Research Study Group: Data Assimilation for the 2D Navier–Stokes Equations

Charles Brett, Andrew Lam, David McCormick, Michael Scott
Mathematics and Statistics Centre for Doctoral Training (MASDOC), University of Warwick

Introduction

- **Data assimilation** is a technique used to make predictions of the future state of a system using information from both a model of the system and observations.
- It has many applications in fields such as geoscience, weather forecasting and hydrology. We focus on the simple case of applying 3DVar to 2D fluid flow with periodic boundary conditions, as there are still open questions here.
- Let $\{u_n\}$ represent the state of the system at timestep n. Suppose we have **model** which can predict the state of the system at each timestep, and a sequence of **noisy observations** $\{y_n\}$.
- In an ideal world, the model and the observations agree; how do we reconcile the model and the data when they do not agree?
- **Goal**: **Estimate** u_n given $\{y_1, \cdots, y_n\}$, and only uncertain knowledge of u_0.

Notation

- Let X and Y be Hilbert spaces. If L is a self-adjoint, positive-definite operator on either X or Y, we define the L-inner product and L-norm as

$$\langle \cdot, \cdot \rangle_L := \langle \cdot, L^{-1} \cdot \rangle, \quad \| \cdot \|_L := \| L^{-1/2} \cdot \|.$$

- Let $\Psi \colon X \to X$ be the operator which **evolves** a system at time nh to the system at time $(n+1)h$, with initial condition $u_0 \in X$. We define $\{u_n\} \subset X$ by

$$u_{n+1} := \Psi(u_n).$$

- Let $\{\bar{\xi}_{n+1}\}$ denote a sequence of i.i.d. Y-valued random variables representing the **observation error**. We define the observations $\{y_n\} \subset Y$ by

$$y_{n+1} := u_{n+1} + \bar{\xi}_{n+1}.$$

3DVar

- Let $\hat{u}_n$ denote our **estimate** of u_n. Given $\hat{u}_n$ we define $\hat{u}_{n+1}$ by

$$\hat{u}_{n+1} := \operatorname*{arg\,min}_{u \in X} \left(\frac{1}{2} \| u - y_{n+1} \|_\Gamma^2 + \frac{1}{2} \| u - \Psi(\hat{u}_n) \|_C^2 \right).$$

- This encodes our belief that the **estimators should be guided by the observations** (the $\frac{1}{2}\|u - y_{n+1}\|_\Gamma^2$ term) **and the model** (the $\frac{1}{2}\|u - \Psi(\hat{u}_n)\|_C^2$ term).
- The estimator $\hat{u}_{n+1}$ is the solution to

$$(C^{-1} + \Gamma^{-1}) \hat{u}_{n+1} = C^{-1} \Psi(\hat{u}_n) + \Gamma^{-1} y_{n+1}. \tag{1}$$

The forward model

- We consider the 2D **Navier-Stokes equations** (NSE) on the torus $\mathbb{T}^2$:

$$\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = f \quad \text{on } \mathbb{T}^2 \times (0, \infty) \tag{2a}$$
$$\nabla \cdot u = 0 \quad \text{on } \mathbb{T}^2 \times (0, \infty) \tag{2b}$$
$$u(x, 0) = u_0(x) \quad \text{on } \mathbb{T}^2 \tag{2c}$$

- We work in the space

$$H := \left\{ u \in (L^2(\mathbb{T}^2))^2 : \nabla \cdot u = 0, \int_{\mathbb{T}^2} u(x) \, \mathrm{d}x = 0 \right\},$$

which projects the NSE into their divergence-free form, and use Fourier analysis. The NSE can be simplified to:

$$\frac{du}{dt} + \nu A u + B(u, u) = f, \quad u(0) = u_0,$$

where A is the projection of $-\Delta$ onto H, $B(u,u)$ is the projection of $u \cdot \nabla u$ onto H, and, abusing notation, f is the forcing projected onto H.

Analysis

We consider the case when C^{-1} **and** Γ^{-1} **are both powers of** A, i.e.,

$$C^{-1} = \frac{1}{\delta^2} A^\gamma, \quad \Gamma^{-1} = \frac{1}{\varepsilon^2} A^\beta.$$

Substituting this into equation (1) and multiplying through by $\varepsilon^2 A^{-\beta}$ we obtain

$$\hat{u}_{n+1} = \eta^2 (I + \eta^2 A^\alpha)^{-1} A^\alpha \Psi(\hat{u}_n) + (I + \eta^2 A^\alpha)^{-1} y_{n+1}, \tag{3}$$

where $\alpha = \gamma - \beta$ and $\eta = \varepsilon / \delta$.

