Stationary Euler flows and ideal magnetohydrodynamics

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Introduction

- The Navier–Stokes equations and the Euler equations
- Regularity and stationary solutions
- Magnetohydrodynamics and magnetic relaxation
- Limits of the velocity field and the magnetic field
- “Stokes” dynamics
- Two dimensions?

The Navier–Stokes equations

Given a smooth bounded domain $\Omega \subset \mathbb{R}^n$ (for $n = 2$ or 3), the Navier–Stokes equations for Ω are:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \nabla p + \boldsymbol{f}, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{1b}$$

Here:

- $\boldsymbol{u} \colon \Omega \times [0, \infty) \to \mathbb{R}^n$ is the (time-dependent) velocity field,
- $p \colon \Omega \times [0, \infty) \to \mathbb{R}$ is the (time-dependent) pressure,
- $\boldsymbol{f} \colon \Omega \times [0, \infty) \to \mathbb{R}^n$ is the (time-dependent) forcing, and
- ν is the fluid viscosity.

The Navier–Stokes equations

Rather than consider specific boundary conditions, we insist only that the following boundary integrals vanish:

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \, \mathrm{d}S = \int_{\partial\Omega} |\boldsymbol{u}|^2 \, (\boldsymbol{u} \cdot \boldsymbol{n}) \, \mathrm{d}S = \int_{\partial\Omega} p(\boldsymbol{u} \cdot \boldsymbol{n}) \, \mathrm{d}S = 0. \quad (2)$$

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Energy evolution law

A smooth solution $\boldsymbol{u}$ of the Navier–Stokes equations (1), subject to boundary conditions (2), satisfies

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{u}\|_2^2 = -\nu \|\nabla \boldsymbol{u}\|_2^2 + \langle \boldsymbol{f}, \boldsymbol{u} \rangle.$$

The Euler equations

The Euler equations are the special case of the Navier–Stokes equations when the viscosity $\nu = 0$:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \boldsymbol{f}, \tag{3a}$$

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Energy conservation

A smooth solution $\boldsymbol{u}$ of the Euler equations (3), subject to boundary conditions (2), satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_2^2 = \langle \boldsymbol{f}, \boldsymbol{u} \rangle.$$

Regularity of solutions of the Euler equations

No global existence or uniqueness results for the Euler equations in 3D. The most important "conditional regularity" theorem is as follows:

Beale–Kato–Majda Theorem (1984)

There exists a global solution of the 3D Euler equations $\boldsymbol{u} \in C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-1})$ for $s \geq 3$ if, for every $T > 0$,

$$\int_0^T \|\nabla \times \boldsymbol{u}(\tau)\|_\infty \, \mathrm{d}\tau < \infty.$$

Stationary solutions of the Euler equations

We study the long-time behaviour of the Euler equations by considering stationary solutions of the Euler equations (3), which satisfy

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0, \quad (4a)$$

$$\nabla \cdot \boldsymbol{u} = 0. \quad (4b)$$

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In particular, we consider the approach of magnetic relaxation: formally, magnetic fields arising as stationary solutions of the magnetohydrodynamics (MHD) equations ought to solve the stationary Euler equations.

Magnetohydrodymanics

The MHD equations for a perfectly conducting fluid in a domain $\Omega \subset \mathbb{R}^3$ can be written in the following form:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p_* = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} + \boldsymbol{f}, \tag{5a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}, \tag{5b}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{5c}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{5d}$$

Here:

- $\boldsymbol{u}, \boldsymbol{B} \colon \Omega \times [0, \infty) \to \mathbb{R}^n$ are the velocity and magnetic fields;
- $p \colon \Omega \times [0, \infty) \to \mathbb{R}$ is the pressure, and $p_* = p + \frac{1}{2} |\boldsymbol{B}|^2$;
- $\boldsymbol{f} \colon \Omega \times [0, \infty) \to \mathbb{R}^n$ is the forcing, and
- ν is the fluid viscosity.

