Stationary Euler flows and magnetohydrodynamics

by

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Contents

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Finally, I wish to dedicate this dissertation to the memory of my grandfather,

Thomas Jackson McCormick (1921–2010),

who would have been proud.

Declarations

Chapters 1 and 2 of this dissertation are introductory; sections 3.1, 4.1 and 4.2 expound on the work of Núñez (2007); sections 3.2 and 4.3 present original adaptations of the preceding sections to a simplified model.

Except as noted above, I declare that, to the best of my knowledge, the material contained in this dissertation is original and my own work except where otherwise indicated, cited, or commonly known.

The material in this dissertation is submitted to the University of Warwick for the degree of Master of Science, and has not been submitted to any other university or for any other degree.

Abstract

In this dissertation, we study stationary solutions of the Euler equations through the approach of magnetic relaxation, where a viscous, non-resistive, incompressible fluid is evolved under the laws of magnetohydrodynamics to a state of equilibrium, in which the limiting magnetic field ought to satisfy the stationary Euler equations.

After introducing the "analogy" — first proposed by Moffatt (1985) — between stationary Euler flows and magnetostatics, we expound the rigorous analysis of Núñez (2007) in showing that, while the kinetic energy decays to zero, the magnetic field may fail to have a weak limit, unless an additional condition on the velocity holds.

We also introduce a reduced model, suggested by Moffatt (2009), in which some of the terms in the equations are neglected, and present an original approach to prove that the kinetic energy also decays to zero for this model; however, we also show that the magnetic field may still fail to have a weak limit.

Chapter 1

Introduction

The Navier–Stokes equations are the fundamental mathematical model for fluid flow: they describe the velocity field and pressure of a moving fluid. At first glance, they seem to be a simple application of Newton's second law of motion, but their key feature is a quadratic nonlinear term that describes the convective acceleration of the fluid. The sheer variety of phenomena described by the Navier–Stokes equations — from laminar flow to chaotic turbulence — is due fundamentally to the nonlinear term.

The nonlinear term together with the requirement of incompressibility have confounded all attempts at proving existence, uniqueness and regularity for solutions of the Navier–Stokes equations in three (space) dimensions. One need only look as far as the seminal books of Constantin & Foias (1988) and Temam (2001) to see the wealth of research in recent years into various aspects of the behaviour of the Navier–Stokes equations. While partial results — such as existence of weak solutions, and uniqueness of strong solutions — are known, the Clay Mathematics Institute declared conclusively proving existence and uniqueness of solutions of the three-dimensional Navier–Stokes equations as one of their Millennium Prize Problems, with a $1 million prize for a successful proof (or counterexample!).

The Euler equations are a special case of the Navier–Stokes equations when the viscosity $\nu = 0$; they thus describe inviscid flow, an idealised case where the fluid has no viscosity. One sees from study of the Navier–Stokes equations that the lower the viscosity, the greater the turbulence. As such, using the Euler equations to study fluids with *zero* viscosity represents an altogether more difficult task. In contrast to the Navier–Stokes equations, no global existence or uniqueness results are known for the Euler equations in three dimensions, though much work has been done on establishing criteria which guarantee either existence for all time, or the formation of a singularity in finite time.

A key method in studying the formation of singularities, or lack thereof, is the consideration of stationary solutions of the Euler equations, ones which do not depend on time, and are therefore candidates for the limiting behaviour of the Euler equations as time $t \to \infty$. This dissertation focusses on one particular approach to stationary solutions of the Euler equations, namely *magnetic relaxation*.

In a classical paper, Moffatt (1985) describes the laws of magnetohydrodynamics as applied to a perfectly conducting, viscous, incompressible fluid, and reasons that the magnetic field should evolve to a static state as $t \to \infty$, and the limiting magnetic field should satisfy the Euler equations. Moffatt's approach to magnetic relaxation is fascinating, and combines intuition with well-chosen examples to illustrate the mechanisms involved. However, some of his statements, while extremely plausible, lack rigorous proof.

More recently, Núñez (2007) has shown that, while the limit of the magnetohydrodynamics equations is static (in some sense), the magnetic field may in fact have *no* weak limit as $t \to \infty$. This does require the presence of an artificial forcing term; nonetheless, it is a stark reminder that the process of magnetic relaxation is not as simple as it first seems.

In this dissertation, we begin by introducing the fundamental models of fluid mechanics, the Navier–Stokes and the Euler equations; in particular, we explain in section 1.3 the significance of topology to the formation of singularities. In chapter 2, we explain the fundamental ideas behind magnetic relaxation; then, in chapters 3 and 4, we expound the work of Núñez (2007), and we also consider a simplified model of magnetic relaxation which turns out to be mathematically more difficult. We use some of the ideas of Núñez to justify the use of the simplified model, but we also show that the same kind of problems can arise and the magnetic field can again fail to have a weak limit.

At the outset, let us fix our (fairly standard) notation: given a function $\boldsymbol{u}\colon \Omega \to \mathbb{R}^n$, defined on some domain $\Omega \subset \mathbb{R}^m$, we define the L^p norm by

$$\|\boldsymbol{u}\|_p := \left(\int_\Omega |\boldsymbol{u}(x)|^p \,\mathrm{d}x \right)^{1/p},$$

and the L^∞ norm by

$$\|\boldsymbol{u}\|_\infty := \operatorname*{ess\,sup}_{x \in \Omega} |\boldsymbol{u}(x)|,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. The square of the L^2 norm, $\|\boldsymbol{u}\|_2^2$, is often called the *energy* of $\boldsymbol{u}$. Furthermore, we denote the L^2 inner product of two functions $\boldsymbol{u}, \boldsymbol{v} \colon \Omega \to \mathbb{R}^n$ by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \int_\Omega \boldsymbol{u}(x) \cdot \boldsymbol{v}(x) \, \mathrm{d}x,$$

where $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^n$.

1.1 The Navier–Stokes equations

Given a smooth bounded domain $\Omega \subset \mathbb{R}^n$ (for $n = 2$ or 3), the Navier–Stokes equations for Ω are:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = 0 \tag{1.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1.1b}$$

Here, $\boldsymbol{u} \colon \Omega \times [0, \infty) \to \mathbb{R}^n$ is the (time-dependent) fluid velocity field, $p \colon \Omega \times [0, \infty) \to \mathbb{R}$ is the (time-dependent) fluid pressure, and ν is the fluid viscosity. For ease of presentation, we will assume throughout that the fluid has unit density.

One can require a variety of different boundary conditions for the Navier–Stokes equations. Often, if using techniques from harmonic analysis, the equations are considered on the whole of $\mathbb{R}^n$; alternatively, if one wants a compact setting in which to use Fourier series, one can use periodic boundary conditions (which, as Temam (1995) remarked, have "no physical meaning"): one requires that, for some $L > 0$, $\Omega = [0, L]^n$ and $\boldsymbol{u}(x + Le_j, t) = \boldsymbol{u}(x, t)$ for all $(x, t) \in \Omega \times [0, \infty)$, with $j = 1, \ldots, n$.

In this dissertation, we choose to keep things more general, and not impose specific boundary conditions, requiring only that the following boundary integrals vanish:

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \, \mathrm{d}S = 0, \tag{1.2a}$$

$$\int_{\partial\Omega} |\boldsymbol{u}|^2 \, (\boldsymbol{u} \cdot \boldsymbol{n}) \, \mathrm{d}S = 0, \tag{1.2b}$$

$$\int_{\partial\Omega} p(\boldsymbol{u} \cdot \boldsymbol{n}) \, \mathrm{d}S = 0. \tag{1.2c}$$

Here, $\boldsymbol{n}$ is the outward unit normal vector to the boundary $\partial\Omega$. These hold with periodic boundary conditions, but they also have the virtue of holding for Dirichlet boundary conditions, and a number of other choices of boundary conditions. (Most of the time it is not the specific boundary conditions that matter, but rather our ability to apply various integral theorems, such as integration by parts or the divergence theorem; ensuring that these boundary integrals vanish will suffice for these purposes.)

One fundamental property of the Navier–Stokes equations is that they dissipate energy. In this proposition, as in the rest of this dissertation, we will assume that a smooth solution of the Navier–Stokes equations exists for all time; as a result, much of the analysis we do will be formal.

Proposition 1.1 (Energy evolution law)**.** *A smooth solution* $\boldsymbol{u}$ *of the Navier–Stokes equations* (1.1)*, subject to boundary conditions* (1.2)*, satisfies*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_2^2 = -\nu\|\nabla\boldsymbol{u}\|_2^2$$

First, a convenient lemma:

Lemma 1.2. *Let* $\Omega \subset \mathbb{R}^n$*, and let* $\boldsymbol{v}, \boldsymbol{w} \colon \Omega \to \mathbb{R}^n$ *be vector fields such that* $\nabla \cdot \boldsymbol{v} = 0$ *everywhere in* Ω*, and such that*

$$\int_{\partial\Omega} |\boldsymbol{w}|^2 \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S = 0.$$

Then $\langle(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}, \boldsymbol{w}\rangle = 0$*.*

Proof. First, observe that

$$\langle(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}, \boldsymbol{w}\rangle = \sum_{i,j=1}^{n} \int_\Omega v_j \frac{\partial w_i}{\partial x_j} w_i \, \mathrm{d}x = \int_\Omega \boldsymbol{v} \cdot \nabla \left(\tfrac{1}{2}|\boldsymbol{w}|^2\right) \, \mathrm{d}x.$$

Since $\boldsymbol{v}$ is divergence-free, we see that

$$\nabla \cdot \left(\tfrac{1}{2}|\boldsymbol{w}|^2\boldsymbol{v}\right) = \tfrac{1}{2}|\boldsymbol{w}|^2 \nabla\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\left(\tfrac{1}{2}|\boldsymbol{w}|^2\right) = \boldsymbol{v}\cdot\nabla\left(\tfrac{1}{2}|\boldsymbol{w}|^2\right).$$

Hence, using the divergence theorem shows that

$$\langle(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}, \boldsymbol{w}\rangle = \int_\Omega \nabla\cdot\left(\tfrac{1}{2}|\boldsymbol{w}|^2\boldsymbol{v}\right) \mathrm{d}x = \int_{\partial\Omega} \tfrac{1}{2}|\boldsymbol{w}|^2 \boldsymbol{v}\cdot\boldsymbol{n} \, \mathrm{d}S = 0. \qquad \square$$

From this, proposition 1.1 follows easily:

Proof of proposition 1.1. Taking the inner product of (1.1a) with $\boldsymbol{u}$ yields

$$\left\langle \frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{u} \right\rangle + \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{u}\rangle - \nu\langle\Delta\boldsymbol{u}, \boldsymbol{u}\rangle + \langle\nabla p, \boldsymbol{u}\rangle = 0.$$

Taking each term in turn, we see first that $\left\langle \frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{u} \right\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_2^2$. The boundary conditions mean that $\langle (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{u}\rangle = 0$ (by lemma 1.2) and that $-\langle \Delta\boldsymbol{u}, \boldsymbol{u}\rangle = \|\nabla\boldsymbol{u}\|_2^2$ (after an integration by parts). Furthermore, since $\boldsymbol{u}$ is divergence free, we see that

$$\nabla \cdot (p_* \boldsymbol{u}) = p_* \nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla p_*,$$

so that

$$\langle \nabla p_*, \boldsymbol{u} \rangle = \int_\Omega \nabla \cdot (p_* \boldsymbol{u}) \,\mathrm{d}x = \int_{\partial\Omega} p_* \boldsymbol{u} \cdot \boldsymbol{n} \,\mathrm{d}S = 0,$$

thanks to the boundary conditions. Hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_2^2 = -\nu\|\nabla\boldsymbol{u}\|_2^2. \qquad \square$$

1.2 The Euler equations

We turn now to the main focus of this dissertation: the Euler equations. These are the special case of the Navier–Stokes equations when the viscosity $\nu = 0$:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0 \tag{1.3a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1.3b}$$

It is often convenient to reformulate the Euler equations using the *vorticity*: we define the vorticity $\boldsymbol{\omega} := \nabla \times \boldsymbol{u}$, whereupon the equations become

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} \tag{1.4a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1.4b}$$

Note that this eliminates the pressure from the equations (which, among other things, often makes numerical simulations easier).

