloured but doing so forces the ground floor to be coloured with 4 new colours, giving 8 in total, 3 more than the 5-colouring that is shown.

Considering only cuboid shaped offices seems to be quite a restrictive condition. When different shapes are allowed it is fairly easy to produce examples of 2 storey buildings that require all 8 colours. Figure 3 is an example of such a building. This uses floor plans consisting of two rectangles and two L shapes. It requires all 8 colours as each office shares some portion of its walls, floor or ceiling with those of each of the other seven offices.

Returning to the original problem, this is by no means an unreasonable layout for a portion of an office building. The problem of a layered building using cuboid offices, though a little contrived given the original problem, is an interesting one. We conjecture that 6 colours are sufficient for such a building, the sixth colour seems to provide just enough freedom to prevent you from requiring a seventh colour.

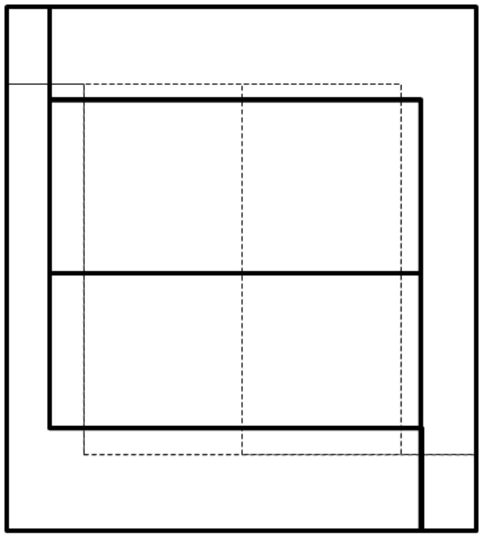


Figure 3: A two storey building requiring 8 colours. The ground floor is shown in bold and the first floor in dotted lines.

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References

Reed, B. & Allwright, D. (2008), 'Painting the office', *Mathematics-in-Industry Case Studies Journal* **1**, 1–8.