A generalised Ladyzhenskaya inequality and a coupled parabolic-elliptic problem

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A coupled parabolic-elliptic MHD system

We consider the following modified system of equations for magnetohydrodynamics on a bounded domain $\Omega \subset \mathbb{R}^2$:

$$
\begin{aligned}
-\Delta u + \nabla p &= (B \cdot \nabla) B \\
\partial_t B - \varepsilon \Delta B + (u \cdot \nabla) B &= (B \cdot \nabla) u,
\end{aligned}
$$

with $\nabla \cdot u = \nabla \cdot B = 0$ and Dirichlet boundary conditions. This is like the standard MHD system, but with the terms $\partial_t u + (u \cdot \nabla) u$ removed.

Theorem

Given $u_0, B_0 \in L^2(\Omega)$ with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$, for any $T > 0$ there exists a unique weak solution (u, B) with

$$u \in L^\infty(0, T; L^{2,\infty}) \cap L^2(0, T; H^1)$$

and

$$B \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1).$$

We prove this using both a generalisation of Ladyzhenskaya's inequality, and some elliptic regularity theory for L^1 forcing.

Weak solutions of the Navier–Stokes equations

Consider the Navier–Stokes equations on a domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3:

$$\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = 0,$$

with $\nabla \cdot u = 0$, and Dirichlet boundary conditions.

Theorem (Leray, 1934, and Hopf, 1951)

Given $u_0 \in L^2(\Omega)$ with $\nabla \cdot u_0 = 0$, there exists a weak solution u of the Navier–Stokes equations satisfying

$$u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1).$$

Moreover, if $n = 2$, this weak solution is unique.

The same is true if $\Omega = \mathbb{R}^n$, or if $\Omega = [0, 1]^n$ with periodic boundary conditions.

Weak solutions of NSE: existence

Let u^m be the mth Galerkin approximation: i.e., the solution of

$$\partial_t u^m + P^m[(u^m \cdot \nabla)u^m] - \Delta u^m + \nabla p^m = 0.$$

Taking the L^2 inner product with u^m, we get

$$\langle \partial_t u^m, u^m \rangle + \langle (u^m \cdot \nabla)u^m, u^m \rangle - \langle \Delta u^m, u^m \rangle + \underbrace{\langle \nabla p^m, u^m \rangle}_{=0} = 0.$$

Fact

If $\nabla \cdot u = 0$, and $u, v, w = 0$ on $\partial\Omega$, then

$$\langle (u \cdot \nabla)v, w \rangle = -\langle (u \cdot \nabla)w, v \rangle.$$

Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^m\|^2 + \|\nabla u^m\|^2 = 0,$$

so integrating in time yields

$$\|u^m(t)\|^2 + \int_0^t \|\nabla u^m(s)\|^2 \,\mathrm{d}s = \|u^m(0)\|^2 \le \|u_0\|^2.$$

Hence u^m are uniformly bounded in $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$.

Ladyzhenskaya's inequality

To get uniform bounds on $\partial_t u^m = \Delta u^m - P^m[(u^m \cdot \nabla)u^m]$, one uses:

Ladyzhenskaya's inequality in 2D (1958)

$$\|u\|_{L^4} \le c\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}.$$

Ladyzhenskaya's inequality yields a priori bounds on the nonlinear term $(u^m \cdot \nabla)u^m$:

$$\left|\int (u^m \cdot \nabla)u^m \cdot \phi\right| = \left|-\int (u^m \cdot \nabla)\phi \cdot u^m\right| \le \|u^m\|_{L^4}^2 \|\nabla\phi\|_{L^2},$$

so

$$\|(u^m \cdot \nabla)u^m\|_{H^{-1}} \le \|u^m\|_{L^4}^2 \le c\|u^m\|_{L^2}\|\nabla u\|_{L^2},$$

and thus $(u^m \cdot \nabla)u^m \in L^2(0,T;H^{-1})$, and hence $\partial_t u^m \in L^2(0,T;H^{-1})$.

Theorem (Aubin, 1963, and Lions, 1969)

If $u^m \in L^2(0,T;H^1)$ and $\partial_t u^m \in L^2(0,T;H^{-1})$ uniformly, then a subsequence $u^{m_k} \to u \in L^2(0,T;L^2)$ (strongly).