One of the most obvious contrasts is that, while the Navier–Stokes equations dissipate energy, under the Euler equations energy is conserved. We saw in proposition 1.1 that, for the Navier–Stokes equations, the $\nu\Delta\boldsymbol{u}$ term dissipates energy: as long as $\boldsymbol{u} \neq 0$, the energy of $\boldsymbol{u}$ will decrease. Without this term, however, the Euler equations *conserve* energy:

Proposition 1.3 (Energy conservation)**.** *A smooth solution* $\boldsymbol{u}$ *of the Euler equations*

(1.3)*, subject to boundary conditions* (1.2)*, satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_2^2 = 0.$$

This result is proved in exactly the same manner as proposition 1.1.

Another invariant of the Euler equations is the *helicity.* Given a velocity field $\boldsymbol{u}$, we define its helicity by

$$\mathscr{H}(\boldsymbol{u}) := \langle \boldsymbol{u}, \boldsymbol{\omega} \rangle = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\omega} \, \mathrm{d}x.$$

It was first shown by Moffatt (1969) that the helicity is also conserved by Euler flows: it measures how knotted, or tangled, vortex lines in the fluid are. Arnold & Khesin (1998), chapter III, contains a proof using differential forms, and explains in detail the properties of helicity and a number of other topological invariants. Here, however, we present a formal proof, assuming that we have a smooth solution, and that the vorticity satisfies various boundary conditions:

Proposition 1.4 (Helicity conservation)**.** *Suppose that, in addition to boundary conditions* (1.2)*, a smooth solution* $\boldsymbol{u}$ *of* (1.3) *satisfies*

$$\int_{\partial\Omega} |\boldsymbol{u}|^2 \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}S = 0,$$
$$\int_{\partial\Omega} p\boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}S = 0.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(\boldsymbol{u}) = 0.$$

Proof. By differentiating under the integral sign, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(\boldsymbol{u}) = \left\langle \frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{\omega} \right\rangle + \left\langle \boldsymbol{u}, \frac{\partial \boldsymbol{\omega}}{\partial t} \right\rangle.$$

Taking the inner product of equation (1.3a) with $\boldsymbol{\omega}$, we obtain

$$\left\langle \frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{\omega} \right\rangle + \langle (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{\omega} \rangle + \langle \nabla p, \boldsymbol{\omega} \rangle = 0.$$

Since $\boldsymbol{\omega}$ is divergence free, we see that

$$\langle \nabla p, \boldsymbol{\omega} \rangle = \int_\Omega \nabla \cdot (p\boldsymbol{\omega}) \, \mathrm{d}x = \int_{\partial\Omega} p\boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}S = 0$$

by the boundary conditions. Similarly, taking the inner product of equation (1.4a) with $\boldsymbol{u}$, we obtain

$$\left\langle \frac{\partial \boldsymbol{\omega}}{\partial t}, \boldsymbol{u} \right\rangle + \langle (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}, \boldsymbol{u} \rangle = \langle (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}, \boldsymbol{u} \rangle = 0$$

by lemma 1.2. An integration by parts shows that $\langle (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{\omega} \rangle = -\langle (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}, \boldsymbol{u} \rangle$, so adding the two equations yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(\boldsymbol{u}) = \left\langle \frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{\omega} \right\rangle + \left\langle \boldsymbol{u}, \frac{\partial \boldsymbol{\omega}}{\partial t} \right\rangle = 0. \qquad \square$$

Entire industries are devoted to the study of the properties of solutions of the Euler equations, and of the Navier–Stokes equations, and much more can be said about them; we refer the interested reader to Acheson (1990), an elementary introduction to the physical properties of fluids, and Batchelor et al. (2000), a "collective introduction to current research" in a broad spectrum of fluid dynamics, ranging from blood flow to lava flow.

1.3 Formation of singularities

The question of whether the incompressible 3D Euler equations develop singularities – specifically, singularities in the vorticity field $\boldsymbol{\omega}$ — in finite time remains one of the most important open questions in mathematical fluid dynamics. That the vorticity grows rapidly under certain initial conditions is not in doubt, but whether that growth is sufficient to develop singularities, or is instead merely exponential, or perhaps double exponential, is not yet known.

The question is connected intimately with global existence of solutions: if we were assured that the solution existed for all time, then we would know that no singularities were present; conversely, the presence of singularities precludes the continuation of a solution past the time at which the singularity forms. A vast quantity of numerical simulations have been performed in order to gather evidence (either in favour of or against the existence of singularities), many of which are summarised in a survey paper of Gibbon (2008).

Theoretically speaking, the most important result on existence of solutions for the 3D Euler equations is the Beale–Kato–Majda theorem, which gives a condition for global existence of a solution in terms of the L^∞ norm of the vorticity:

Theorem 1.5 (Beale, Kato & Majda (1984))**.** *There exists a global solution of the 3D Euler equations* $\boldsymbol{u} \in C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-1})$ *for* $s \geq 3$ *if, for every*

$T > 0$,

$$\int_0^T \|\boldsymbol{\omega}(\tau)\|_\infty \,\mathrm{d}\tau < \infty.$$

One way to make further progress in understanding the behaviour of the 3D Euler equations is to focus on particular types of singularities. Córdoba, Fefferman & de la Llave (2004) defined a "squirt" singularity as one where some portion of material is ejected from a set of positive measure. They show that if a volume-preserving vector field $\boldsymbol{u}$ has a squirt singularity at time T, then

$$\int_0^T \|\boldsymbol{u}(t)\|_\infty \,\mathrm{d}t = \infty.$$

They then proceed to show that if $\boldsymbol{u}$ satisfies the 2D or 3D Navier–Stokes equations (or the 2D or 3D Boussinesq equations with positive viscosity), then

$$\int_0^T \|\boldsymbol{u}(t)\|_\infty \,\mathrm{d}t < \infty.$$

Therefore, solutions of those equations cannot experience squirt singularities. Their proof does not extend to the Euler equations, though earlier work (Córdoba & Fefferman 2001, 2002) in two dimensions did apply to the Euler equations.

Nonetheless, squirt singularities are an interesting example of how the topology of a fluid can evolve in time, and illustrates the fact that studying the topological aspects of fluid dynamics is worthwhile in and of itself. The area of "topological fluid mechanics" is relatively young, and its state is summarised in Khesin (2005).

A more traditional approach to understanding the formation of singularities is by considering the long-time behaviour, through *stationary solutions* of the Euler equations, i.e. solutions of the Euler equations which do not depend on time. For such a solution $\boldsymbol{u}$, we necessarily have $\frac{\partial \boldsymbol{u}}{\partial t} = 0$, and hence stationary solutions of the unforced Euler equations satisfy

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0 \tag{1.5a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1.5b}$$

From a dynamical systems viewpoint, we can think of stationary solutions of the Euler equations as being fixed points in the function space in which solutions of the Euler equations evolve. Even if these fixed points turn out to be unstable, their location in the function space may go some way to explaining the structure of turbulence in fluid flow.

Chapter 2

Magnetic relaxation

In this chapter, we introduce the technique of *magnetic relaxation*, where the equations of magnetohydrodynamics — which describe the movement of a magnetically or electrically conducting fluid — are exploited in order to study stationary solutions of the Euler equations. In essence, in the absence of forcing a magnetohydrodynamical system should settle down to equilibrium, in which the fluid is not moving, and the magnetic field is in some static (but possibly non-zero) state. It turns out that the limiting magnetic field ought to satisfy the stationary Euler equations, and thus there is an "analogy" between magnetostatics and stationary Euler flows, first introduced by Moffatt (1985).

2.1 Magnetohydrodynamics

Magnetohydrodynamics, sometimes abbreviated as MHD, concerns the study of a fluid which has an associated magnetic field: the equations governing MHD are derived from a combination of Newton's second law and Maxwell's equations. Magnetohydrodynamics is a flourishing subject in its own right, and we do not have space here to go into detail; chapter 7 of Batchelor et al. (2000) provides a useful précis.

In a seminal paper, Moffatt (1985) introduces an "analogy" between the steady Euler equations and magnetostatics (which he attributes to Arnol′d (1974)). Under the laws of magnetohydrodynamics, a viscous, perfectly conducting, incompressible fluid with a magnetic field will evolve under the following equations:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu \Delta \boldsymbol{v} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} + \boldsymbol{f}, \tag{2.1a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{2.1b}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{2.1c}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{2.1d}$$

Here, $\boldsymbol{v}$ is the fluid velocity, p is the fluid pressure, ν is the fluid viscosity, $\boldsymbol{B}$ is the magnetic field, and $\boldsymbol{J} = \nabla \times \boldsymbol{B}$ is the current density. Once again, the fluid has been assumed to have unit density (with minor loss of generality). We include here a forcing term $\boldsymbol{f}$: this is not strictly required according to the laws of magnetohydrodynamics, and we will usually set $\boldsymbol{f} \equiv 0$; nonetheless, there will be occasions on which including the forcing $\boldsymbol{f}$ will be mathematically convenient.

We see that the fluid velocity $\boldsymbol{v}$ is evolving under the Navier–Stokes equations, with a forcing term $\boldsymbol{J} \times \boldsymbol{B}$: in essence, the magnetic effects cause the fluid to move. Since the magnetic field $\boldsymbol{B}$ also depends on the fluid velocity $\boldsymbol{v}$, the two equations are strongly coupled. We consider these equations on a smooth bounded domain Ω, which will usually be a subset of $\mathbb{R}^3$; however, we will see that some of the analysis will also apply in two dimensions.

In order to simplify the equations, we use some standard vector calculus identities: first, observe that, since $\boldsymbol{v}$ and $\boldsymbol{B}$ are divergence-free,

$$\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B},$$

and also that

$$\boldsymbol{J} \times \boldsymbol{B} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \tfrac{1}{2}\nabla(|\boldsymbol{B}|^2).$$

So, setting $p_* = p + \frac{1}{2}|\boldsymbol{B}|^2$ to be the total pressure, we obtain the following equivalent form of the equations:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu \Delta \boldsymbol{v} + \nabla p_* = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} + \boldsymbol{f}, \tag{2.2a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v}, \tag{2.2b}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{2.2c}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.2d}$$

Much as with the Navier–Stokes equations, there is no proof of global existence of smooth solutions to the MHD equations. Therefore, in order to perform any rigorous analysis on equations (2.2), we must make certain assumptions; we follow here the paper of Núñez (2007). We assume that there exists a smooth solution $(\boldsymbol{v}, \boldsymbol{B})$ for all time (since otherwise it makes no sense to study the limit as $t \to \infty$). We also make the physically reasonable assumption that the magnetic field remains bounded in the supremum norm for all time: that is, $\|\boldsymbol{B}(t)\|_\infty \leq M$ for all $0 < t < \infty$.