Effects of α

- Equation (3) shows that $\hat{u}_{n+1}$ is a **convex combination** of the dynamics $\Psi(\hat{u}_n)$ and the observations y_{n+1}.
- α controls the **weightings** we put on each of these components:
  - If $\alpha = -1$ the observations have a larger weight than the dynamics for large wavenumbers.
  - If $\alpha = 1$ the dynamics have a larger weight than the observations for large wavenumbers.
- We prove **two theorems** for the case $\alpha = -1$, and perform **numerical simulations** for $\alpha = \pm 1$.

Theorem 1 (Convergence of estimators with same observations)

Suppose the true solution u_n and the observation noise $\bar{\xi}_n$ are uniformly bounded in $H^s := \mathcal{D}(A^{s/2}) \subset H$. Given initial estimators $\hat{u}_0, \hat{v}_0 \in H^s$, for all sufficiently small η, there exists a constant $\lambda < 1$ such that, for all $n \in \mathbb{N}$,

$$\| \hat{u}_n - \hat{v}_n \|_s \leq \lambda^n \| \hat{u}_0 - \hat{v}_0 \|_s.$$

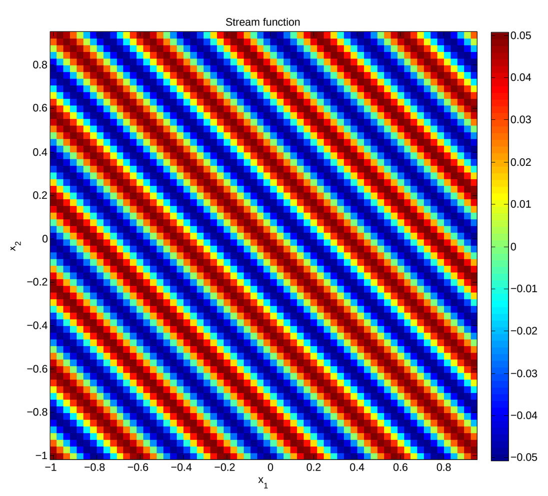
Theorem 2 (Convergence of estimators to the truth)

Suppose the truth u_n and the observation noise $\bar{\xi}_n$ are uniformly bounded in H^s; set $M := \sup \| \bar{\xi}_n \|_s$. Given initial estimators $\hat{u}_0, \hat{v}_0 \in H^s$, for all sufficiently small η, there exists a constant $\lambda < 1$ such that, for all $n \in \mathbb{N}$,

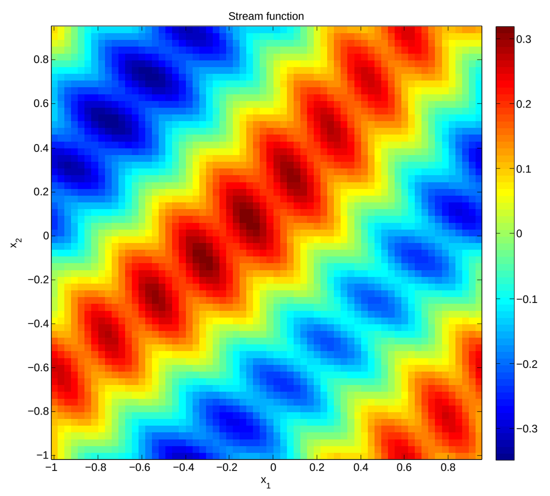
$$\| u_n - \hat{u}_n \|_s \leq \lambda^n \| u_0 - \hat{u}_0 \|_s + \frac{M}{1 - \lambda}.$$

Numerical simulations

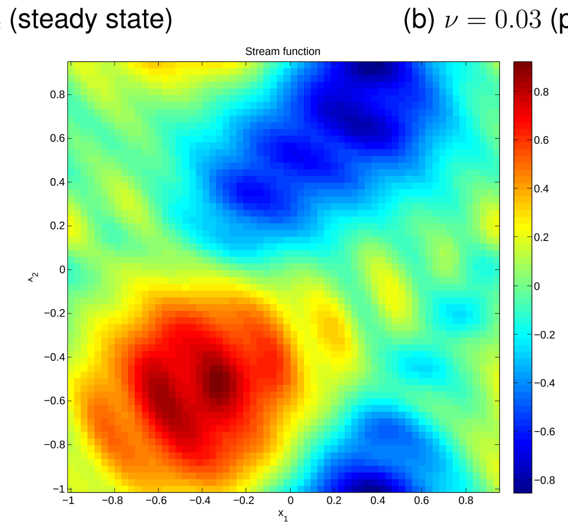
We used MATLAB, with code written by Kody Law, to simulate the NSE with different viscosity values ν. Here we exhibit typical stream functions from numerical simulations to show the limiting behaviour in three different regimes: steady, periodic, and turbulent dynamics.