Magnetic relaxation (Moffatt, 1985)

Aim to construct stationary solutions of the Euler equations with non-trivial topology: $(u \cdot \nabla)u + \nabla p = 0$.
Consider the MHD equations with zero magnetic resistivity

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= (B \cdot \nabla)B \\ B_t + (u \cdot \nabla)B &= (B \cdot \nabla)u \end{aligned}$$

and assume that smooth solutions exist for all $t \geq 0$ (open even in 2D).
Energy equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2 + \|B\|^2\right) + \|\nabla u\|^2 = 0.$$

So $\|B\|$ decreases while $\|\nabla u\| \neq 0$.
Since the 'magnetic helicity' $\mathscr{H} = \int A \cdot B$ is preserved, where $B = \nabla \times A$ and $\nabla \cdot A = 0$, $\|B\|$ is bounded below:

$$c\|B\|^4 \geq \|B\|^2\|A\|^2 \geq \left(\int A \cdot B\right)^2 = \mathscr{H}^2.$$

"So" $u(t) \to 0$ as $t \to \infty$ (Nuñez, 2007) and $B(t) \to B$ with $\nabla p = (B \cdot \nabla)B$.

Magnetic relaxation (Moffatt, 2009)

The dynamics are arbitrary, so consider instead the 'simpler' model

$$-\Delta u + \nabla p = (B \cdot \nabla) B$$
$$B_t + (u \cdot \nabla) B = (B \cdot \nabla) u.$$

The new energy equation is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|B\|^2 + \|\nabla u\|^2 = 0.$$

"So" $\|\nabla u\|^2 \to 0$ " $\Longrightarrow$ " $u(t) \to 0$ and $B(t) \to B^*$ as $t \to \infty$.
Open: does $u(t) \to 0$ *as* $t \to \infty$ *in this case?*
To address existence of solutions we make two simplifications: we consider 2D, and regularise the B equation:

$$-\Delta u + \nabla p = (B \cdot \nabla) B$$
$$B_t - \varepsilon \Delta B + (u \cdot \nabla) B = (B \cdot \nabla) u.$$

A priori estimates

We consider the 2D system

$$-\Delta u + \nabla p = (B\cdot\nabla)B \qquad \nabla\cdot u = 0$$
$$B_t - \varepsilon\Delta B + (u\cdot\nabla)B = (B\cdot\nabla)u \qquad \nabla\cdot B = 0.$$

'Toy version' of 3D Navier–Stokes, which in vorticity form ($\omega = \nabla\times u$) is

$$\omega_t - \nu\Delta\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u.$$

Take inner product with u in the first equation, with B in the second equation

$$\|\nabla u\|^2 = \langle (B\cdot\nabla)B, u\rangle = -\langle (B\cdot\nabla)u, B\rangle$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|B\|^2 + \varepsilon\|\nabla B\|^2 = \langle (B\cdot\nabla)u, B\rangle$$

and add:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|B\|^2 + \varepsilon\|\nabla B\|^2 + \|\nabla u\|^2 = 0.$$

We get:

$$B\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1), \qquad u\in L^2(0,T;H^1).$$

A priori estimates

What can we say about u? We know that $B \in L^\infty(0,T;L^2)$.
Note that $[(u \cdot \nabla)v]_i = u_j \partial_j v_i = \partial_j(u_j v_i) =: \nabla \cdot (u \otimes v)$, since $\nabla \cdot v = 0$.
We can write the equation for u as

$$-\Delta u + \nabla p = (B \cdot \nabla)B = \nabla \cdot (\underbrace{B \otimes B}_{L^1}).$$

Elliptic regularity works for $p > 1$:

$$-\Delta u + \nabla p = f, \qquad f \in L^p \implies u \in W^{2,p}$$

(e.g. Temam, 1979).
If we could take $p = 1$ then we would expect, for RHS ∂f with $f \in L^1$, to get $u \in W^{1,1} \subset L^2$.
In fact for RHS ∂f with $f \in L^1$ we get $u \in L^{2,\infty}$, where $L^{2,\infty}$ is the weak-L^2 space.