As with the Navier–Stokes equations, we do not require particular boundary conditions, instead requiring that the following boundary integrals all vanish:

$$\int_{\partial\Omega} \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \, \mathrm{d}S = 0, \tag{2.3a}$$

$$\int_{\partial\Omega} (\boldsymbol{B} \cdot \boldsymbol{v})(\boldsymbol{B} \cdot \boldsymbol{n}) \, \mathrm{d}S = 0, \tag{2.3b}$$

$$\int_{\partial\Omega} |\boldsymbol{v}|^2 \, (\boldsymbol{v} \cdot \boldsymbol{n}) \, \mathrm{d}S = 0, \tag{2.3c}$$

$$\int_{\partial\Omega} |\boldsymbol{B}|^2 \, (\boldsymbol{v} \cdot \boldsymbol{n}) \, \mathrm{d}S = 0, \tag{2.3d}$$

$$\int_{\partial\Omega} p(\boldsymbol{v} \cdot \boldsymbol{n}) \, \mathrm{d}S = 0. \tag{2.3e}$$

Again, $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$. This holds both with Dirichlet boundary conditions (i.e., $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega$) or with periodic boundary conditions. Finally, we assume that a Poincaré inequality holds: that is, there exists a constant c_p such that

$$\|\boldsymbol{u}\|_2 \leq c_p \|\nabla \boldsymbol{u}\|_2;$$

this holds, for example, whenever the functions have zero integral in Ω, or when $\boldsymbol{u} \cdot \boldsymbol{n} = 0$.

2.2 Motivation: the stationary equations

The fundamental idea behind the analogy of Moffatt (1985) is that, in the limit as $t \to \infty$, the magnetic forces on the fluid should come to equilibrium, so that the fluid velocity $\boldsymbol{v} \to 0$, and we are left with a steady magnetic field $\boldsymbol{B}$ which satisfies the stationary Euler equations.

If we have a stationary solution of equations (2.2) — that is, one that does not depend on time — then, formally, $\frac{\partial \boldsymbol{v}}{\partial t}$ and $\frac{\partial \boldsymbol{B}}{\partial t}$ should be identically zero, and we should be left with the following equations:

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \boldsymbol{f} \tag{2.4a}$$

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{v} \tag{2.4b}$$

$$\nabla\cdot\boldsymbol{v} = 0 \tag{2.4c}$$

$$\nabla\cdot\boldsymbol{B} = 0 \tag{2.4d}$$

where $p_* = p + \frac{1}{2}|\boldsymbol{B}|^2$ is the total pressure, and $\boldsymbol{f} \in L^2(\Omega)$.

If the magnetised fluid is unforced — that is, if $\boldsymbol{f} = 0$ — then we expect that the fluid should settle down to an equilibrium state, and thus that $\boldsymbol{v} = 0$ in the limit as $t \to \infty$. In this case, equations (2.4) reduce to the following equations for $\boldsymbol{B}$:

$$(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \nabla p_* = 0, \tag{2.5a}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{2.5b}$$

which (up to a change of sign for the pressure) are the stationary unforced Euler equations (1.5). Therefore, magnetic fields which are stationary solutions of the MHD equations (2.2) ought (formally) to solve the stationary Euler equations (1.5).

In fact, it is not too hard to see that any solution of equations (2.4) must have $\boldsymbol{v} = 0$: the following formal argument is based on more rigorous work by Núñez (2007) (indeed, we present that argument later in proposition 3.1). Let us assume that we have a smooth solution of equations (2.4): if we take the inner product of (2.4a) with $\boldsymbol{v}$ and the inner product of (2.4b) with $\boldsymbol{B}$, we obtain

$$\begin{aligned} \langle(\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{v}\rangle - \nu\langle\Delta\boldsymbol{v},\boldsymbol{v}\rangle + \langle\nabla p_*,\boldsymbol{v}\rangle &= \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{B},\boldsymbol{v}\rangle + \langle\boldsymbol{f},\boldsymbol{v}\rangle, \\ \langle(\boldsymbol{v}\cdot\nabla)\boldsymbol{B},\boldsymbol{B}\rangle &= \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{v},\boldsymbol{B}\rangle. \end{aligned}$$

Under boundary conditions (2.3), lemma 1.2 tells us that $\langle(\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{v}\rangle = 0$ and $\langle(\boldsymbol{v}\cdot\nabla)\boldsymbol{B},\boldsymbol{B}\rangle = 0$, since $\boldsymbol{v}$ and $\boldsymbol{B}$ are divergence-free. As in the proof of proposition 1.1, the boundary conditions tell us that

$$\langle\nabla p,\boldsymbol{v}\rangle = \int_{\partial\Omega} p\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S = 0.$$

An integration by parts shows that $-\langle\Delta\boldsymbol{v},\boldsymbol{v}\rangle = \|\nabla\boldsymbol{v}\|_2^2$, and that $\langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{v},\boldsymbol{B}\rangle = -\langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{B},\boldsymbol{v}\rangle$. Thus, adding the two equations together, we obtain

$$\nu\|\nabla\boldsymbol{v}\|_2^2 = \langle\boldsymbol{f},\boldsymbol{v}\rangle.$$

This tells us that, if $\boldsymbol{f} = 0$, then $\nabla \boldsymbol{v} = 0$; assuming also that a Poincaré inequality holds, this tells that $\boldsymbol{v}$ is, indeed, identically zero. Of course, all this is only formal, and in reality we must consider how the solutions of the MHD equations (2.2) behave as $t \to \infty$.

2.3 Magnetic relaxation and steady Euler flows

The preceding discussion tells us that states of magnetostatic equilibria should give rise to solutions of the steady Euler equations, but fails to tell us how or why such an analogy is useful; for that we must turn to questions of topology.

In equations (2.2), one observes that while there is a viscous dissipation term $\nu\Delta\boldsymbol{v}$ for the fluid velocity, there is no such term for the magnetic field: this is due to the assumption of perfect conductivity, i.e. the lack of magnetic diffusion. Due to the perfect conductivity assumption, the induction equation (2.2b) implies that the magnetic field lines (or $\boldsymbol{B}$-lines) are transported by the flow, and are effectively "frozen" in the fluid. Hence, the topological structure of the magnetic field $\boldsymbol{B}(t)$ at any finite time $t > 0$ will be homeomorphic to the initial field $\boldsymbol{B}_0$.

In Moffatt (1985), it is suggested that this topological equivalence can be exploited as follows. Suppose we want to know about the long-time behaviour of the Euler equations in a given domain Ω, where Ω has non-trivial topological structure (such as a multiply-holed torus). By taking the MHD equations (2.2) and running them for long time, we will have a magnetic field which is homeomorphic to the initial magnetic field, and in the limit should satisfy the Euler equations.

Moffatt thus uses this procedure to *construct* (or, at least, prove the existence of) solutions of the Euler equations on a domain with arbitrarily prescribed topology; the fact that the magnetic field is homeomorphic to the initial field at any positive time means that any knots or links in the $\boldsymbol{B}$-lines are preserved, which may prevent (especially on domains of complex topology) the magnetic field simply decaying to zero as $t \to \infty$. This technique therefore generates non-trivial steady solutions of the Euler equations.

In his paper, Moffatt (1985) chooses a number of examples to show how the field line configuration may obstruct the decay of the magnetic field to zero. One of the simplest is a "simple linkage": a domain Ω consisting of two linked tori in $\mathbb{R}^3$. In the limit $t \to \infty$, the magnetic field from one of the tori spreads out and is "wrapped around" the other torus, thus preventing the $\boldsymbol{B}$-lines from collapsing to zero.

There are two obvious questions that arise. The first is that, while the magnetic field is homeomorphic to the initial field for any *finite* time, there is no guarantee that the magnetic field will be homeomorphic in the limit $t \to \infty$; indeed, it may (and usually does) possess discontinuities. Physically speaking, this is welcome: contact discontinuities, such as current sheets, help to explain many phenomena in which magnetic energy is converted rapidly into kinetic energy.

Mathematically speaking, however, we must be more careful in order to have any sense in which the limiting magnetic field has "the same" topology as the initial field. Moffatt (1985) proposes a weaker form of topological equivalence: a magnetic field is *topologically accessible* from an initial field $\boldsymbol{B}_0$ if it can be obtained by the convective action on $\boldsymbol{B}_0$ of a velocity field $\boldsymbol{v}$ which is smooth and has finite total viscous energy dissipation; that is,

$$\int_0^\infty \|\nabla \boldsymbol{v}(t)\|_2^2 \,\mathrm{d}t < \infty.$$

We will see in section 3.1 that solutions of the MHD equations (2.2) always satisfy this requirement.

More important, however, than whether the magnetic field in the limit $t \to \infty$ has the same topology, is whether the magnetic field actually *has* a limit at all as $t \to \infty$: it is quite conceivable that, even if we assume the solution to exist for all time, that the magnetic field has no limit, even in a weak sense.

Moreover, even if the magnetic field does have a limit, there is no guarantee that this limit will correspond to a state of magnetostatic equilibrium. In order to ensure that the limit was in magnetostatic equilibrium, we would require that, as $t \to \infty$, that $\frac{\partial \boldsymbol{v}}{\partial t}$ and $\frac{\partial \boldsymbol{B}}{\partial t}$ should tend to zero, in order that the system should satisfy the steady state equations (2.4). Furthermore, we would require that $\boldsymbol{v} \to 0$ as $t \to \infty$ in order to assert that the limit magnetic field really is a steady solution of the Euler equations, as in equations (2.5).

Actually making everything rigorous, however, is quite difficult. In a series of papers, Nishiyama (2002, 2003*a*,*b*, 2007) used Galerkin approximations and finite-difference approximations to prove go some way to proving rigorously that approximations to the solutions of the MHD equations (2.2) also solve the stationary Euler equations, but little progress was made with the full system of equations until the work of Núñez (2007).

In the following two chapters, we present the work of Núñez (2007) in rigorously considering the asymptotic behaviour of the velocity field and the magnetic field. In particular, we see that the velocity field tends to zero in the L^2 sense, i.e. the kinetic energy $\|\boldsymbol{v}(t)\|_2 \to 0$ as $t \to \infty$. However, Núñez constructs a solution to equations (2.2) such that, while the magnetic energy $\|\boldsymbol{B}(t)\|_2 \to M$ for some constant M, the magnetic field $\boldsymbol{B}$ has no weak limit as $t \to \infty$.

2.4 Alternative models of magnetic relaxation

While equations (2.2) are the standard model for magnetohydrodynamics, there are other plausible models for the evolution of magnetic fluids which might also produce stationary solutions to the Euler equations. In a talk given at the University of Warwick, Moffatt (2009) suggests that "we are free to make whatever dynamical model we like, provided it dissipates energy". In particular, he suggests considering "Stokes" dynamics, under which we neglect the terms $\frac{\partial \boldsymbol{v}}{\partial t}$ and $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$. This gives rise to the following simplified system of equations:

$$-\nu\Delta\boldsymbol{v} + \nabla p_* = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} + \boldsymbol{f} \tag{2.6a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} \tag{2.6b}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{2.6c}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2.6d}$$

where once again $p_* = p + \frac{1}{2}|\boldsymbol{B}|^2$ is the total pressure, and $\boldsymbol{f} \in L^2((0,\infty), L^2(\Omega))$.

Over the following two chapters, we will see that this model has the same problems as the standard MHD model: we adapt the counterexample of Núñez to these equations to show that the magnetic field may have no weak limit.

Moreover, we require additional hypotheses on the magnetic field in order to show that the velocity field decays to zero in the L^2 norm: the approach of Núñez to proving that the kinetic energy decays to zero no longer applies, since that fundamentally depends on the presence of the $\frac{\partial \boldsymbol{v}}{\partial t}$ term in the equations. Instead, a new and original approach is required, and we see that further hypothesis are required to guarantee the decay of the kinetic energy to zero.