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Again we assume the following boundary integrals vanish:

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \,\mathrm{d}S = \int_{\partial\Omega} (\boldsymbol{B} \cdot \boldsymbol{u})(\boldsymbol{B} \cdot \boldsymbol{n}) \,\mathrm{d}S = \int_{\partial\Omega} |\boldsymbol{u}|^2 \,(\boldsymbol{u} \cdot \boldsymbol{n}) \,\mathrm{d}S = \int_{\partial\Omega} |\boldsymbol{B}|^2 \,(\boldsymbol{u} \cdot \boldsymbol{n}) \,\mathrm{d}S = \int_{\partial\Omega} p(\boldsymbol{u} \cdot \boldsymbol{n}) \,\mathrm{d}S = 0. \tag{6}$$

Energy evolution law

A smooth solution $\boldsymbol{u}$, $\boldsymbol{B}$ of equations (5), subject to boundary conditions (6), satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}(t)\|_2^2 + \|\boldsymbol{B}(t)\|_2^2\right) = -\nu\|\nabla\boldsymbol{u}\|_2^2 + \langle \boldsymbol{f}, \boldsymbol{u}\rangle.$$

The stationary MHD equations

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- We thus expect that the fluid should settle down to an equilibrium state, and that $\boldsymbol{u} \to 0$ in some sense as $t \to \infty$.
- Formally, we should thus be left with a stationary magnetic field, and equations (5) should reduce to the following equations for $\boldsymbol{B}$:

$$(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla p_* = 0, \tag{7a}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{7b}$$

This "analogy" is originally due to Moffatt (1985).

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Kinetic energy decay

- Energy evolution law tells us that $\nabla \boldsymbol{u} \in L^2((0,\infty), L^2(\Omega))$, but that doesn't guarantee that $\|\boldsymbol{u}(t)\|_2 \to 0$ as $t \to \infty$.
- Núñez (2007) showed that the kinetic energy does indeed decay to zero:

Kinetic energy decay (Núñez, 2007)

Suppose we have a smooth solution $\boldsymbol{u}$, $\boldsymbol{B}$ of equations (5), subject to boundary conditions (6), such that $\|\boldsymbol{B}(t)\|_\infty \leq M$ for all $t > 0$. If $\boldsymbol{f} \in L^2((0,\infty), L^2(\Omega))$, then

$$\lim_{t\to\infty} \|\boldsymbol{u}(t)\|_2 = 0.$$

Sketch of proof

A Gronwall-type argument shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{rt}\|\boldsymbol{u}(t)\|_2^2) \leq 2e^{rt}\|\nabla\boldsymbol{u}(t)\|_2(\underbrace{\|\boldsymbol{B}(t)\|_\infty\|\boldsymbol{B}(t)\|_2}_{\leq M} + c_p\|\boldsymbol{f}(t)\|_2)$$

so integrating between s and t yields

$$\begin{aligned}
&\|\boldsymbol{u}(t)\|_2^2 - e^{r(s-t)}\|\boldsymbol{u}(s)\|_2^2 \\
&\qquad \leq 2Me^{-rt}\left(\int_s^t e^{2r\tau}\,\mathrm{d}\tau\right)^{1/2}\left(\int_s^t \|\nabla\boldsymbol{u}(\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/2} \\
&\qquad\qquad + 2c_p\left(\int_s^t \|\boldsymbol{f}(\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/2}\left(\int_s^t \|\nabla\boldsymbol{u}(\tau)\|_2^2\,\mathrm{d}\tau\right)^{1/2}.
\end{aligned}$$

Weak limits of the magnetic field

- As $\|\boldsymbol{u}(t)\|_2 \to 0$ as $t \to \infty$, the energy evolution law tells us that $\|\boldsymbol{B}(t)\|_2$ converges to a limit as $t \to \infty$.
- Hence $\{\boldsymbol{B}(t) : t \geq 0\}$ is weakly precompact (by the Banach–Alaoglu theorem): for every sequence $t_n \to \infty$, there exists a subsequence $t_{n_j} \to \infty$ such that $\boldsymbol{B}(t_{n_j}) \rightharpoonup \boldsymbol{B}_\infty \in L^2(\Omega)$.
- We would like the limit to be unique: if $t_n, t'_n \to \infty$, then we want $\boldsymbol{B}(t_n)$ and $\boldsymbol{B}(t'_n)$ to tend to the same weak limit; if so, the whole function $\boldsymbol{B}(t) \rightharpoonup \boldsymbol{B}_\infty$.