$L^{p,\infty}$: weak L^p spaces

For $f: \mathbb{R}^n \to \mathbb{R}$ define

$$d_f(\alpha) = \mu\{x : |f(x)| > \alpha\}.$$

Note that

$$\|f\|_{L^p}^p = \int_{\mathbb{R}^n} |f(x)|^p \geq \int_{\{x:\ |f(x)|>\alpha\}} |f(x)|^p \geq \alpha^p d_f(\alpha).$$

For $1 \leq p < \infty$ set

$$\|f\|_{L^{p,\infty}} = \inf\left\{C :\ d_f(\alpha) \leq \frac{C^p}{\alpha^p}\right\} = \sup\{\gamma d_f(\gamma)^{1/p} :\ \gamma > 0\}.$$

The space $L^{p,\infty}(\mathbb{R}^n)$ consists of all those f such that $\|f\|_{L^{p,\infty}} < \infty$.

- $L^p \subset L^{p,\infty}$
- $|x|^{-n/p} \in L^{p,\infty}(\mathbb{R}^n)$ but $\notin L^p(\mathbb{R}^n)$.
- if $f \in L^{p,\infty}(\mathbb{R}^n)$ then $d_f(\alpha) \leq \|f\|_{L^{p,\infty}}^p \alpha^{-p}$.

$L^{p,\infty}$: weak L^p spaces

Just as with strong L^p spaces, we can interpolate between weak L^p spaces:

Weak L^p interpolation

Take $p < r < q$. If $f \in L^{p,\infty} \cap L^{q,\infty}$ then $f \in L^r$ and

$$\|f\|_{L^r} \leq c_{p,r,q} \|f\|_{L^{p,\infty}}^{p(q-r)/r(q-p)} \|f\|_{L^{q,\infty}}^{q(r-p)/r(q-p)}.$$

Recall Young's inequality for convolutions: if $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$ then

$$\|E * f\|_{L^p} \leq \|E\|_{L^q} \|f\|_{L^r}.$$

There is also a weak form, which requires stronger conditions on p, q, r:

Weak form of Young's inequality for convolutions

If $1 \leq r < \infty$ and $1 < p, q < \infty$, and $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$ then

$$\|E * f\|_{L^{p,\infty}} \leq \|E\|_{L^{q,\infty}} \|f\|_{L^r}.$$

Elliptic regularity in L^1

Fundamental solution of Stokes operator on $\mathbb{R}^2$ is

$$E_{ij}(x) = -\delta_{ij} \log |x| + \frac{x_i x_j}{|x|^2},$$

i.e. solution of $-\Delta u + \nabla p = f$ is $u = E * f$.
Solution of $-\Delta u + \nabla p = \partial f$ is $u = E * (\partial f) = (\partial E) * f$. Note that

$$\partial_k E_{ij} = \delta_{ij} \frac{x_k}{|x|^2} + \frac{\delta_{ik} x_j + \delta_{jk} x_i}{|x|^2} - \frac{x_i x_j x_k}{|x|^4} \sim \frac{1}{|x|}.$$

Thus $\partial E \in L^{2,\infty}$ and so

$$f \in L^1 \implies u = \partial E * f \in L^{2,\infty}.$$

If we consider the problem in a bounded domain we have the same regularity. We replace the fundamental solution E by the Dirichlet Green's function G satisfying

$$-\Delta G = \delta(x-y) \qquad G|_{\partial\Omega} = 0.$$

Mitrea & Mitrea (2011) showed that in this case we still have $\partial G \in L^{2,\infty}$.
So on our bounded domain, $u \in L^\infty(0, T; L^{2,\infty})$.

Estimates on time derivatives: $\partial_t B \in L^2(0,T;H^{-1})$?