Chapter 3

Asymptotic behaviour of the velocity field

In this chapter and the following chapter, we consider the rigorous analysis of Núñez (2007) for the model of magnetic relaxation under the MHD equations (2.2): in this chapter we consider the behaviour of the velocity field in the limit $t \to \infty$, and in the next chapter we consider the limiting behaviour of the magnetic field.

In both chapters, after expounding the work of Núñez, we undertake an original adaptation of that analysis to magnetic relaxation under the "Stokes" dynamics of equations (2.6).

3.1 Asymptotic behaviour under standard MHD

In this section, we present the work of Núñez (2007) in considering the behaviour of the velocity field as $t \to \infty$ under the MHD equations (2.2). In a similar vein to proposition 1.1, we first consider a simple energy identity for the MHD equations, originally derived by Moffatt (1985) (see equation (3.11) there).

Proposition 3.1. *The solutions* $\boldsymbol{v}$*,* $\boldsymbol{B}$ *of equations* (2.2) *satisfy*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{v}(t)\|_2^2 + \|\boldsymbol{B}(t)\|_2^2\right) = -\nu\|\nabla\boldsymbol{v}\|_2^2 + \langle \boldsymbol{f}, \boldsymbol{v}\rangle.$$

Proof. The proof is similar to that of proposition 1.1: taking the inner product of

(2.2a) with $\boldsymbol{v}$, and the inner product of (2.2b) with $\boldsymbol{B}$, yields

$$\left\langle \frac{\partial \boldsymbol{v}}{\partial t}, \boldsymbol{v} \right\rangle + \langle (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{v}\rangle - \nu \langle \Delta \boldsymbol{v}, \boldsymbol{v}\rangle + \langle \nabla p_*, \boldsymbol{v}\rangle = \langle (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}, \boldsymbol{v}\rangle + \langle \boldsymbol{f}, \boldsymbol{v}\rangle,$$
$$\left\langle \frac{\partial \boldsymbol{B}}{\partial t}, \boldsymbol{B} \right\rangle + \langle (\boldsymbol{v} \cdot \nabla)\boldsymbol{B}, \boldsymbol{B}\rangle = \langle (\boldsymbol{B} \cdot \nabla)\boldsymbol{v}, \boldsymbol{B}\rangle.$$

Under boundary conditions (2.3), lemma 1.2 tells us that $\langle (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{v}\rangle = 0$ and $\langle (\boldsymbol{v} \cdot \nabla)\boldsymbol{B}, \boldsymbol{B}\rangle = 0$, since $\boldsymbol{v}$ and $\boldsymbol{B}$ are divergence-free. As in the proof of proposition 1.1, the boundary conditions tell us that

$$\langle \nabla p, \boldsymbol{v}\rangle = \int_{\partial\Omega} p\boldsymbol{v} \cdot \boldsymbol{n} \,\mathrm{d}S = 0.$$

An integration by parts shows that $-\langle \Delta \boldsymbol{v}, \boldsymbol{v}\rangle = \|\nabla \boldsymbol{v}\|_2^2$, and that $\langle (\boldsymbol{B} \cdot \nabla)\boldsymbol{v}, \boldsymbol{B}\rangle = -\langle (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}, \boldsymbol{v}\rangle$. Thus, adding the two equations together, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{v}(t)\|_2^2 + \|\boldsymbol{B}(t)\|_2^2\right) = -\nu \|\nabla \boldsymbol{v}\|_2^2 + \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \qquad \square$$

From this, Moffatt (1985) argues as follows. As long as $\boldsymbol{v}$ is not identically zero (and assuming the system is unforced), the total energy of the system must decrease. This decrease proceeds by contraction of $\boldsymbol{B}$-lines, releasing magnetic energy for so long as they are able to contract unimpeded. However, if the topology of the initial field $\boldsymbol{B}_0$ is nontrivial, then there will come a point where $\boldsymbol{B}$-lines cannot contract without cutting each other, and thus there will be a minimum energy below which the system cannot decrease; therefore, there exists some $E \geq 0$ such that

$$\|\boldsymbol{v}(t)\|_2^2 + \|\boldsymbol{B}(t)\|_2^2 \to E$$

as $t \to \infty$. If it is indeed the case that the kinetic energy $\|\boldsymbol{v}(t)\|_2^2 \to 0$ as $t \to \infty$, then we will have that the magnetic energy $\|\boldsymbol{B}(t)\|_2^2 \to E \geq 0$.

In order to measure the "degree of linkage" of the $\boldsymbol{B}$-lines, Moffatt introduces a counterpart of helicity for MHD, which we introduced in section 1.2 for the Euler equations. Given a magnetic field $\boldsymbol{B}$, let $\boldsymbol{A}$ be a vector field such that $\nabla \cdot \boldsymbol{A} = 0$ and $\nabla \times \boldsymbol{A} = \boldsymbol{B}$. We then define the *magnetic helicity* as

$$\mathscr{H}_{\mathrm{M}} := \langle \boldsymbol{A}, \boldsymbol{B}\rangle = \int_{\Omega} \boldsymbol{A} \cdot \boldsymbol{B} \,\mathrm{d}x.$$

Using the Cauchy–Schwarz inequality, we find that

$$|\mathscr{H}_{\mathrm{M}}| = |\langle \boldsymbol{A}, \boldsymbol{B}\rangle| \leq \|\boldsymbol{A}\|_2 \|\boldsymbol{B}\|_2,$$

and using the Poincaré inequality, we see that

$$\|\boldsymbol{A}\|_2 \leq c_p \|\nabla \times \boldsymbol{A}\|_2 = c_p \|\boldsymbol{B}\|_2.$$

From these, we see that

$$\|\boldsymbol{B}\|_2^2 \geq \frac{|\mathscr{H}_{\mathrm{M}}|}{\|\boldsymbol{A}\|_2} \cdot \frac{\|\boldsymbol{A}\|_2}{c_p} = \frac{|\mathscr{H}_{\mathrm{M}}|}{c_p},$$

which provides a positive lower bound on the magnetic energy, so long as the magnetic helicity is nonzero.

As a first step towards proving that the kinetic energy decays to zero, using the energy equality in proposition 3.1, and applying the Cauchy–Schwarz and Young inequalities, we obtain

$$|\langle \boldsymbol{f}, \boldsymbol{v}\rangle| \leq \|\boldsymbol{f}\|_2 \|\boldsymbol{v}\|_2 \leq c_p \|\boldsymbol{f}\|_2 \|\nabla \boldsymbol{v}\|_2 \leq \frac{\nu}{2}\|\nabla \boldsymbol{v}\|_2^2 + \frac{c_p^2}{2\nu}\|\boldsymbol{f}\|_2^2,$$

hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{v}\|_2^2 + \|\boldsymbol{B}\|_2^2) \leq -\frac{\nu}{2}\|\nabla \boldsymbol{v}\|_2^2 + \frac{c_p^2}{2\nu}\|\boldsymbol{f}\|_2^2.$$

Integrating in time yields

$$\begin{aligned} \frac{1}{2}&\left[(\|\boldsymbol{v}(t)\|_2^2 + \|\boldsymbol{B}(t)\|_2^2) - (\|\boldsymbol{v}(0)\|_2^2 + \|\boldsymbol{B}(0)\|_2^2)\right] \\ &\leq -\frac{\nu}{2}\int_0^t \|\nabla \boldsymbol{v}(s)\|_2^2 \,\mathrm{d}s + \frac{c_p^2}{2\nu}\int_0^t \|\boldsymbol{f}(s)\|_2^2 \,\mathrm{d}s. \end{aligned} \tag{3.1}$$

Since we have assumed that $\boldsymbol{f} \in L^2((0,\infty), L^2(\Omega))$, we may bound the last integral uniformly in time. This implies that $\|\boldsymbol{v}(t)\|_2$ and $\|\boldsymbol{B}(t)\|_2$ are bounded uniformly in time, and that $\nabla \boldsymbol{v} \in L^2((0,\infty), L^2(\Omega))$.

However, a function being square-integrable over $(0,\infty)$ does not necessarily imply that it converges to zero as $t \to \infty$. Therefore, while this means that $\|\nabla \boldsymbol{v}(t)\|_2$ (and, by the Poincaré inequality, $\|\boldsymbol{v}(t)\|_2$) is small for large times, it may be that the velocity field has arbitrarily high jumps, provided that the time over which they occur decreases to zero sufficiently fast. Fortunately, we show now that this does not occur:

Theorem 3.2. *The solution $\boldsymbol{v}$ of equations* (2.2) *satisfies*

$$\lim_{t\to\infty} \|\boldsymbol{v}(t)\|_2 = 0.$$

Núñez's proof is based upon one by Agapito & Schonbek (2007) where they consider the MHD equations on the whole of $\mathbb{R}^3$, rather than some bounded domain $\Omega \subset \mathbb{R}^3$; their proof is rather more sophisticated, splitting the solution into low and high Fourier modes, since they have no Poincaré inequality. However, unlike the proof of Agapito & Schonbek, this proof applies equally to two dimensions.

Proof. Using the method of proposition 3.1, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}(t)\|_2^2 = -\nu\|\nabla\boldsymbol{v}(t)\|_2^2 - \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}, \boldsymbol{B}\rangle + \langle\boldsymbol{f}, \boldsymbol{v}\rangle.$$

So, fix $r > 0$ such that $rc_p^2 \leq 2\nu$. Differentiating yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(e^{rt}\|\boldsymbol{v}(t)\|_2^2) &= re^{rt}\|\boldsymbol{v}(t)\|_2^2 + e^{rt}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}(t)\|_2^2\\ &= re^{rt}\|\boldsymbol{v}(t)\|_2^2 - 2\nu e^{rt}\|\nabla\boldsymbol{v}(t)\|_2^2\\ &\qquad - 2e^{rt}\langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}, \boldsymbol{B}\rangle + 2e^{rt}\langle\boldsymbol{f}, \boldsymbol{v}\rangle,\end{aligned}$$

so using Poincaré's inequality, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(e^{rt}\|\boldsymbol{v}(t)\|_2^2) &\leq rc_p^2 e^{rt}\|\nabla\boldsymbol{v}(t)\|_2^2 - 2\nu e^{rt}\|\nabla\boldsymbol{v}(t)\|_2^2\\ &\qquad + 2e^{rt}\|\boldsymbol{B}(t)\|_\infty\|\nabla\boldsymbol{v}(t)\|_2\|\boldsymbol{B}(t)\|_2 + 2c_p e^{rt}\|\boldsymbol{f}(t)\|_2\|\nabla\boldsymbol{v}(t)\|_2\\ &\leq 2e^{rt}\|\nabla\boldsymbol{v}(t)\|_2\left(\|\boldsymbol{B}(t)\|_\infty\|\boldsymbol{B}(t)\|_2 + c_p\|\boldsymbol{f}(t)\|_2\right).\end{aligned}$$

By hypothesis, $\|\boldsymbol{B}\|_\infty$ is bounded, and by the energy inequality $\|\boldsymbol{B}\|_2$ is bounded. Thus it follows that, for some constant M,

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{rt}\|\boldsymbol{v}(t)\|_2^2) \leq 2e^{rt}\|\nabla\boldsymbol{v}(t)\|_2\left(M + c_p\|\boldsymbol{f}(t)\|_2\right).$$

Integrating between s and t, and multiplying by e^{-rt}, we find

$$\begin{aligned}
&\|\boldsymbol{v}(t)\|_2^2 - e^{r(s-t)}\|\boldsymbol{v}(s)\|_2^2 \\
&\quad \leq 2Me^{-rt}\int_s^t e^{r\tau}\|\nabla\boldsymbol{v}(\tau)\|_2 \,\mathrm{d}\tau + 2c_p\int_s^t e^{r(\tau-t)}\|\boldsymbol{f}(\tau)\|_2\|\nabla\boldsymbol{v}(\tau)\|_2 \,\mathrm{d}\tau \\
&\quad \leq 2Me^{-rt}\left(\int_s^t e^{2r\tau}\,\mathrm{d}\tau\right)^{1/2}\left(\int_s^t \|\nabla\boldsymbol{v}(\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/2} \\
&\qquad + 2c_p\left(\int_s^t \|\boldsymbol{f}(\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/2}\left(\int_s^t \|\nabla\boldsymbol{v}(\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/2}
\end{aligned}$$

since $e^{r(\tau-t)} \leq 1$.

