(Magnetic) Fluid Dynamics from a PDE-Theoretical Point of View

Dave McCormick

MASDOC

Mathematics and Statistics Centre for Doctoral Training
University of Warwick

May 9th, 2012

Introduction

- What are the Navier–Stokes equations?
- Weak derivatives and Sobolev spaces
- Laplace's equation
- Heat equation and the Galerkin method
- Navier–Stokes equations
- Magnetohydrodynamics (MHD)

The Navier–Stokes equations

Let $\boldsymbol{u}$ and p represent the velocity and pressure fields of a fluid. By applying Newton's second law to a small packet of fluid, we obtain the following PDE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t, x(t)) = \tag{1a}$$

$$\tag{1b}$$

The Navier–Stokes equations

Let $\boldsymbol{u}$ and p represent the velocity and pressure fields of a fluid. By applying Newton's second law to a small packet of fluid, we obtain the following PDE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t, x(t)) = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \tag{1a}$$

$$\tag{1b}$$

There are three forcing terms:

The Navier–Stokes equations

Let $\boldsymbol{u}$ and p represent the velocity and pressure fields of a fluid. By applying Newton's second law to a small packet of fluid, we obtain the following PDE:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} \tag{1a}$$

$$\tag{1b}$$

There are three forcing terms:

- $\nu \Delta \boldsymbol{u}$ represents the viscous friction of the fluid "rubbing against itself" (where the coefficient ν is the viscosity);

The Navier–Stokes equations

Let $\boldsymbol{u}$ and p represent the velocity and pressure fields of a fluid. By applying Newton's second law to a small packet of fluid, we obtain the following PDE:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \nabla p \tag{1a}$$

$$\tag{1b}$$

There are three forcing terms:

- $\nu \Delta \boldsymbol{u}$ represents the viscous friction of the fluid "rubbing against itself" (where the coefficient ν is the viscosity);
- ∇p ensures that fluid moves from areas of high pressure to areas of low pressure,

The Navier–Stokes equations

Let $\boldsymbol{u}$ and p represent the velocity and pressure fields of a fluid. By applying Newton's second law to a small packet of fluid, we obtain the following PDE:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \nabla p \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1b}$$

There are three forcing terms:

- $\nu \Delta \boldsymbol{u}$ represents the viscous friction of the fluid "rubbing against itself" (where the coefficient ν is the viscosity);
- ∇p ensures that fluid moves from areas of high pressure to areas of low pressure, thus enforcing the divergence-free condition $\nabla \cdot \boldsymbol{u} = 0$;

The Navier–Stokes equations

Let $\boldsymbol{u}$ and p represent the velocity and pressure fields of a fluid. By applying Newton's second law to a small packet of fluid, we obtain the following PDE:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \nabla p + \boldsymbol{f}, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{1b}$$

There are three forcing terms:

- $\nu \Delta \boldsymbol{u}$ represents the viscous friction of the fluid "rubbing against itself" (where the coefficient ν is the viscosity);
- ∇p ensures that fluid moves from areas of high pressure to areas of low pressure, thus enforcing the divergence-free condition $\nabla \cdot \boldsymbol{u} = 0$;
- $\boldsymbol{f}$ represents an external forcing.

Complications

The Navier–Stokes equations are difficult to deal with rigorously, due to a combination of two factors:

Complications

The Navier–Stokes equations are difficult to deal with rigorously, due to a combination of two factors:

- the nonlinear term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, and

Complications

The Navier–Stokes equations are difficult to deal with rigorously, due to a combination of two factors:

- the nonlinear term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, and
- the pressure gradient ∇p.

Complications

The Navier–Stokes equations are difficult to deal with rigorously, due to a combination of two factors:

- the nonlinear term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, and
- the pressure gradient ∇p.

So let's start by considering a simplified equation without either of these difficult terms:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = f,$$

which is the heat equation.

Complications

The Navier–Stokes equations are difficult to deal with rigorously, due to a combination of two factors:

- the nonlinear term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, and
- the pressure gradient ∇p.