Take $v \in H^1$ with $\|v\|_{H^1} = 1$. Then

$$\begin{aligned} |\langle \partial_t B, v\rangle| &= |\langle \varepsilon\Delta B - (u\cdot\nabla)B + (B\cdot\nabla)u, v\rangle| \\ &\le \varepsilon\|\nabla B\|\|\nabla v\| + 2\|u\|_{L^4}\|B\|_{L^4}\|\nabla v\|_{L^2}. \end{aligned}$$

so

$$\boxed{\|\partial_t B\|_{H^{-1}} \le \varepsilon\|\nabla B\| + 2\|u\|_{L^4}\|B\|_{L^4}.}$$

Standard 2D Ladyzhenskaya inequality gives

$$\|B\|_{L^4} \le c\|B\|^{1/2}\|\nabla B\|^{1/2};$$

but we only have uniform bounds on u in $L^{2,\infty}$.
If $\|f\|_{L^4} \le c\|f\|_{L^{2,\infty}}^{1/2}\|\nabla f\|^{1/2}$ then

$$\|\partial_t B\|_{H^{-1}} \le \varepsilon\|\nabla B\| + c\|u\|_{L^{2,\infty}}^{1/2}\|B\|^{1/2}\|\nabla u\|^{1/2}\|\nabla B\|^{1/2}$$

which would yield

$$\partial_t B \in L^2(0,T;H^{-1}).$$

Generalised Ladyzhenskaya inequality

In 2D,

$$\|f\|_{L^4}^2 \leq c\|f\|_{L^2}\|\nabla f\|_{L^2}.$$

Proof:
(i) write $f^2 = 2\int f\partial_j f\,\mathrm{d}x_j$ and integrate $(f^2)^2$.
(ii) use the Sobolev embedding $\dot{H}^{1/2} \subset L^4$ and interpolation in $\dot{H}^s$:

$$\|f\|_{L^4} \leq c\|f\|_{\dot{H}^{1/2}} \leq c\|f\|_{L^2}^{1/2}\|f\|_{\dot{H}^1}^{1/2}.$$

In fact, using the theory of interpolation spaces:

$$\|f\|_{L^4} \leq c\|f\|_{L^{2,\infty}}^{1/2}\|f\|_{\mathrm{BMO}}^{1/2}.$$

Since $\dot{H}^1 \subset \mathrm{BMO}$ in 2D, this yields

$$\boxed{\|f\|_{L^4} \leq c\|f\|_{L^{2,\infty}}^{1/2}\|f\|_{\dot{H}^1}^{1/2}.}$$

Besides the proof using interpolation spaces, we can also prove this directly using Fourier transforms.

Bounded mean oscillation

Let $f_Q := \frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x$ denote the average of f over a cube $Q \subset \mathbb{R}^n$. Define

$$\|f\|_{\mathrm{BMO}} := \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Let $\mathrm{BMO}(\mathbb{R}^n)$ denote the set of functions $f\colon \mathbb{R}^n \to \mathbb{R}$ for which $\|f\|_{\mathrm{BMO}} < \infty$.

- $\|f\|_{\mathrm{BMO}} = 0 \implies f$ is constant (almost everywhere).
- $L^\infty \subsetneq \mathrm{BMO}$ and $\|f\|_{\mathrm{BMO}} \le 2\|f\|_\infty$; $\log|x| \in \mathrm{BMO}$ but is unbounded.
- $\dot{H}^{n/2} \subset \mathrm{BMO}$ and $\|f\|_{\mathrm{BMO}} \le c\|f\|_{\dot{H}^{n/2}}$ in $\mathbb{R}^n$, even though $\dot{H}^{n/2} \not\subset L^\infty$.
- $W^{1,n} \subset \mathrm{BMO}$, by Poincaré's inequality: let B be a ball of radius r, then

$$\frac{1}{|B|}\int_B |f(x) - f_B|\,\mathrm{d}x \le \frac{cr}{r^n}\int_B |\mathrm{D}f|\,\mathrm{d}x \le c\left(\int_B |\mathrm{D}f|^n\,\mathrm{d}x\right)^{1/n} \le c\|f\|_{W^{1,n}},$$

  so $\|f\|_{\mathrm{BMO}} \le c\|f\|_{W^{1,n}}$.

Interpolation spaces

For $0 \leq \theta \leq 1$ one can define an interpolation space $X_\theta := [X^0, X^1]_\theta$ in such a way that $\|f\|_{X_\theta} \leq c\|f\|_{X^0}^{1-\theta}\|f\|_{X^1}^{\theta}$. (Note that $\|f\|_{X_1} \leq c\|f\|_{X^1}$.)