Recall that, by the energy inequality (3.1), there exists a constant N such that $\|v(s)\|_2 \leq N$ for all $s \in (0,\infty)$. Since

$$e^{-rt}\left(\int_s^t e^{2r\tau}\,\mathrm{d}\tau\right)^{1/2} = e^{-rt}\left(\frac{e^{2rt}-e^{2rs}}{2r}\right)^{1/2} = \left(\frac{1-e^{2r(s-t)}}{2r}\right)^{1/2} \leq \frac{1}{\sqrt{2r}},$$

and since $f \in L^2((0,\infty), L^2(\Omega))$ we have that

$$\left(\int_s^t \|\boldsymbol{f}(\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/2} \leq \left(\int_0^\infty \|\boldsymbol{f}(\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/2} =: F < \infty,$$

so we see that

$$\|\boldsymbol{v}(t)\|_2^2 \leq Ne^{r(s-t)} + \left[\frac{2M}{\sqrt{2r}} + 2c_pF\right]\left(\int_s^t \|\nabla\boldsymbol{v}(\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/2}.$$

Given $\varepsilon > 0$, take s large enough so that $e^{-rs} < \varepsilon/2N$, and so that

$$\int_s^\infty \|\nabla\boldsymbol{v}(\tau)\|_2^2\,\mathrm{d}\tau < \frac{\varepsilon^2}{4\left(\frac{2M}{\sqrt{2r}} + 2c_pF\right)^2};$$

then, for any $t > 2s$, we see that $\|\boldsymbol{v}(t)\|_2^2 < \varepsilon$, as required. □

We are therefore assured that the system will, in the limit $t \to \infty$, become stationary, in the sense that the kinetic energy of the system will tend to zero. Hence, the magnetic energy of the system will decrease to some constant $E \geq 0$. As we will see in the next chapter, however, this is not sufficient to ensure that the magnetic field itself actually has a limit.

3.2 Asymptotic behaviour under "Stokes" dynamics

In this section, we now consider the alternative model outlined in section 2.4, in which we neglect the $\partial \boldsymbol{v}/\partial t$ and $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ terms and consider the "Stokes equations" for the evolution of the velocity. While parts of the analysis are based on the approach of Núñez (2007), parts of Núñez's arguments do not apply, and so much of our approach is original.

Using the same method of proof as proposition 3.1, we obtain the analogous result for equations (2.6):

Proposition 3.3. *The solutions* $\boldsymbol{v}$*,* $\boldsymbol{B}$ *of equations* (2.6) *satisfy*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}\|_2^2 = -\nu\|\nabla\boldsymbol{v}\|_2^2 + \langle \boldsymbol{f}, \boldsymbol{v}\rangle.$$

Using the Cauchy–Schwarz and Young inequalities, we obtain

$$|\langle \boldsymbol{f}, \boldsymbol{v}\rangle| \le \|\boldsymbol{f}\|_2\|\boldsymbol{v}\|_2 \le c_p\|\boldsymbol{f}\|_2\|\nabla\boldsymbol{v}\|_2 \le \frac{\nu}{2}\|\nabla\boldsymbol{v}\|_2^2 + \frac{c_p^2}{2\nu}\|\boldsymbol{f}\|_2^2,$$

hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}\|_2^2 \le -\frac{\nu}{2}\|\nabla\boldsymbol{v}\|_2^2 + \frac{c_p^2}{2\nu}\|\boldsymbol{f}\|_2^2.$$

Integrating in time yields

$$\frac{1}{2}\left(\|\boldsymbol{B}(t)\|_2^2 - \|\boldsymbol{B}(0)\|_2^2\right) \le -\frac{\nu}{2}\int_0^t \|\nabla\boldsymbol{v}(s)\|_2^2\,\mathrm{d}s + \frac{c_p^2}{2\nu}\int_0^t \|\boldsymbol{f}(s)\|_2^2\,\mathrm{d}s. \qquad (3.2)$$

Since $f \in L^2((0,\infty), L^2(\Omega))$, we see that $\|\boldsymbol{B}(t)\|_2$ is bounded for all time, and that $\int_0^\infty \|\nabla\boldsymbol{v}(t)\|_2^2\,\mathrm{d}t < \infty$, so that $\nabla\boldsymbol{v} \in L^2((0,\infty), L^2(\Omega))$.

Unfortunately, attempting the same line of proof as Núñez (2007) to achieve an analogue of theorem 3.2 seems impossible, due to the absence of the $\frac{\partial \boldsymbol{v}}{\partial t}$ term from the equations.

Instead, we take a new approach to the issue of whether the kinetic energy decays to zero. We know that $\nabla\boldsymbol{v} \in L^2((0,\infty), L^2(\Omega))$, so that

$$\int_t^\infty \|\nabla\boldsymbol{v}(s)\|_2^2\,\mathrm{d}s \to 0,$$

but even if $\|\nabla\boldsymbol{v}(s)\|_2 \le C$ for all s, this does not guarantee that $\|\nabla\boldsymbol{v}(s)\|_2 \to 0$ as $s \to \infty$. However, the only way in which we can have $\nabla\boldsymbol{v} \in L^2((0,\infty), L^2(\Omega))$ but $\nabla\boldsymbol{v} \not\to 0$ as $t \to \infty$ is if the velocity "spikes" infinitely often, where the duration of the spikes decays sufficiently rapidly. We can ensure that this cannot happen by controlling $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{v}\|_2^2$, so we seek a bound on this. We start by finding an elementary bound on the kinetic energy itself:

Lemma 3.4. *The solution* $\boldsymbol{v}$ *of equations* (2.6) *satisfies*

$$\nu\|\nabla \boldsymbol{v}\|_2 \leq \|\boldsymbol{B}\|_\infty \|\boldsymbol{B}\|_2 + c_p\|\boldsymbol{f}\|_2.$$

Proof. Taking the inner product of (2.6a) with $\boldsymbol{v}$ and integrating by parts, we obtain

$$\begin{aligned}\nu\|\nabla \boldsymbol{v}\|_2^2 &= \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \boldsymbol{v}\rangle + \langle \boldsymbol{f}, \boldsymbol{v}\rangle \\ &\leq -\langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}, \boldsymbol{B}\rangle + \|\boldsymbol{f}\|_2\|\boldsymbol{v}\|_2 \\ &\leq \|\boldsymbol{B}\|_\infty\|\nabla \boldsymbol{v}\|_2\|\boldsymbol{B}\|_2 + c_p\|\boldsymbol{f}\|_2\|\nabla \boldsymbol{v}\|_2,\end{aligned}$$

hence

$$\nu\|\nabla \boldsymbol{v}\|_2 \leq \|\boldsymbol{B}\|_\infty \|\boldsymbol{B}\|_2 + c_p\|\boldsymbol{f}\|_2. \qquad \square$$

We use this to derive a new bound on $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{v}\|_2^2$:

Proposition 3.5. *Suppose that, in addition to boundary conditions* (2.3)*, the following boundary integral vanishes:*

$$\int_{\partial\Omega} \frac{\partial p_*}{\partial t}(\boldsymbol{v}\cdot n)\,\mathrm{d}S = 0. \tag{3.3}$$

Then

$$\begin{aligned}\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{v}\|_2^2 \leq \|\boldsymbol{B}\|_\infty &\left(\left\|\frac{\partial^2 \boldsymbol{B}}{\partial t^2}\right\|_1 + \|\nabla \boldsymbol{v}\|_2\left\|\frac{\partial \boldsymbol{B}}{\partial t}\right\|_2\right) \\ &+ c_p\|\nabla \boldsymbol{B}\|_\infty\left\|\frac{\partial \boldsymbol{B}}{\partial t}\right\|_2\|\nabla \boldsymbol{v}\|_2 + c_p\|\boldsymbol{f}\|_2\|\nabla \boldsymbol{v}\|_2.\end{aligned}$$

Proof. First, note that

$$\begin{aligned}\left\langle \nabla p_*, \frac{\partial \boldsymbol{v}}{\partial t}\right\rangle &= \int_\Omega \nabla\cdot\left(p_*\frac{\partial \boldsymbol{v}}{\partial t}\right)\,\mathrm{d}x \\ &= \int_\Omega \left[\frac{\partial}{\partial t}\left(\nabla\cdot(p_*\boldsymbol{v})\right) - \nabla\cdot\left(\frac{\partial p_*}{\partial t}\boldsymbol{v}\right)\right]\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \nabla\cdot(p_*\boldsymbol{v})\,\mathrm{d}x\right) - \int_\Omega \nabla\cdot\left(\frac{\partial p_*}{\partial t}\boldsymbol{v}\right)\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\partial\Omega} p_*\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S\right) - \int_{\partial\Omega}\frac{\partial p_*}{\partial t}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S \\ &= 0.\end{aligned}$$

Thus, taking the inner product of (2.6a) with $\frac{\partial \boldsymbol{v}}{\partial t}$ yields

$$0 = -\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{v}\|_2^2 + \left\langle (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \frac{\partial \boldsymbol{v}}{\partial t}\right\rangle + \langle \boldsymbol{f}, \boldsymbol{v}\rangle.$$

Differentiating (2.6b) with respect to t yields

$$\frac{\partial^2 \boldsymbol{B}}{\partial t^2} + \left(\frac{\partial \boldsymbol{v}}{\partial t}\cdot\nabla\right)\boldsymbol{B} + (\boldsymbol{v}\cdot\nabla)\left(\frac{\partial \boldsymbol{B}}{\partial t}\right) = \left(\frac{\partial \boldsymbol{B}}{\partial t}\cdot\nabla\right)\boldsymbol{v} + (\boldsymbol{B}\cdot\nabla)\left(\frac{\partial \boldsymbol{v}}{\partial t}\right).$$

By lemma 1.2 we have that $\left\langle\left(\frac{\partial \boldsymbol{v}}{\partial t}\cdot\nabla\right)\boldsymbol{B}, \boldsymbol{B}\right\rangle = 0$, so taking the inner product of this with respect to $\boldsymbol{B}$ yields

$$\left\langle \frac{\partial^2 \boldsymbol{B}}{\partial t^2}, \boldsymbol{B}\right\rangle + \left\langle (\boldsymbol{v}\cdot\nabla)\left(\frac{\partial \boldsymbol{B}}{\partial t}\right), \boldsymbol{B}\right\rangle = \left\langle \left(\frac{\partial \boldsymbol{B}}{\partial t}\cdot\nabla\right)\boldsymbol{v}, \boldsymbol{B}\right\rangle + \left\langle (\boldsymbol{B}\cdot\nabla)\left(\frac{\partial \boldsymbol{v}}{\partial t}\right), \boldsymbol{B}\right\rangle.$$