So let's start by considering a simplified equation without either of these difficult terms:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = f,$$

which is the heat equation. To study this, we must first study the time-independent version of this equation:

$$-\Delta \boldsymbol{u} = f,$$

which is Laplace's equation.

Laplace's equation

On regions with simple geometry, we can solve Laplace's equation explicitly. For example, Laplace's equation on a ball of radius r:

$$\begin{aligned} \Delta u &= 0 && \text{in } B(0,r) \\ u &= g && \text{on } \partial B(0,r) \end{aligned}$$

has the solution formula

$$u(x) = \frac{r^2 - |x|^2}{r} \frac{\Gamma(\frac{n}{2}+1)}{n\pi^{n/2}} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \, \mathrm{d}S(y).$$

Laplace's equation

What if the domain we want to solve the equation on looks like this?

Laplace's equation

What if the domain we want to solve the equation on looks like this?

We need a more general theory that can cope with any (reasonable) domain Ω, and doesn't depend on special geometries.

Laplace's equation

We need a more general theory that can cope with any (reasonable) domain Ω, and doesn't depend on special geometries.

Aim

We wish to use the language of functional analysis — Hilbert and Banach spaces — to study PDEs. The first task is to find the "right" spaces to work in.

Weak derivatives

Let $C_c^\infty(\Omega)$ denote the set of infinitely differentiable functions $\phi\colon \Omega \to \mathbb{R}$ with compact support in Ω — that is,

$$\operatorname{spt}(\phi) := \overline{\{x \in \Omega : \phi(x) \neq 0\}} \subset \Omega^\circ.$$

Weak derivatives

Let $C_c^\infty(\Omega)$ denote the set of infinitely differentiable functions $\phi\colon \Omega \to \mathbb{R}$ with compact support in Ω — that is,

$$\operatorname{spt}(\phi) := \overline{\{x \in \Omega : \phi(x) \neq 0\}} \subset \Omega^\circ.$$

Suppose we are given a function $u \in C^1(\Omega)$, and $\phi \in C_c^\infty(\Omega)$. Then we may integrate by parts as follows:

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega \frac{\partial u}{\partial x_i} \phi.$$

Weak derivatives

Let $C_c^\infty(\Omega)$ denote the set of infinitely differentiable functions $\phi\colon \Omega \to \mathbb{R}$ with compact support in Ω — that is,

$$\operatorname{spt}(\phi) := \overline{\{x \in \Omega : \phi(x) \neq 0\}} \subset \Omega^\circ.$$

Suppose we are given a function $u \in C^1(\Omega)$, and $\phi \in C_c^\infty(\Omega)$. Then we may integrate by parts as follows:

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega \frac{\partial u}{\partial x_i}\phi.$$

More generally, if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $u \in C^k(\Omega)$, then we may integrate by parts $|\alpha| = \alpha_1 + \cdots + \alpha_n$ times to obtain:

$$\int_\Omega u(D^\alpha \phi) = (-1)^{|\alpha|} \int_\Omega (D^\alpha u)\phi.$$

Weak derivatives

Notice that the left-hand side of this formula makes sense even if u is not C^k. The problem is that if u is not C^k then $D^\alpha u$ has no obvious meaning. We circumvent this by using the above expression to *define* $D^\alpha u$:

Notice that the left-hand side of this formula makes sense even if u is not C^k. The problem is that if u is not C^k then $D^\alpha u$ has no obvious meaning. We circumvent this by using the above expression to *define* $D^\alpha u$:

Definition

Suppose $u, v \in L^1_{\text{loc}}(\Omega)$, and $\alpha \in \mathbb{N}_0^n$ is a multi-index. We say that v is the α^{th} weak partial derivative of u, written $D^\alpha u$, provided that

$$\int_\Omega u(D^\alpha \phi) = (-1)^{|\alpha|} \int_\Omega v\phi$$

for all $\phi \in C_c^\infty(\Omega)$.

Definition

Given $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) := \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \begin{array}{l} \text{for all } \alpha \in \mathbb{N}_0^n \text{ such that} \\ |\alpha| \leq k,\ D^\alpha u \in L^p(\Omega) \end{array} \right\},$$

where we identify functions which agree almost everywhere.