Theorem (Bennett & Sharpley, 1988)

$L^{p,\infty} = [L^1, \mathrm{BMO}]_{1-(1/p)}$ *for* $1 < p < \infty$*; so* $L^{2,\infty} = [L^1, \mathrm{BMO}]_{1/2}$.

Write $\mathfrak{B} = [L^1, \mathrm{BMO}]_1$ and note that $\|f\|_{\mathfrak{B}} \leq c\|f\|_{\mathrm{BMO}}$.

Reiteration Theorem

If $A_0 = [X_0, X_1]_{\theta_0}$ and $A_1 = [X_0, X_1]_{\theta_1}$ then for $0 < \theta < 1$

$$[A_0, A_1]_\theta = [X_0, X_1]_{(1-\theta)\theta_0 + \theta\theta_1}.$$

So $L^{3,\infty} = [L^{2,\infty}, \mathfrak{B}]_{1/3}$ and $L^{6,\infty} = [L^{2,\infty}, \mathfrak{B}]_{2/3}$, and hence

$$\begin{aligned}
\|f\|_{L^4} &\leq c\|f\|_{L^{3,\infty}}^{1/2}\|f\|_{L^{6,\infty}}^{1/2} \\
&\leq c[c\|f\|_{L^{2,\infty}}^{2/3}\|f\|_{\mathfrak{B}}^{1/3}]^{1/2}[c\|f\|_{L^{2,\infty}}^{1/3}\|f\|_{\mathfrak{B}}^{2/3}]^{1/2} \\
&= c\|f\|_{L^{2,\infty}}^{1/2}\|f\|_{\mathfrak{B}}^{1/2} \leq c\|f\|_{L^{2,\infty}}^{1/2}\|f\|_{\mathrm{BMO}}^{1/2}.
\end{aligned}$$

Generalised Ladyzhenskaya inequality: direct method

There exists a Schwartz function ϕ such that if $\hat{f}$ is supported in $B(0,R)$,

$$f = \phi^{1/R} * f, \qquad \text{where } \phi^{1/R}(x) = R^n \phi(Rx).$$

(Take ϕ with $\hat{\phi} = 1$ on $B(0,1)$; then $\mathscr{F}[\phi^{1/R} * f] = \mathscr{F}[\phi^{1/R}]\hat{f} = \hat{f}$.)

Lemma (Weak-strong Bernstein inequality)

Suppose that $\operatorname{supp}(\hat{f}) \subset B(0,R)$. *Then for* $1 \leq p < q < \infty$

$$\|f\|_{L^q} \leq cR^{n(1/p-1/q)} \|f\|_{L^{p,\infty}}.$$

Using the weak form of Young's inequality, for $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$,

$$\|f\|_{L^{q,\infty}} = \|\phi^{1/R} * f\|_{L^{q,\infty}} \leq c\|\phi^{1/R}\|_{L^r} \|f\|_{L^{p,\infty}},$$

and noting that $\|\phi^{1/R}\|_{L^r} = cR^{n(1-1/r)}$, it follows that

$$\|f\|_{L^{1,\infty}} \leq cR^{n(1/p-1)} \|f\|_{L^{p,\infty}} \quad \text{and} \quad \|f\|_{L^{2q,\infty}} \leq cR^{n(1/p-1/2q)} \|f\|_{L^{p,\infty}}.$$

Finally interpolate L^q between $L^{1,\infty}$ and $L^{2q,\infty}$.

Generalised Ladyzhenskaya inequality: direct method

To show

$$\|f\|_{L^4} \leq c\|f\|_{L^{2,\infty}}^{1/2}\|f\|_{\dot{H}^1}^{1/2}$$

write

$$f(x) = \underbrace{\int_{|k|\leq R} \hat{f}(k)e^{2\pi ik\cdot x}\,\mathrm{d}k}_{f_-} + \underbrace{\int_{|k|\geq R} \hat{f}(k)e^{2\pi ik\cdot x}\,\mathrm{d}k}_{f_+}.$$