Integrating by parts shows that

$$\left\langle (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \frac{\partial \boldsymbol{v}}{\partial t}\right\rangle = -\left\langle (\boldsymbol{B}\cdot\nabla)\left(\frac{\partial \boldsymbol{v}}{\partial t}\right), \boldsymbol{B}\right\rangle,$$

and that

$$\left\langle (\boldsymbol{v}\cdot\nabla)\left(\frac{\partial \boldsymbol{B}}{\partial t}\right), \boldsymbol{B}\right\rangle = -\left\langle (\boldsymbol{v}\cdot\nabla)\boldsymbol{B}, \left(\frac{\partial \boldsymbol{B}}{\partial t}\right)\right\rangle,$$

hence

$$\begin{aligned}
\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{v}\|_2^2 &= \left\langle (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \frac{\partial \boldsymbol{v}}{\partial t}\right\rangle + \langle \boldsymbol{f}, \boldsymbol{v}\rangle \\
&= -\left\langle (\boldsymbol{B}\cdot\nabla)\left(\frac{\partial \boldsymbol{v}}{\partial t}\right), \boldsymbol{B}\right\rangle + \langle \boldsymbol{f}, \boldsymbol{v}\rangle \\
&= \left\langle \left(\frac{\partial \boldsymbol{B}}{\partial t}\cdot\nabla\right)\boldsymbol{v}, \boldsymbol{B}\right\rangle - \left\langle \frac{\partial^2 \boldsymbol{B}}{\partial t^2}, \boldsymbol{B}\right\rangle + \left\langle (\boldsymbol{v}\cdot\nabla)\boldsymbol{B}, \left(\frac{\partial \boldsymbol{B}}{\partial t}\right)\right\rangle + \langle \boldsymbol{f}, \boldsymbol{v}\rangle \\
&\le \left\|\frac{\partial \boldsymbol{B}}{\partial t}\right\|_2 \|\nabla \boldsymbol{v}\|_2 \|\boldsymbol{B}\|_\infty + \left\|\frac{\partial^2 \boldsymbol{B}}{\partial t^2}\right\|_1 \|\boldsymbol{B}\|_\infty \\
&\qquad + \|\boldsymbol{v}\|_2 \|\nabla \boldsymbol{B}\|_\infty \left\|\frac{\partial \boldsymbol{B}}{\partial t}\right\|_2 + \|\boldsymbol{f}\|_2 \|\boldsymbol{v}\|_2 \\
&\le \|\boldsymbol{B}\|_\infty \left(\left\|\frac{\partial^2 \boldsymbol{B}}{\partial t^2}\right\|_1 + \|\nabla \boldsymbol{v}\|_2 \left\|\frac{\partial \boldsymbol{B}}{\partial t}\right\|_2\right) \\
&\qquad + c_p \|\nabla \boldsymbol{B}\|_\infty \left\|\frac{\partial \boldsymbol{B}}{\partial t}\right\|_2 \|\nabla \boldsymbol{v}\|_2 + c_p \|\boldsymbol{f}\|_2 \|\nabla \boldsymbol{v}\|_2.
\end{aligned}$$

□

Since lemma 3.4 shows that control of $\|\boldsymbol{B}\|_\infty$, and hence of $\|\boldsymbol{B}\|_2$, gives us control of $\|\nabla \boldsymbol{v}\|_2$, it appears that, in order to get a bound on $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{v}\|_2^2$, we must control four quantities: $\|\boldsymbol{B}\|_\infty$, $\|\nabla\boldsymbol{B}\|_\infty$, $\left\|\frac{\partial\boldsymbol{B}}{\partial t}\right\|_2$ and $\left\|\frac{\partial^2\boldsymbol{B}}{\partial t^2}\right\|_1$.

However, the next lemma shows that we can bound $\left\|\frac{\partial\boldsymbol{B}}{\partial t}\right\|_2$ in terms of $\|\nabla\boldsymbol{B}\|_\infty$, and thus reduce by one the number of assumptions required to use proposition 3.5:

Lemma 3.6. *The solution* $\boldsymbol{B}$ *of equations* (2.6) *satisfies*

$$\left\|\frac{\partial\boldsymbol{B}}{\partial t}\right\|_2 \leq \|\nabla\boldsymbol{v}\|_2\left(\|\boldsymbol{B}\|_\infty + c_p\|\nabla\boldsymbol{B}\|_\infty\right).$$

Proof. By taking the inner product of equation (2.6b) with $\frac{\partial\boldsymbol{B}}{\partial t}$, we obtain

$$\begin{aligned}\left\|\frac{\partial\boldsymbol{B}}{\partial t}\right\|_2^2 &= \left\langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}, \frac{\partial\boldsymbol{B}}{\partial t}\right\rangle - \left\langle(\boldsymbol{v}\cdot\nabla)\boldsymbol{B}, \frac{\partial\boldsymbol{B}}{\partial t}\right\rangle \\ &\leq \|\boldsymbol{B}\|_\infty\|\nabla\boldsymbol{v}\|_2\left\|\frac{\partial\boldsymbol{B}}{\partial t}\right\|_2 + \|\boldsymbol{v}\|_2\|\nabla\boldsymbol{B}\|_\infty\left\|\frac{\partial\boldsymbol{B}}{\partial t}\right\|_2,\end{aligned}$$

so dividing through and applying Poincaré's inequality yields

$$\left\|\frac{\partial\boldsymbol{B}}{\partial t}\right\|_2 \leq \|\nabla\boldsymbol{v}\|_2\left(\|\boldsymbol{B}\|_\infty + c_p\|\nabla\boldsymbol{B}\|_\infty\right). \qquad \square$$

Suppose now that, for all $t \in (0,\infty)$, the following bounds hold:

$$\|\boldsymbol{B}\|_\infty \leq M_1 \tag{3.4a}$$

$$\|\nabla\boldsymbol{B}\|_\infty \leq M_2, \tag{3.4b}$$

$$\left\|\frac{\partial^2\boldsymbol{B}}{\partial t^2}\right\|_1 \leq M_3. \tag{3.4c}$$

By combining lemma 3.4, proposition 3.5 and lemma 3.6, we may conclude that $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{v}\|_2^2$ is uniformly bounded. The following elementary lemma then tells us that $\|\nabla\boldsymbol{v}(t)\|_2^2 \to 0$ (and hence, by the Poincaré inequality, that $\|\boldsymbol{v}(t)\|_2^2 \to 0$) as $t \to \infty$:

Lemma 3.7. *Suppose that* $f\colon (0,\infty) \to \mathbb{R}$ *is* C^1*, that* $\int_0^\infty |f(t)|\,\mathrm{d}t < \infty$*, and that* $|f'(t)| \leq M$ *for all* $t \in (0,\infty)$*. Then* $f(t) \to 0$ *as* $t \to \infty$*.*

Proof. Suppose not. Then there exists $\varepsilon > 0$ such that, for any $r > 0$, there exists $s > r$ such that $|f(s)| \geq \varepsilon$. By the mean value theorem, for all $t \in (s, s+\varepsilon/2M)$, we have

$$|f(t) - f(s)| \leq (t-s)M \leq \varepsilon/2,$$

so $|f(t)| \geq \varepsilon/2$ for all $t \in (s, s+\varepsilon/2M)$. Hence

$$\int_r^\infty |f(t)|\,\mathrm{d}t \geq \int_s^{s+\varepsilon/2M} |f(t)|\,\mathrm{d}t \geq \frac{\varepsilon^2}{4M}.$$

Since r was arbitrary, this means that $\int_r^\infty |f(t)| \,\mathrm{d}t \not\to 0$ as $r \to \infty$, which contradicts the fact that $\int_0^\infty |f(t)| \,\mathrm{d}t < \infty$. $\square$

We have therefore proved that the kinetic energy of the simplified system (2.6) does indeed decay to zero, though with more hypotheses than required by Núñez for the original system. However, since we are assuming the existence of a unique smooth solution to the equations, demanding that $\nabla \boldsymbol{B}$ and $\frac{\partial^2 \boldsymbol{B}}{\partial t^2}$ are suitably bounded does not seem wholly unreasonable.

Nonetheless, obtaining a priori bounds on the time derivatives of $\boldsymbol{B}$ seems quite difficult. For a bound on $\frac{\partial^2 \boldsymbol{B}}{\partial t^2}$, one obvious way to proceed is to differentiate equation (2.6b) with respect to t, which yields

$$\frac{\partial^2 \boldsymbol{B}}{\partial t^2} + \left(\frac{\partial \boldsymbol{v}}{\partial t} \cdot \nabla\right) \boldsymbol{B} + (\boldsymbol{v} \cdot \nabla) \left(\frac{\partial \boldsymbol{B}}{\partial t}\right) = \left(\frac{\partial \boldsymbol{B}}{\partial t} \cdot \nabla\right) \boldsymbol{v} + (\boldsymbol{B} \cdot \nabla) \left(\frac{\partial \boldsymbol{v}}{\partial t}\right).$$

Unfortunately this still leaves us with a $\frac{\partial \boldsymbol{v}}{\partial t}$ term on the right-hand side, which cannot easily be eliminated. Taking the inner product with $\boldsymbol{B}$ again yields

$$\left\langle \frac{\partial^2 \boldsymbol{B}}{\partial t^2}, \boldsymbol{B} \right\rangle + \left\langle (\boldsymbol{v} \cdot \nabla) \left(\frac{\partial \boldsymbol{B}}{\partial t}\right), \boldsymbol{B} \right\rangle = \left\langle \left(\frac{\partial \boldsymbol{B}}{\partial t} \cdot \nabla\right) \boldsymbol{v}, \boldsymbol{B} \right\rangle + \left\langle (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}, \frac{\partial \boldsymbol{v}}{\partial t} \right\rangle.$$

One could try integrating in time, and then integrating in parts in time to get a bound on $\left\| \frac{\partial \boldsymbol{B}}{\partial t} \right\|_2$, but this still involves $\frac{\partial \boldsymbol{v}}{\partial t}$, as well as having a time integral and some nasty time boundary terms which are not likely to be helpful.

While it is a more plausible assumption, actually obtaining bounds on $\|\nabla \boldsymbol{B}\|_\infty$ seems just as difficult. Alternatively, by omitting the last integration by parts in proposition 3.5, we can obtain

$$\frac{\nu}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \boldsymbol{v}\|_2^2 \le \|\boldsymbol{B}\|_\infty \left(\left\| \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \right\|_1 + 2c_p \|\nabla \boldsymbol{v}\|_2 \left\| \nabla \left(\frac{\partial \boldsymbol{B}}{\partial t}\right) \right\|_2 \right) + c_p \|\boldsymbol{f}\|_2 \|\nabla \boldsymbol{v}\|_2,$$

which removes the need to bound $\|\nabla \boldsymbol{B}\|_\infty$, at the price of requiring a bound on $\left\| \nabla \left(\frac{\partial \boldsymbol{B}}{\partial t}\right) \right\|_2$, which seems less plausible and even more difficult; trying a similar approach to lemma 3.6 with $\nabla \left(\frac{\partial \boldsymbol{B}}{\partial t}\right)$ does not get us very far, since we no longer have a norm on the left-hand side, but rather we still have an inner product. Equally, taking the spatial gradient of equation (2.6b) seems to be just as fruitless.

Regrettably, therefore, we cannot yet remove assumptions (3.4) and still conclude that the kinetic energy tends to zero; this deserves further investigation.