Sobolev spaces

Definition

Given $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) := \left\{ u \in L^1_{\text{loc}}(\Omega) : \begin{array}{l} \text{for all } \alpha \in \mathbb{N}_0^n \text{ such that} \\ |\alpha| \leq k,\ D^\alpha u \in L^p(\Omega) \end{array} \right\},$$

where we identify functions which agree almost everywhere. The $W^{k,p}$ norm is defined by

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \right)^{1/p} \qquad \text{if } 1 \leq p < \infty$$

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leq k} \operatorname*{ess\,sup}_\Omega |D^\alpha u|$$

Lemma

$W^{k,p}(\Omega)$ is a separable Banach space.

Sobolev spaces

Lemma

$W^{k,p}(\Omega)$ is a separable Banach space.

Definition

When $p = 2$, we write $H^k(\Omega) := W^{k,p}(\Omega)$, and define the H^k inner product by

$$(u, v)_{H^k(\Omega)} := \sum_{|\alpha| \le k} \int_\Omega (D^\alpha u)(D^\alpha v).$$

Lemma

$W^{k,p}(\Omega)$ is a separable Banach space.

Definition

When $p = 2$, we write $H^k(\Omega) := W^{k,p}(\Omega)$, and define the H^k inner product by

$$(u, v)_{H^k(\Omega)} := \sum_{|\alpha| \leq k} \int_\Omega (D^\alpha u)(D^\alpha v).$$

Lemma

$H^k(\Omega)$ is a separable Hilbert space.

Sobolev spaces with zero trace

It often proves useful to approximate Sobolev functions by smooth test functions. It is an easy fact from measure theory that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.

Sobolev spaces with zero trace

It often proves useful to approximate Sobolev functions by smooth test functions. It is an easy fact from measure theory that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$. One might think, therefore, that $C_c^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ for $k \geq 1$, but this is in fact not true: in fact, the closure of $C_c^\infty(\Omega)$ is a new, smaller space:

Sobolev spaces with zero trace

It often proves useful to approximate Sobolev functions by smooth test functions. It is an easy fact from measure theory that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$. One might think, therefore, that $C_c^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ for $k \geq 1$, but this is in fact not true: in fact, the closure of $C_c^\infty(\Omega)$ is a new, smaller space:

Definition

For $1 < p < \infty$, $W_0^{k,p}(\Omega)$ is defined to be the completion of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$; i.e., $u \in W_0^{k,p}(\Omega)$ if and only if there exist functions $u_m \in C_c^\infty(\Omega)$ such that $u_m \to u$ in $W^{k,p}(\Omega)$.

Sobolev spaces with zero trace

It often proves useful to approximate Sobolev functions by smooth test functions. It is an easy fact from measure theory that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$. One might think, therefore, that $C_c^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ for $k \geq 1$, but this is in fact not true: in fact, the closure of $C_c^\infty(\Omega)$ is a new, smaller space:

Definition

For $1 < p < \infty$, $W_0^{k,p}(\Omega)$ is defined to be the completion of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$; i.e., $u \in W_0^{k,p}(\Omega)$ if and only if there exist functions $u_m \in C_c^\infty(\Omega)$ such that $u_m \to u$ in $W^{k,p}(\Omega)$.

Heuristically, $W_0^{k,p}(\Omega)$ comprises those functions $u \in W^{k,p}(\Omega)$ such that "$D^\alpha u = 0$ on $\partial\Omega$" for all $|\alpha| \leq k - 1$. Again, we write $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

Sobolev spaces with zero trace

Definition

For $1 < p < \infty$, we define the norm $W_0^{k,p}(\Omega)$ by

$$\|u\|_{W_0^{k,p}(\Omega)} := \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^p \right)^{1/p},$$

Sobolev spaces with zero trace

Definition

For $1 < p < \infty$, we define the norm $W_0^{k,p}(\Omega)$ by

$$\|u\|_{W_0^{k,p}(\Omega)} := \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^p \right)^{1/p},$$

and the $H_0^k(\Omega)$ inner product by

$$(u, v)_{H_0^k(\Omega)} := \sum_{|\alpha|=k} \int_\Omega (D^\alpha u)(D^\alpha v).$$

Poincaré's inequality

Let $1 < p < \infty$, and suppose Ω is bounded. Then there exists a constant c_p such that

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{W_0^{k,p}(\Omega)}$$

for all $u \in W_0^{k,p}(\Omega)$.