(a) $\nu = 0.04$ (steady state)



(b) $\nu = 0.03$ (periodic state)

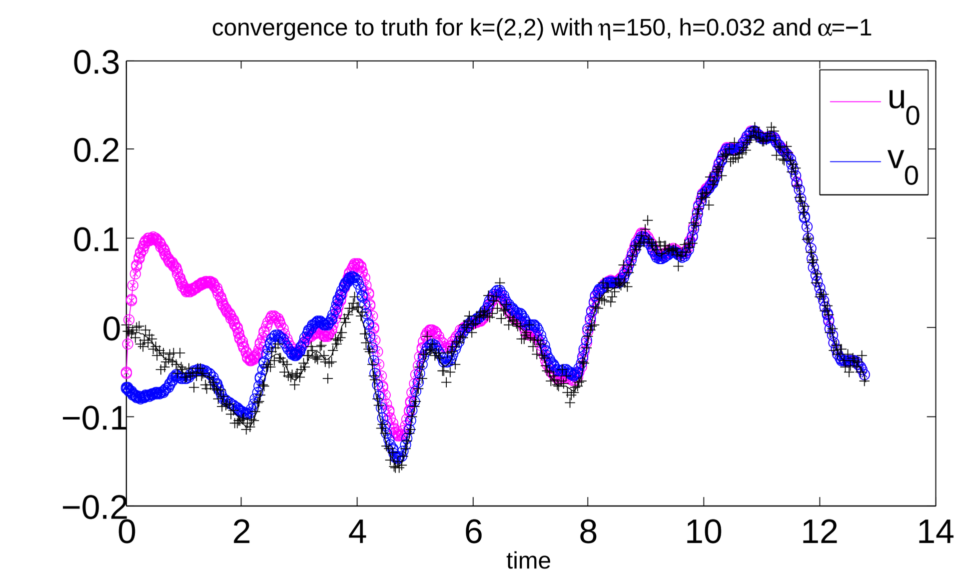


(c) $\nu = 0.016$ (turbulent state)

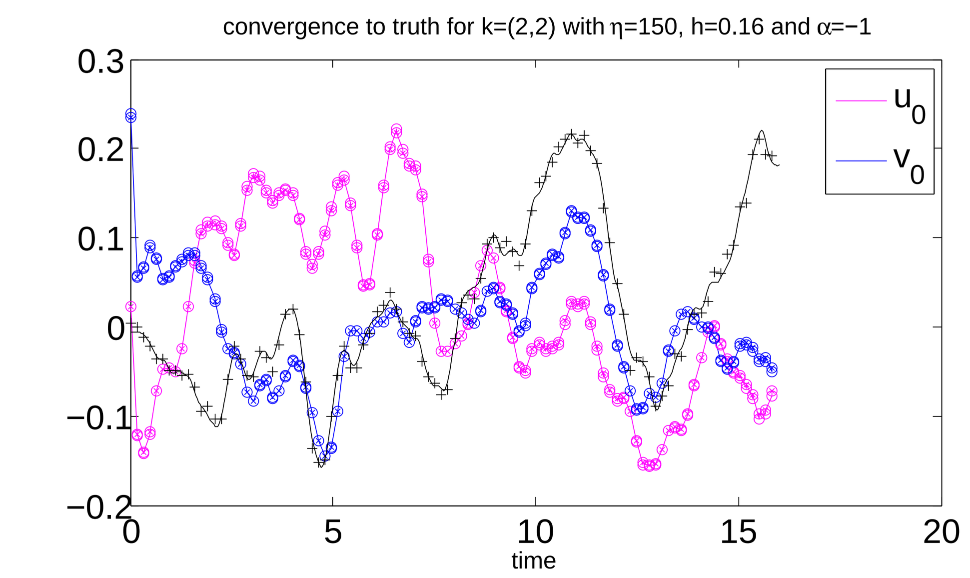
Figure 1: Snapshots of the stream function of the NSE at one point in time, with different viscosity values.