Chapter 4

Asymptotic behaviour of the magnetic field

In this chapter, having shown that, for both the standard and the simplified models of MHD, the kinetic energy decays to zero, we consider the magnetic field itself. Since we are trying to construct solutions of the Euler equations as magnetic fields which are steady states of the MHD equations, we had better make sure not just that the magnetic energy converges to a finite limit, but that the field itself converges — at least in some weak sense — to a magnetostatic state.

Regrettably, this is not always the case. We demonstrate in section 4.1 that one extra condition on the velocity field will ensure that the magnetic field has a weak limit; however, in section 4.2, we give an example from Núñez (2007) in which this extra condition does not hold, and the magnetic field has a weak limit. In section 4.3, we present a new adaptation of Núñez's counterexample to the reduced system (2.6) of "Stokes" dynamics.

4.1 Conditions to ensure a weak limit of $\boldsymbol{B}$

Here we present the arguments of Núñez (2007) on conditions to ensure that the magnetic field has a weak limit. As we saw in the previous chapter, theorem 3.2 tells us that, for the standard MHD equations (2.2), the velocity decays to zero in the L^2 norm. Taking the energy identity from proposition 3.1 and integrating in time yields

$$\|\boldsymbol{v}(t)\|_2^2 + \|\boldsymbol{B}(t)\|_2^2 = \|\boldsymbol{v}(0)\|_2^2 + \|\boldsymbol{B}(0)\|_2^2 - 2\nu \int_0^t \|\nabla \boldsymbol{v}(s)\|_2^2 \,\mathrm{d}s + 2\int_0^t \langle \boldsymbol{f}, \boldsymbol{v} \rangle \,\mathrm{d}s;$$

hence, taking the limit as $t \to \infty$, we see that

$$\lim_{t\to\infty} \|\boldsymbol{B}(t)\|_2^2 = \|\boldsymbol{v}(0)\|_2^2 + \|\boldsymbol{B}(0)\|_2^2 - 2\nu \int_0^\infty \|\nabla \boldsymbol{v}(s)\|_2^2 \,\mathrm{d}s + 2\int_0^\infty \langle \boldsymbol{f}, \boldsymbol{v}\rangle \,\mathrm{d}s.$$

Thus the magnetic energy $\|\boldsymbol{B}\|_2$ has a limit as $t \to \infty$, so the set $\{\boldsymbol{B}(t) : t \geq 0\}$ in $L^2(\Omega)$ is bounded, and hence weakly precompact, according to the Banach–Alaoglu theorem: that is, for every sequence $t_n \to \infty$, there exists a subsequence t_{n_j} such that $\boldsymbol{B}(t_{n_j})$ converges weakly to some $\boldsymbol{B}_\infty \in L^2(\Omega)$ (that is, for every $\boldsymbol{w} \in L^2(\Omega)$, $\langle \boldsymbol{B}(t_{n_j}), \boldsymbol{w}\rangle \to \langle \boldsymbol{B}_\infty, \boldsymbol{w}\rangle$).

If this limit were unique it would mean that the whole function $\boldsymbol{B}(t)$ tends to $\boldsymbol{B}_\infty$ weakly as $t \to \infty$, which would go some way towards satisfying our demand for the existence of a limit state. However, it turns out that proving the weak limit is unique requires an extra condition. Let t_n and t'_n be two sequences tending to infinity; we want to show that $\boldsymbol{B}(t_n)$ and $\boldsymbol{B}(t'_n)$ will have the same weak limit. To do so, taking the inner product of equation (2.1b) with a test function $\boldsymbol{w} \in C_c^\infty(\Omega)$ (which does not depend on time), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{B}, \boldsymbol{w}\rangle = \left\langle \frac{\partial \boldsymbol{B}}{\partial t}, \boldsymbol{w} \right\rangle = \langle \nabla \times (\boldsymbol{v} \times \boldsymbol{B}), \boldsymbol{w}\rangle.$$

Consider the standard identity

$$\nabla \cdot ((\boldsymbol{v} \times \boldsymbol{B}) \times \boldsymbol{w}) = \boldsymbol{w} \cdot (\nabla \times (\boldsymbol{v} \times \boldsymbol{B})) - (\boldsymbol{v} \times \boldsymbol{B}) \cdot (\nabla \times \boldsymbol{w});$$

since $\boldsymbol{w} = 0$ on $\partial\Omega$ the divergence theorem shows that the (space) integral of the left-hand side of the identity, and hence we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{B}, \boldsymbol{w}\rangle = \langle \boldsymbol{v} \times \boldsymbol{B}, \nabla \times \boldsymbol{w}\rangle.$$

Integrating in time from t_n to t'_n yields

$$\langle \boldsymbol{B}(t'_n) - \boldsymbol{B}(t_n), \boldsymbol{w}\rangle = \int_{t_n}^{t'_n} \langle \boldsymbol{v} \times \boldsymbol{B}, \nabla \times \boldsymbol{w}\rangle \,\mathrm{d}\tau.$$

If $\boldsymbol{v} \in L^1((0,\infty), L^1(\Omega))$, then by applying Hölder's inequality we obtain

$$\langle \boldsymbol{B}(t'_n) - \boldsymbol{B}(t_n), \boldsymbol{w}\rangle \leq \|\nabla \times \boldsymbol{w}\|_\infty \left(\sup_{t\in(0,\infty)} \|\boldsymbol{B}(t)\|_\infty \right) \int_{t_n}^{t'_n} \|\boldsymbol{v}(\tau)\|_1 \,\mathrm{d}\tau.$$

By taking n large enough, the last integral can be made arbitrarily small, because

$\boldsymbol{v} \in L^1((0,\infty), L^1(\Omega))$. Since the set $C_c^\infty(\Omega)$ of test functions is dense in $L^2(\Omega)$, any weak limit of (a subsequence of) $\boldsymbol{B}(t'_n)$ must be the same as the weak limit of $\boldsymbol{B}(t_n)$. Hence the sequential limit is unique, so the whole function $\boldsymbol{B}(t) \to \boldsymbol{B}_\infty$ weakly.

In principle, however, we only know that $\boldsymbol{v} \in L^2((0,\infty), L^2(\Omega))$. If, however, we have two sequences t_n and t'_n which do not separate too much from each other — that is, there exists a fixed constant T such that $|t_n - t'_n| \leq T$ for all n — then we could proceed as follows:

$$\begin{aligned}
\langle \boldsymbol{B}(t_n) - \boldsymbol{B}(t'_n), \boldsymbol{w}\rangle &\leq \|\nabla \times \boldsymbol{w}\|_\infty \int_{t_n}^{t'_n} \|\boldsymbol{v}(\tau)\|_2 \|\boldsymbol{B}(\tau)\|_2 \, \mathrm{d}\tau \\
&\leq \|\nabla \times \boldsymbol{w}\|_\infty \left(\sup_{t\in(0,\infty)} \|\boldsymbol{B}(t)\|_2 \right) \int_{t_n}^{t'_n} \|\boldsymbol{v}(\tau)\|_2 \, \mathrm{d}\tau \\
&\leq \|\nabla \times \boldsymbol{w}\|_\infty \left(\sup_{t\in(0,\infty)} \|\boldsymbol{B}(t)\|_2 \right) \sqrt{T} \left(\int_{t_n}^{t'_n} \|\boldsymbol{v}(\tau)\|_2^2 \, \mathrm{d}\tau \right)^{1/2},
\end{aligned}$$

then, since $\boldsymbol{v} \in L^2((0,\infty), L^2(\Omega))$, the last integral becomes arbitrarily small for sufficiently large n, and we may conclude that $\boldsymbol{B}(t_n)$ and $\boldsymbol{B}(t'_n)$ have the same weak limit. Note that the assumption that there is a constant T is fundamental: we cannot guarantee that sequences at widely varying times have the same limit.

Indeed, Núñez gives some plausible intuitive arguments which suggest that the condition that $\boldsymbol{v} \in L^1((0,\infty), L^1(\Omega))$ is not just sufficient for the existence of a unique weak limit, but also necessary. Nonetheless, not every velocity field which solves equations (2.2) will necessarily be $L^1((0,\infty), L^1(\Omega))$, and so we cannot expect that every solution of equations (2.2) will have a magnetic field with a weak limit.

4.2 Example where $\boldsymbol{B}$ has no weak limit

In this section, we present an example from Núñez (2007) where the magnetic field has no weak limit. Consider a domain of the form $\Omega = U \times (0, R)$, a plane velocity field of the form $\boldsymbol{v} = (v_1(x,y,t), v_2(x,y,t), 0)$, satisfying Dirichlet or periodic boundary conditions in U, and a vertical magnetic field $\boldsymbol{B} = (0, 0, b(x,y,t))$. Notice that all the boundary conditions in (2.3) are satisfied, since $\boldsymbol{B} \cdot \boldsymbol{v} = 0$ everywhere, and $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = \boldsymbol{0}$ in the horizontal boundaries $U \times \{0, R\}$.

Calculations reveal that $(\boldsymbol{B} \cdot \nabla)\boldsymbol{v} = 0$,and so the induction equation for $\boldsymbol{B}$ simply reduces to

$$\frac{\partial b}{\partial t} + (\boldsymbol{v} \cdot \nabla) b = 0.$$

Furthermore, $(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} = \boldsymbol{0}$, so the momentum equation for $\boldsymbol{v}$ simply reduces to

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \nabla p_* = \boldsymbol{f},$$

the Navier–Stokes equations but with the total pressure $p_* = p + b^2/2$. Since the velocity field is two-dimensional, we are assured that a unique solution exists for all time: this may be found by projecting into the space of divergence-free functions appropriate to the boundary conditions (see, for example, Temam (2001)). This projection kills the pressure gradient, and therefore we can effectively ignore the ∇p_* term in all that follows.

If we could find an example in this setup where $\boldsymbol{v} \notin L^1((0,\infty), L^1(\Omega))$, then we will be able to find a magnetic field without limit as $t \to \infty$. While (in a bounded domain) unforced solutions of the Navier–Stokes equations decay as rapidly as one could wish, in the presence of a nonzero forcing this is not necessarily the case. We will take a nonzero — but decaying — forcing $\boldsymbol{f} \in L^2((0,\infty), L^2(\Omega))$, which will be sufficient to obtain an example.

So, let $U = (0, 2\pi) \times (0, 2\pi)$, and take functions of the form $\boldsymbol{v} = (v(y,t), 0, 0)$, $\boldsymbol{B} = (0, 0, b(x,y,t))$, $\boldsymbol{f} = (f(y,t), 0, 0)$, which are smooth, periodic, and of mean zero in U. The equations reduce to

$$\frac{\partial v}{\partial t} - \nu\frac{\partial^2 v}{\partial y^2} = f, \tag{4.1a}$$

$$\frac{\partial b}{\partial t} + v\frac{\partial b}{\partial x} = 0, \tag{4.1b}$$

since $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = 0$. Letting $W(y,t) = \int_0^t v(y,s)\,\mathrm{d}s$ be a primitive of v, we see that, given the initial condition

$$b(x,y,0) = \cos x,$$

the solution of (4.1b) is

$$b(x,y,t) = \cos(x - W(y,t)),$$

since

$$\frac{\partial b}{\partial t} = -\sin(x - W(y,t))\cdot\frac{\partial W}{\partial t} = -v\sin(x - W(y,t)),$$

and

$$\frac{\partial b}{\partial x} = -\sin(x - W(y,t)).$$

Let us take a velocity of the form

$$v(y,t) = \lambda(t)\phi(y),$$

where ϕ is a smooth periodic function of mean zero such that it takes some non-zero constant value (e.g., $\phi = 1$) on some subinterval I of $(0, 2\pi)$, and λ is a positive function such that $\lambda, \lambda' \in L^2(0,\infty)$ but $\lambda \notin L^1(0,\infty)$. (One such example is $\lambda(t) = \frac{1}{1+t}$.) From this we see that

$$\frac{\partial v}{\partial t} = \lambda'(t)\phi(y),$$

and

$$\frac{\partial^2 v}{\partial y^2} = \lambda(t)\phi_{yy}(y).$$

So, in order to solve (4.1a), we set

$$f(y,t) = \lambda'(t)\phi(y) - \nu\lambda(t)\phi_{yy}(y).$$

Thanks to the restrictions on λ, we see that $f \in L^2((0,\infty), L^2(\Omega))$, but that $f \notin L^1((0,\infty), L^1(\Omega))$, and the solution of the momentum equation (4.1a) with initial condition $v(y,0) = \lambda(0)\phi(y)$ is nothing but $v(y,t) = \lambda(t)\phi(y)$. In this case, $W(y,t) = \Lambda(t)\phi(y)$, where $\Lambda(t) = \int_0^s \lambda(s)\,\mathrm{d}s$ is a primitive of λ (so, in our example, $\Lambda(t) = \log(1+t)$). Since $\lambda \notin L^1(0,\infty)$, $\Lambda(t) \to \infty$ as $t \to \infty$, though the convergence may be slow.