Poincaré's inequality

Let $1 < p < \infty$, and suppose Ω is bounded. Then there exists a constant c_p such that

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{W_0^{k,p}(\Omega)}$$

for all $u \in W_0^{k,p}(\Omega)$.

In other words, on $W_0^{k,p}(\Omega)$, the norms $\|\cdot\|_{W_0^{k,p}(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$ are equivalent.

Laplace's equation

We can now reformulate Laplace's equation in Sobolev spaces. Take Laplace's equation with Dirichlet boundary conditions:

$$\begin{aligned} \Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Laplace's equation

We can now reformulate Laplace's equation in Sobolev spaces. Take Laplace's equation with Dirichlet boundary conditions:

$$\begin{aligned} \Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Suppose we have a smooth solution u of Laplace's equation, and let $v \in C_c^\infty(\Omega)$ be a smooth test function. Multiplying the equation by v and integrating over Ω, we get

$$-\int_\Omega (\Delta u)v = \int_\Omega fv.$$

Using the fact that $v = 0$ on $\partial\Omega$, we may integrate by parts and obtain

$$(u, v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv.$$

Laplace's equation

Let $H^{-1}(\Omega)$ denote the dual space to $H_0^1(\Omega)$ (that is, the space of all bounded linear functionals from $H_0^1(\Omega)$ into $\mathbb{R}$), and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Laplace's equation

Let $H^{-1}(\Omega)$ denote the dual space to $H_0^1(\Omega)$ (that is, the space of all bounded linear functionals from $H_0^1(\Omega)$ into $\mathbb{R}$), and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Definition

Given $f \in H^{-1}(\Omega)$, $u \in H_0^1(\Omega)$ is a weak solution of Laplace's equation if it satisfies

$$(u, v)_{H_0^1(\Omega)} = \langle f, v \rangle \tag{2}$$

for all $v \in H_0^1(\Omega)$.

Laplace's equation

Let $H^{-1}(\Omega)$ denote the dual space to $H_0^1(\Omega)$ (that is, the space of all bounded linear functionals from $H_0^1(\Omega)$ into $\mathbb{R}$), and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Definition

Given $f \in H^{-1}(\Omega)$, $u \in H_0^1(\Omega)$ is a weak solution of Laplace's equation if it satisfies

$$(u, v)_{H_0^1(\Omega)} = \langle f, v \rangle \tag{2}$$

for all $v \in H_0^1(\Omega)$.

If $f \in H^{-1}(\Omega)$ then, by the Riesz representation theorem applied to the linear functional $\ell(v) := \int_\Omega fv$, there exists a unique $u \in H_0^1(\Omega)$ such that (2) holds. Furthermore, if f is more regular, then so is u.

Eigenfunctions of the Laplacian

By the Hilbert–Schmidt theorem we may prove the following:

Theorem

Given a domain Ω, there exists a countably infinite sequence of $C^\infty(\Omega)$ eigenfunctions $(w_n)_{n\in\mathbb{N}}$ satisfying

$$\begin{aligned} -\Delta w_n &= \lambda_n w_n && \text{in } \Omega \\ w_n &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $\lambda_n \to \infty$ as $n \to \infty$. These eigenfunctions $(w_n$ form an orthonormal basis of $L^2(\Omega)$.

Heat equation

We now consider the heat equation:

$$\frac{\partial u}{\partial t} - \Delta u = f(t),$$

subject to the boundary condition $u = 0$ on $\partial\Omega$ and the initial condition $u(x, 0) = u_0(x)$.

Heat equation

We now consider the heat equation:

$$\frac{\partial u}{\partial t} - \Delta u = f(t),$$

subject to the boundary condition $u = 0$ on $\partial\Omega$ and the initial condition $u(x, 0) = u_0(x)$.