Numerical results for $\alpha = -1$

- Here we present numerical simulations for 3DVar in the turbulent regime, with $\nu = 0.016$, in the case $\alpha = -1$.
- For frequent observations (i.e. for small h), we observe that two estimators with different initial conditions become asymptotically equal to each other and to the truth, even for large η (figure 2(a)).
- However, if we increase h from 0.032 to 0.16, we fail to get convergence unless η is made smaller (figure 2(b)).
- The spatial L^2 difference between two estimators with different initial conditions converges exponentially to zero (figure 3(a)), which confirms theorem 1.
- After a time, the spatial L^2 error between an estimator and the truth remains below a small threshold (figure 3(b)), which confirms theorem 2.

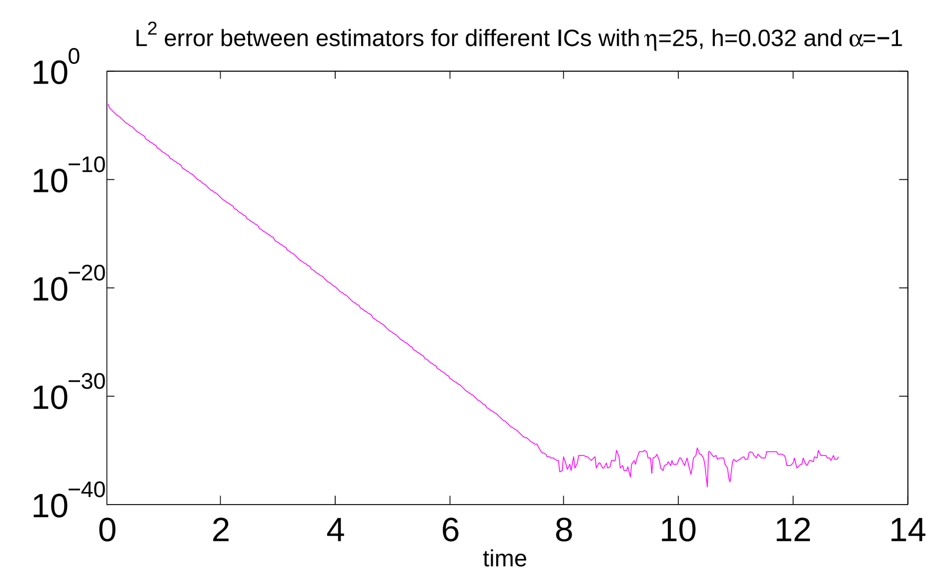


(a) The two estimators are asymptotically equal: for sufficiently large times they are almost indistinguishable.

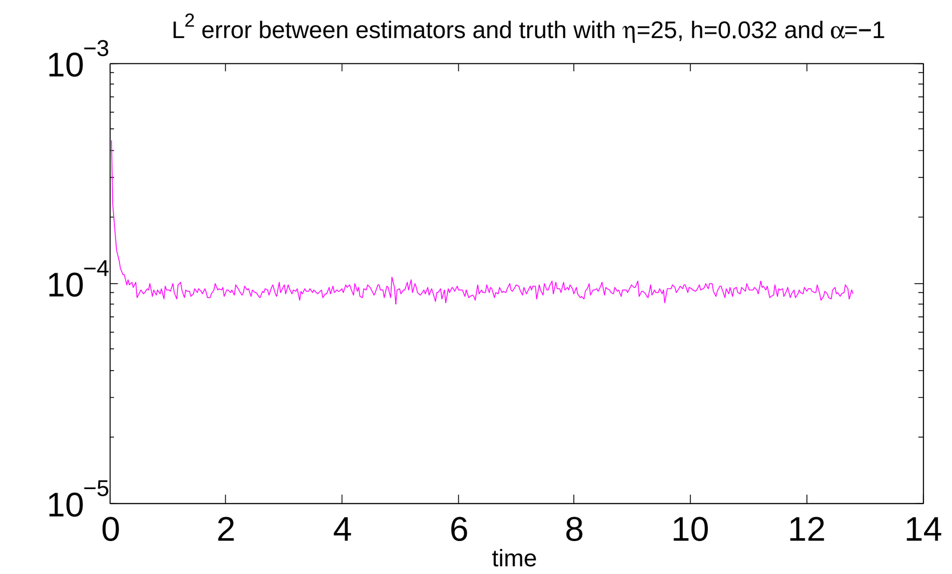


(b) Infrequent observations prevent the estimators locking onto the truth.

Figure 2: Estimators exposed to the same observations but with two different initial conditions u_0 and v_0. ⊗ denotes the estimators $\hat{u}_n$ and ○ denotes $\Psi(\hat{u}_n)$, the estimators evolved forward by the model one time step. $+$ denote the observations y_{n+1} that are being assimilated, and the black line is the truth.



(a) The squared L^2 difference between estimators for two different initial conditions decreases exponentially; eventually we are limited by machine precision.