We now show that the magnetic field

$$b(x,y,t) = \cos(x - W(y,t)) = \cos(x - \Lambda(t)\phi(y))$$

has no weak limit. First, note that, using the change of variables $\xi = x - \Lambda(t)\phi(y)$ (and $y = y$), we see that

$$\begin{aligned}
\iint_U b^2(x,y,t)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi}\int_0^{2\pi} \cos^2(x - \Lambda(t)\phi(y))\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2\pi}\int_{-\Lambda(t)\phi(y)}^{2\pi-\Lambda(t)\phi(y)} \cos^2\xi\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \int_0^{2\pi}\int_0^{2\pi} \cos^2\xi\,\mathrm{d}\xi\,\mathrm{d}y \\
&= 2\pi^2,
\end{aligned}$$

since the integral of a 2π-periodic function over an interval of length 2π is always the same. Hence the magnetic energy $\|\boldsymbol{B}\|_2 = \|b\|_2 = \pi\sqrt{2}$ is constant in time, which obviously has a limit as $t \to \infty$. However, we assert that the field itself has no limit, even in a weak sense. Since

$$\cos(x - W(y,t)) = \cos x \cos W(y,t) + \sin x \sin W(y,t),$$

it suffices to show that neither $\cos W(y,t)$ nor $\sin W(y,t)$ have weak limits in $L^2(\Omega)$ as $t \to \infty$. To do so, take a test function ψ whose support is contained in the interval I; there, $\phi = 1$, and hence $W(y,t) = \Lambda(t)$ in I. Therefore

$$\int_0^{2\pi} (\cos W(y,t))\psi(y)\,\mathrm{d}y = \int_I (\cos W(y,t))\psi(y)\,\mathrm{d}y = \cos(\Lambda(t)) \cdot \int_I \psi(y)\,\mathrm{d}y.$$

As $t \to \infty$, $\Lambda(t) \to \infty$, hence $\cos(\Lambda(t))$ (and $\sin(\Lambda(t))$) oscillates ad infinitum, and never tends to a limit; since ψ need not have mean zero, the previous integral has no limit, and thus the magnetic field $\boldsymbol{B}$ can have no weak limit.

This example depends crucially on the presence of the forcing term $\boldsymbol{f}$: with this particular configuration of $\boldsymbol{v}$, $\boldsymbol{B}$ and Ω, zero forcing yields exponential decay of the velocity, and hence uniqueness of weak limits of $\boldsymbol{B}$. Nonetheless, since the forcing itself decays to zero it seems plausible that an example in which $\boldsymbol{f} = 0$ but $v \notin L^1((0,\infty), L^1(\Omega))$ can be found.

4.3 The situation under "Stokes" dynamics

Here, we present a new adaptation of the example of Núñez to the situation of "Stokes" dynamics under equations (2.6). Using proposition 3.3, integrating in time and taking the limit as $t \to \infty$ yields

$$\lim_{t\to\infty} \|\boldsymbol{B}(t)\|_2^2 = \|\boldsymbol{B}(0)\|_2^2 - 2\nu \int_0^\infty \|\nabla \boldsymbol{v}(s)\|_2^2\,\mathrm{d}s + 2\int_0^\infty \langle \boldsymbol{f}, \boldsymbol{v}\rangle\,\mathrm{d}s.$$

Thus the magnetic energy $\|\boldsymbol{B}\|_2$ has a limit as $t \to \infty$, so the set $\{\boldsymbol{B}(t) : t \geq 0\}$ in $L^2(\Omega)$ is bounded; therefore, the same arguments as in section 4.1 shows that $\boldsymbol{v} \in L^1((0,\infty), L^1(\Omega))$ suffices to ensure that the magnetic field solving equations (2.6) has a weak limit.

However, by adapting the construction of Núñez, as discussed in section 4.2, we can show that even the simplified system of equations (2.6) can have a magnetic field with no weak limit. Again, consider a domain of the form $\Omega = U \times (0, R)$, a plane velocity field of the form $\boldsymbol{v} = (v_1(x,y,t), v_2(x,y,t), 0)$, satisfying Dirichlet or periodic boundary conditions in U, and a vertical magnetic field $\boldsymbol{B} = (0, 0, b(x,y,t))$.

Again, the induction equation for $\boldsymbol{B}$ reduces to

$$\frac{\partial b}{\partial t} + (\boldsymbol{v} \cdot \nabla) b = 0.$$

and the momentum equation for $\boldsymbol{v}$ reduces to

$$-\nu \Delta \boldsymbol{v} + \nabla p_* = \boldsymbol{f},$$

the stationary Stokes equations but with the total pressure $p_* = p + b^2/2$. Since $\boldsymbol{f}$ depends on t, we may solve this for each time by projecting onto the space of divergence-free functions, which again kills the pressure gradient (see, for example, Temam (2001)).

Now, let $U = (0, 2\pi) \times (0, 2\pi)$, and take functions of the form $\boldsymbol{v} = (v(y,t), 0, 0)$, $\boldsymbol{B} = (0, 0, b(x,y,t))$, $\boldsymbol{f} = (f(y,t), 0, 0)$ which are smooth, periodic, and of mean zero in U. The equations reduce to

$$-\nu \frac{\partial^2 v}{\partial y^2} = f, \tag{4.2a}$$

$$\frac{\partial b}{\partial t} + v \frac{\partial b}{\partial x} = 0. \tag{4.2b}$$

Letting $W(y,t) = \int_0^t v(y,s)\,\mathrm{d}s$ be a primitive of v, we see that, given the initial conditions

$$b(x,y,0) = \cos x,$$

the solution of (4.2b) is again

$$b(x,y,t) = \cos(x - W(y,t)).$$

We take a velocity of the form

$$v(y,t) = \lambda(t)\phi(y),$$

where ϕ is a smooth periodic function of mean zero such that it takes some non-zero constant value (e.g., $\phi = 1$) on some subinterval I of $(0, 2\pi)$, and λ is a positive function such that $\lambda, \lambda' \in L^2(0,\infty)$ but $\lambda \notin L^1(0,\infty)$. (One such example is $\lambda(t) = \frac{1}{1+t}$, where $\lambda'(t) = \log(1+t)$.) From this we see that

$$\frac{\partial^2 v}{\partial y^2} = \lambda(t)\phi_{yy}(y).$$

So, in order to solve (4.2a), this time we set

$$f(y,t) = -\nu\lambda(t)\phi_{yy}(y).$$

Thanks to the restrictions on λ, we see that $f \in L^2((0,\infty), L^2(\Omega))$, but that $f \notin L^1((0,\infty), L^1(\Omega))$, and the solution of the momentum equation (4.2a) with initial condition $v(y,0) = \lambda(0)\phi(y)$ is nothing but $v(y,t) = \lambda(t)\phi(y)$. Since we have the same velocity $\boldsymbol{v}$, this yields the same magnetic field $\boldsymbol{B}$ as in section 4.2, and thus again the magnetic field can have no weak limit.

Chapter 5

Conclusion

It has long been assumed that evolving a viscous, non-resistive, incompressible fluid under the model of magnetohydrodynamics, as introduced in chapter 2, will lead to a static plasma, in which the magnetic field satisfies the stationary Euler equations, as first suggested by Moffatt (1985). More recently, Moffatt (2009) has suggested that any model for magnetic relaxation, as long as it dissipates energy, should have similar limiting behaviour.

In this dissertation, we studied the work of Núñez (2007) in analysing the classical model of MHD in equations (2.2), and then adapted it to the reduced model in equations (2.6) suggested by Moffatt. We saw in chapters 3 and 4 that there are a number of issues to be contended with.

Firstly, we would like that the limit actually is static, in the sense that $\boldsymbol{v} \to 0$ as $t \to \infty$: in section 3.1 we presented Núñez's proof that, even in the presence of a forcing that, indeed, $\boldsymbol{v} \to 0$ for the full MHD model (see theorem 3.2).

Adapting this result to the reduced model of equations (2.6) required an entirely new approach, as seen in section 3.2, and necessitated additional hypotheses on various derivatives of the magnetic field; see assumptions 3.4. It would be rather more satisfying if we could eliminate at least one of these extra hypotheses — the hypothesis on $\frac{\partial^2 \boldsymbol{B}}{\partial t^2}$ being particularly unreasonable — but further work is needed to do so.

The second issue to be contended with is whether the magnetic field actually has a limit as $t \to \infty$, and whether such limits are indeed stationary solutions of the Euler equations. In section 4.1, we saw that, while the existence of weak sequential limits is always assured, to ensure that the magnetic field has a unique weak limit requires an additional condition on the velocity, namely that $\boldsymbol{v} \in L^1((0, \infty), L^1(\Omega))$.

There is a good reason for the requirement of an extra condition on the velocity:

in section 4.2, we saw an example due to Núñez (2007) where there is no weak limit of the magnetic field as $t \to \infty$. While this example does require an artificial forcing term, since the forcing decays to zero as $t \to \infty$, it seems possible that examples also occur in its absence. We saw in section 4.3 that a simple modification of this example applies to the reduced system (2.6).

There are a number of avenues for further research in this topic. Firstly, more work is needed to reduce the number of assumptions required to show that the limit is static for the reduced system (2.6); while assumptions (3.4) are not too demanding, they are nonetheless more than we would like.

Secondly, it would be particularly interesting if an example could be found in which the magnetic field has no weak limit even in the absence of forcing; it is clear, however, that a new approach (probably one in which the magnetic field is not simply transported as a passive scalar) is required.

Thirdly, no attempt has been made to ensure, even if the magnetic field has a weak limit, that the weak limit really does solve the stationary Euler equations: it would be necessary to prove that $\frac{\partial \boldsymbol{v}}{\partial t}$ and $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ tend to zero as $t \to \infty$. All three of these problems, however, seem relatively intractable.

More generally, however, it is clear that, while Moffatt's approach to obtaining stationary solutions of the Euler equations with a given topology is somewhat lacking in rigour, the use of topological ideas in fluid dynamics is broader and more useful that it seems at first: the recent work of Córdoba et al. (2004), as contrasted with the classic result of Beale et al. (1984), is an indication of the power of such approaches, when done rigorously. In this case, however, we must conclude that the approach of magnetic relaxation to studying stationary solutions of the Euler equations is not as easy as it first seems.

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