Energy evolution law

A smooth solution u of the heat equation, subject to Dirichlet boundary conditions, satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 = -\nu\|\nabla u\|_2^2 + (f, u).$$

Heat equation

Multiplying the equation by a fixed $v \in C_c^\infty(\Omega)$ and integrating (in space) yields

$$\left(\frac{\partial u}{\partial t}, v\right)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} = \langle f(t), v \rangle.$$

Heat equation

Multiplying the equation by a fixed $v \in C_c^\infty(\Omega)$ and integrating (in space) yields

$$\left(\frac{\partial u}{\partial t}, v\right)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} = \langle f(t), v\rangle.$$

This motivates the following definition:

Definition

Given $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T;H^{-1}(\Omega))$, u is a weak solution of the heat equation if:

- $u \in L^2(0,T;H_0^1(\Omega))$;
- its weak time derivative $\dot{u} \in L^2(0,T;H^{-1}(\Omega))$; and
- u satisfies

$$(\dot{u}, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} = \langle f(t), v\rangle$$

for all $v \in H_0^1(\Omega)$ and almost every $t \in (0,T)$.

Galerkin approximations

To show existence (and uniqueness) of weak solutions of the heat equation, we will use the Galerkin method:

- take the infinite-dimensional PDE problem,
- approximate by a sequence of finite-dimensional ODE problems,
- then take the limit.

Galerkin approximations

To show existence (and uniqueness) of weak solutions of the heat equation, we will use the Galerkin method:

- take the infinite-dimensional PDE problem,
- approximate by a sequence of finite-dimensional ODE problems,
- then take the limit.

Let $(w_n)_{n\in\mathbb{N}}$ be the eigenfunctions of the Laplacian, which as above form an orthonormal basis of $L^2(\Omega)$. Given $u \in L^2(\Omega)$, we can approximate it in the space spanned by the first n eigenfunctions as follows:

$$P_n u = \sum_{j=1}^{n} (u, w_j) w_j.$$

Galerkin approximations

Let $u_n(t)$ be contained in the span of $\{w_1, \ldots, w_n\}$:

$$u_n(t) = \sum_{j=1}^{n} u_{n,j}(t) w_j.$$

Galerkin approximations

Let $u_n(t)$ be contained in the span of $\{w_1, \dots, w_n\}$:

$$u_n(t) = \sum_{j=1}^{n} u_{n,j}(t) w_j.$$

We try to solve

$$(\dot{u}_n, w_j)_{L^2(\Omega)} + (\nabla u_n, \nabla w_j)_{L^2(\Omega)} = \langle f(t), w_j \rangle$$

with $u_n(0) = P_n u_0$, for $j = 1, \dots, n$. There are two ways of thinking of this:

Galerkin approximations

Let $u_n(t)$ be contained in the span of $\{w_1, \ldots, w_n\}$:

$$u_n(t) = \sum_{j=1}^{n} u_{n,j}(t) w_j.$$

We try to solve

$$(\dot{u}_n, w_j)_{L^2(\Omega)} + (\nabla u_n, \nabla w_j)_{L^2(\Omega)} = \langle f(t), w_j \rangle$$

with $u_n(0) = P_n u_0$, for $j = 1, \ldots, n$. There are two ways of thinking of this:

1. as a system of n ODEs for $u_{n,j}$, $j = 1, \ldots, n$:

$$\dot{u}_{n,j} + \lambda_j u_{n,j} = f_j(t) := \langle f(t), w_j \rangle;$$

Galerkin approximations

Let $u_n(t)$ be contained in the span of $\{w_1, \ldots, w_n\}$:

$$u_n(t) = \sum_{j=1}^{n} u_{n,j}(t) w_j.$$

We try to solve

$$(\dot{u}_n, w_j)_{L^2(\Omega)} + (\nabla u_n, \nabla w_j)_{L^2(\Omega)} = \langle f(t), w_j \rangle$$

with $u_n(0) = P_n u_0$, for $j = 1, \ldots, n$. There are two ways of thinking of this:

1. as a system of n ODEs for $u_{n,j}$, $j = 1, \ldots, n$:

$$\dot{u}_{n,j} + \lambda_j u_{n,j} = f_j(t) := \langle f(t), w_j \rangle;$$

2. as a truncated PDE for u_n:

$$\frac{\partial u_n}{\partial t} - \Delta u_n = P_n f(t).$$

Galerkin approximations

- Standard ODE theory gives a unique solution to the system of ODEs

$$\dot{u}_{n,j} + \lambda_j u_{n,j} = f_j(t) := \langle f(t), w_j \rangle;$$

at least for some short time interval.

Galerkin approximations

- Standard ODE theory gives a unique solution to the system of ODEs
$$\dot{u}_{n,j} + \lambda_j u_{n,j} = f_j(t) := \langle f(t), w_j \rangle;$$
at least for some short time interval.
- Problems only occur if the norm blows up, i.e. if $\sum_{j=1}^{n} |u_{n,j}|^2$ becomes infinite. This sum is equal to the L^2 norm of the function u_n, so it suffices to show that $\|u_n\|_{L^2(\Omega)}$ remains bounded independent of n (so that we can take the limit as $n \to \infty$.

Uniform boundedness

To see this, take the inner product of the PDE with u_n:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|_{L^2(\Omega)}^2 + \|\nabla u_n\|_{L^2(\Omega)}^2 \\ &= \langle P_n f(t), u_n\rangle = \langle f(t), P_n u_n\rangle = \langle f(t), u_n\rangle \\ &\leq \|f(t)\|_{H^{-1}(\Omega)}\|u_n\|_{H_0^1(\Omega)} = \|f(t)\|_{H^{-1}(\Omega)}\|\nabla u_n\|_{L^2(\Omega)}\end{aligned}$$

Uniform boundedness

To see this, take the inner product of the PDE with u_n:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2_{L^2(\Omega)} + \|\nabla u_n\|^2_{L^2(\Omega)} \\ &= \langle P_n f(t), u_n\rangle = \langle f(t), P_n u_n\rangle = \langle f(t), u_n\rangle \\ &\leq \|f(t)\|_{H^{-1}(\Omega)}\|u_n\|_{H^1_0(\Omega)} = \|f(t)\|_{H^{-1}(\Omega)}\|\nabla u_n\|_{L^2(\Omega)}\end{aligned}$$

By Young's inequality we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2_{L^2(\Omega)} + \|\nabla u_n\|^2_{L^2(\Omega)} \leq \|f(t)\|^2_{H^{-1}(\Omega)}$$

Uniform boundedness

To see this, take the inner product of the PDE with u_n:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|u_n\|^2_{L^2(\Omega)} + \|\nabla u_n\|^2_{L^2(\Omega)}\\ &= \langle P_n f(t), u_n\rangle = \langle f(t), P_n u_n\rangle = \langle f(t), u_n\rangle\\ &\le \|f(t)\|_{H^{-1}(\Omega)}\|u_n\|_{H^1_0(\Omega)} = \|f(t)\|_{H^{-1}(\Omega)}\|\nabla u_n\|_{L^2(\Omega)}\end{aligned}$$

By Young's inequality we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2_{L^2(\Omega)} + \|\nabla u_n\|^2_{L^2(\Omega)} \le \|f(t)\|^2_{H^{-1}(\Omega)}$$

so integrating in time from 0 to t we get, for all $t \in [0, T]$,

$$\begin{aligned}\|u_n(t)\|^2_{L^2(\Omega)} &+ \int_0^t \|\nabla u_n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\\ &\le \|u_n(0)\|^2_{L^2(\Omega)} + \int_0^t \|f(s)\|^2_{H^{-1}(\Omega)}\,\mathrm{d}s\\ &\le \|u_0\|^2_{L^2(\Omega)} + \int_0^t \|f(s)\|^2_{H^{-1}(\Omega)}\,\mathrm{d}s.\end{aligned}$$

Uniform boundedness

This tells us that (u_n) is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$.