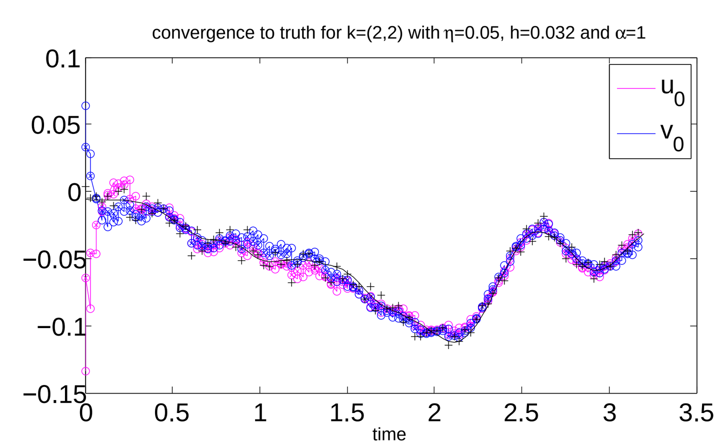


(b) A plot of $e_n^2 := \|\hat{u}_n - u_n\|^2$ against time. The error decreases rapidly to around 10^{-4}.

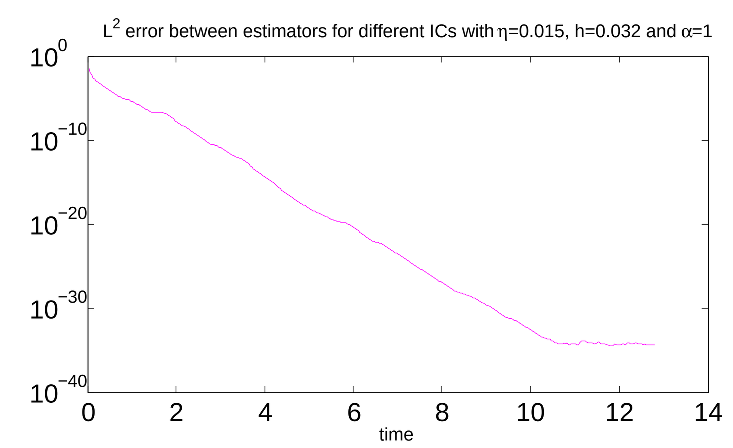
Figure 3: L^2 error plots for $\alpha = -1$.

Numerical predictions for $\alpha = 1$

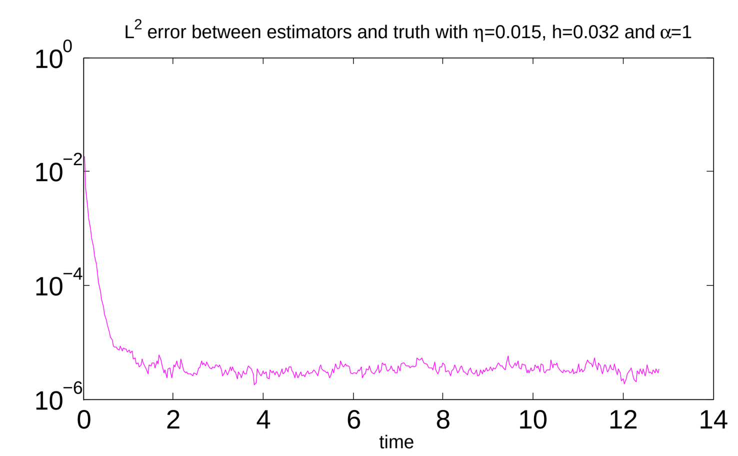
- Here we present numerical simulations for 3DVar in the turbulent regime, with $\nu = 0.016$, in the case $\alpha = 1$.
- For well-chosen parameter values (e.g. $\eta = 0.05$, $h = 0.032$) two estimators with different initial conditions will converge together and to the truth (figure 4(a)). However η and h have to be chosen more carefully than in the $\alpha = -1$ case to get convergence.
- The spatial L^2 error between two estimators with different initial conditions will decrease exponentially to zero (figure 4(b)).
- After a time the spatial L^2 error between an estimator and the truth will remain below a small threshold (figure 4(c)). This threshold appears to be smaller than in the $\alpha = -1$ case.



(a) The estimators become asymptotically equal, as in the $\alpha = -1$ case.



(b) As before, the square of the L^2 difference between the estimators decreases exponentially.



(c) We get a lower bound on the error $e_n^2 := \|\hat{u}_n - u_n\|^2$ than in the $\alpha = -1$ case.

Figure 4: Estimators and L^2 error plots for $\alpha = 1$.