Now using the weak-strong Bernstein inequality

$$\|f_-\|_{L^4} \leq cR^{1/2}\|f\|_{L^{2,\infty}},$$

and using the embedding $\dot{H}^{1/2} \subset L^4$,

$$\begin{aligned}\|f_+\|_{L^4}^2 \leq c\|f_+\|_{\dot{H}^{1/2}}^2 &= c\int_{|k|\geq R} |k||\hat{f}(k)|^2\,\mathrm{d}k \\ &\leq \frac{c}{R}\int_{|k|\geq R} |k|^2|\hat{f}(k)|^2\,\mathrm{d}k \\ &\leq \frac{c}{R}\|f\|_{\dot{H}^1}^2.\end{aligned}$$

Thus

$$\|f\|_{L^4} \leq cR^{1/2}\|f\|_{L^{2,\infty}} + cR^{-1/2}\|f\|_{\dot{H}^1},$$

and choosing $R = \|f\|_{\dot{H}^1}/\|f\|_{L^{2,\infty}}$ yields the inequality.

Generalised Gagliardo–Nirenberg inequalities

It is not hard to generalise this direct proof to prove the following:

Theorem

Take $1 \leq q < p < \infty$ *and* $s > n\left(\frac{1}{2} - \frac{1}{p}\right)$. *There exists a constant* $c_{p,q,s}$ *such that if* $f \in L^{q,\infty} \cap \dot{H}^s$ *then* $f \in L^p$ *and*

$$\|f\|_{L^p} \leq c_{p,q,s} \|f\|_{L^{q,\infty}}^{\theta} \|f\|_{\dot{H}^s}^{1-\theta}$$

for every $f \in L^{q,\infty} \cap \dot{H}^s$, *where* $\frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{s}{n}\right)$.

With a little work, in the case $s = n/2$ we can generalise this:

Theorem

Take $1 \leq q < p < \infty$. *There exists a constant* $c_{p,q}$ *such that if* $f \in L^{q,\infty} \cap \mathrm{BMO}$ *then* $f \in L^p$ *and*

$$\|f\|_{L^p} \leq c_{p,q} \|f\|_{L^{q,\infty}}^{q/p} \|f\|_{\mathrm{BMO}}^{1-q/p}$$

for every $f \in L^{q,\infty} \cap \mathrm{BMO}$.

Global existence of weak solutions

Take $B^m(0) = P^m B(0)$ and consider the Galerkin approximations:

$$\begin{aligned} -\Delta u^m + \nabla p &= P^m (B^m \cdot \nabla) B^m \\ \partial_t B^m - \varepsilon \Delta B^m + P^m (u^m \cdot \nabla) B^m &= P^m (B^m \cdot \nabla) u^m. \end{aligned}$$

The B^m equation is a Lipschitz ODE on a finite-dimensional space, so it has a unique solution. By repeating the *a priori* estimates on these smooth equations (now rigorous) we can obtain estimates uniform in n:

$$B^m \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \quad \partial_t B^m \in L^2(0,T;H^{-1})$$

and

$$u^m \in L^\infty(0,T;L^{2,\infty}) \cap L^2(0,T;H^1).$$

By Aubin–Lions, a subsequence of the Galerkin approximations $B^m \to B$ strongly in $L^2(0,T;L^2)$. Hence, by elliptic regularity,

$$\|u^m - u\|_{L^{2,\infty}} \le \|B^m \otimes B^m - B \otimes B\|_{L^1} \le \|B^m - B\|_{L^2}(\|B^m\|_{L^2} + \|B\|_{L^2}),$$

hence $u^m \to u$ strongly in $L^2(0,T;L^{2,\infty})$. This is enough to show the nonlinear terms converge weak-$*$ in $L^2(0,T;H^{-1})$, and hence that (u,B) is a weak solution (i.e. a solution with equality in $L^2(0,T;H^{-1})$).

Conclusion

Similar arguments to the a priori estimates show uniqueness of weak solutions, and so:

Theorem

Given $u_0, B_0 \in L^2(\Omega)$ with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$, for any $T > 0$ there exists a unique weak solution (u, B) with

$$u \in L^\infty(0, T; L^{2,\infty}) \cap L^2(0, T; H^1)$$

and

$$B \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1).$$

What about $\varepsilon = 0$?

- Try looking at more regular solutions and taking the limit $\varepsilon \to 0$ to get local existence
- Assume regularity and show that $u(t) \to 0$ as $t \to \infty$ (Moffatt)?

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