Uniform boundedness

This tells us that (u_n) is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$. To get a bound on the time derivative, consider that

$$\frac{\partial u_n}{\partial t} = \Delta u_n + P_n f(t).$$

Since (as we showed above) u_n is bounded in $L^2(0,T;H_0^1(\Omega))$, Δu_n is bounded in $L^2(0,T;H^{-1}(\Omega))$; since $f \in L^2(0,T;H^{-1}(\Omega))$, we see that $\frac{\partial u_n}{\partial t}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$ independently of n.

Uniform boundedness

This tells us that (u_n) is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$ and $L^2(0, T; H_0^1(\Omega))$. To get a bound on the time derivative, consider that

$$\frac{\partial u_n}{\partial t} = \Delta u_n + P_n f(t).$$

Since (as we showed above) u_n is bounded in $L^2(0, T; H_0^1(\Omega))$, Δu_n is bounded in $L^2(0, T; H^{-1}(\Omega))$; since $f \in L^2(0, T; H^{-1}(\Omega))$, we see that $\frac{\partial u_n}{\partial t}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$ independently of n. In summary, the sequence (u_n) of approximations satisfies:

- (u_n) is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$;
- (u_n) is uniformly bounded in $L^2(0, T; H_0^1(\Omega))$;
- $(\dot{u}_n)$ is uniformly bounded in $L^2(0, T; H^{-1}(\Omega))$.

Taking the limit of the approximations

- The Banach–Alaoglu compactness theorem tells us that there is a subsequence u_{n_j} such that

$$\begin{aligned} u_{n_j} &\rightharpoonup u && \text{in } L^2(0,T;H_0^1(\Omega)) \\ u_{n_j} &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T;L^2(\Omega)) \\ \dot{u}_{n_j} &\overset{*}{\rightharpoonup} v && \text{in } L^2(0,T;H^{-1}(\Omega)) \end{aligned}$$

Taking the limit of the approximations

- The Banach–Alaoglu compactness theorem tells us that there is a subsequence u_{n_j} such that

$$
\begin{aligned}
u_{n_j} &\rightharpoonup u && \text{in } L^2(0,T;H_0^1(\Omega)) \\
u_{n_j} &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T;L^2(\Omega)) \\
\dot{u}_{n_j} &\overset{*}{\rightharpoonup} v && \text{in } L^2(0,T;H^{-1}(\Omega))
\end{aligned}
$$

- It remains to check that u is actually a weak solution of the equations, and that v is actually its weak time derivative!

Taking the limit of the approximations

- The Banach–Alaoglu compactness theorem tells us that there is a subsequence u_{n_j} such that

$$\begin{aligned} u_{n_j} &\rightharpoonup u && \text{in } L^2(0,T;H_0^1(\Omega)) \\ u_{n_j} &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T;L^2(\Omega)) \\ \dot{u}_{n_j} &\overset{*}{\rightharpoonup} v && \text{in } L^2(0,T;H^{-1}(\Omega)) \end{aligned}$$

- It remains to check that u is actually a weak solution of the equations, and that v is actually its weak time derivative!
- To show uniqueness, we can prove that $u \in L^2(0,T;H_0^1(\Omega))$ and $\dot{u} \in L^2(0,T;H^{-1}(\Omega))$ implies that $u \in C([0,T];L^2(\Omega))$.

Navier–Stokes equations

We return now to the Navier–Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \nabla p + \boldsymbol{f}, \tag{3a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{3b}$$

Navier–Stokes equations

We return now to the Navier–Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \nabla p + \boldsymbol{f}, \tag{3a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{3b}$$

Energy evolution law

A smooth solution $\boldsymbol{u}$ of the Navier–Stokes equations (1), subject to Dirichlet boundary conditions, satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_2^2 = -\nu\|\nabla \boldsymbol{u}\|_2^2 + (\boldsymbol{f}, \boldsymbol{u}).$$

We can again take Galerkin approximations, in which $(\boldsymbol{u}_n)$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$. Bounding the time derivative $\dot{\boldsymbol{u}}$, however, now requires bounding $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ in $H^{-1}(\Omega)$:

We can again take Galerkin approximations, in which $(\boldsymbol{u}_n)$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$. Bounding the time derivative $\dot{\boldsymbol{u}}$, however, now requires bounding $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ in $H^{-1}(\Omega)$:

$$\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{H^{-1}(\Omega)} = \sup_{\substack{\boldsymbol{v}\in H_0^1(\Omega),\\ \|\boldsymbol{v}\|_{H_0^1(\Omega)}=1}} \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{v}\rangle = -\sup\langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{u}\rangle.$$

Now, $\langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{u}\rangle \leq \|\boldsymbol{u}\|_{L^4(\Omega)}^2\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}$, so $\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{H^{-1}(\Omega)} \leq \|\boldsymbol{u}\|_{L^4(\Omega)}^2$.

Navier–Stokes equations

Bounding $\|\boldsymbol{u}\|^2_{L^4(\Omega)}$ depends on the dimension!

- In two dimensions, we have

$$\|\boldsymbol{u}\|^2_{L^4(\Omega)} \leq \|\boldsymbol{u}\|_{L^2(\Omega)} \|\nabla \boldsymbol{u}\|_{L^2(\Omega)},$$

  so $\dot{\boldsymbol{u}} \in L^2(0, T; H^{-1}(\Omega))$.

- In three dimensions, however, we have

$$\|\boldsymbol{u}\|^2_{L^4(\Omega)} \leq \|\boldsymbol{u}\|^{1/2}_{L^2(\Omega)} \|\nabla \boldsymbol{u}\|^{3/2}_{L^2(\Omega)}$$

  so $\dot{\boldsymbol{u}} \in L^{4/3}(0, T; H^{-1}(\Omega))$!

This is the key difference that makes proving uniqueness (and regularity) in 3D so hard.

The Euler equations

The Euler equations are the special case of the Navier–Stokes equations when the viscosity $\nu = 0$:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \boldsymbol{f}, \tag{4a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{4b}$$

The Euler equations

The Euler equations are the special case of the Navier–Stokes equations when the viscosity $\nu = 0$:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \boldsymbol{f}, \tag{4a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{4b}$$

Energy conservation

A smooth solution $\boldsymbol{u}$ of the Euler equations (4) satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_2^2 = (\boldsymbol{f}, \boldsymbol{u}).$$

The MHD equations for a conducting fluid in a domain $\Omega \subset \mathbb{R}^3$ can be written in the following form:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p_* = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} + \boldsymbol{f}, \tag{5a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} - \mu \Delta \boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}, \tag{5b}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{5c}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{5d}$$

Here $\boldsymbol{u}$, $\boldsymbol{B}$, and p are the velocity, magnetic, and pressure fields respectively, $p_* = p + \frac{1}{2}\left|\boldsymbol{B}\right|^2$, $\boldsymbol{f}$ is an external forcing, and ν and μ are the fluid viscosity and magnetic resistivity respectively.

Energy evolution law

A smooth solution $\boldsymbol{u}$, $\boldsymbol{B}$ of equations (5) satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}(t)\|_2^2 + \|\boldsymbol{B}(t)\|_2^2\right) = -\nu\|\nabla\boldsymbol{u}\|_2^2 - \mu\|\nabla\boldsymbol{B}\|_2^2 + (\boldsymbol{f}, \boldsymbol{u}).$$

Energy evolution law

A smooth solution $\boldsymbol{u}$, $\boldsymbol{B}$ of equations (5) satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}(t)\|_2^2 + \|\boldsymbol{B}(t)\|_2^2\right) = -\nu\|\nabla\boldsymbol{u}\|_2^2 - \mu\|\nabla\boldsymbol{B}\|_2^2 + (\boldsymbol{f}, \boldsymbol{u}).$$

- In two dimensions there is existence and uniqueness theory for the case where $\nu > 0$ and $\mu > 0$.
- Kozono (1989) proved global existence (but not uniqueness) of solutions when $\nu = 0$ (but $\mu > 0$).
- When $\mu = 0$, only existence results for short time or small initial data have been proved.