Weak limits of the magnetic field

If $\boldsymbol{u} \in L^1((0,\infty), L^1(\Omega))$, then weak limits *are* unique: for a time-independent test function $\boldsymbol{w} \in C_c^\infty(\Omega)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{B}, \boldsymbol{w}\rangle = \left\langle \frac{\partial \boldsymbol{B}}{\partial t}, \boldsymbol{w} \right\rangle = \langle \nabla \times (\boldsymbol{u} \times \boldsymbol{B}), \boldsymbol{w}\rangle = \langle \boldsymbol{u} \times \boldsymbol{B}, \nabla \times \boldsymbol{w}\rangle.$$

Integrating in time from t_n to t_n' yields

$$\langle \boldsymbol{B}(t_n') - \boldsymbol{B}(t_n), \boldsymbol{w}\rangle = \int_{t_n}^{t_n'} \langle \boldsymbol{u} \times \boldsymbol{B}, \nabla \times \boldsymbol{w}\rangle \,\mathrm{d}\tau.$$

If $\boldsymbol{u} \in L^1((0,\infty), L^1(\Omega))$, then by Hölder's inequality,

$$\langle \boldsymbol{B}(t_n') - \boldsymbol{B}(t_n), \boldsymbol{w}\rangle \leq \|\nabla \times \boldsymbol{w}\|_\infty \left(\sup_{t\in(0,\infty)} \|\boldsymbol{B}(t)\|_\infty \right) \int_{t_n}^{t_n'} \|\boldsymbol{u}(\tau)\|_1 \,\mathrm{d}\tau.$$

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- Consider the following example due to Núñez (2007): consider $\Omega = U \times (0,R)$, a plane velocity field $\boldsymbol{u} = (u_1(x,y,t), u_2(x,y,t), 0)$, and a vertical magnetic field $\boldsymbol{B} = (0,0,b(x,y,t))$. The equations reduce to

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p_* = \boldsymbol{f},$$

$$\frac{\partial b}{\partial t} + (\boldsymbol{u}\cdot\nabla)b = 0.$$

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- So b is just transported around by $\boldsymbol{u}$, and $\boldsymbol{u}$ solves the 2D Navier–Stokes equations.
- If $\boldsymbol{f} = 0$, then $\boldsymbol{u}$ will decay as rapidly as we like.
- However, if we take $\boldsymbol{f} \in L^2((0,\infty), L^2(\Omega)$ (but $\boldsymbol{f} \notin L^1((0,\infty), L^1(\Omega))$, we may construct a magnetic field with no weak limit.

“Stokes” dynamics

In some ways, the specific model that we use for magnetic relaxation doesn’t matter, as long as it dissipates energy. Moffatt (2009) proposed neglecting $\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$ and using:

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \boldsymbol{f} \tag{8a}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} \tag{8b}$$

$$\nabla\cdot\boldsymbol{u} = 0 \tag{8c}$$

$$\nabla\cdot\boldsymbol{B} = 0 \tag{8d}$$

where once again $p_* = p + \frac{1}{2}|\boldsymbol{B}|^2$ is the total pressure, and $\boldsymbol{f} \in L^2((0,\infty), L^2(\Omega))$.

"Stokes" dynamics

Good news: the kinetic energy still decays to zero.

Bad news: we need more hypotheses on $\boldsymbol{B}$.

Worse news: we can adapt the same example to show that $\boldsymbol{B}$ need not have a weak limit here either.

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Kinetic energy decay (DMcC)

Suppose we have a smooth solution $\boldsymbol{u}$, $\boldsymbol{B}$ of equations (8), subject to boundary conditions (6), such that for all $t \in (0, \infty)$,

$$\|\boldsymbol{B}\|_\infty \leq M_1, \qquad \|\nabla \boldsymbol{B}\|_\infty \leq M_2, \qquad \left\| \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \right\|_1 \leq M_3.$$

If $\boldsymbol{f} \in L^2((0, \infty), L^2(\Omega))$, then $\lim_{t \to \infty} \|\boldsymbol{u}(t)\|_2 = 0$.

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\begin{aligned}
\nu \|\nabla \boldsymbol{u}\|_2 &\leq \|\boldsymbol{B}\|_\infty \|\boldsymbol{B}\|_2 + c_p \|\boldsymbol{f}\|_2, \\
\left\| \frac{\partial \boldsymbol{B}}{\partial t} \right\|_2 &\leq \|\nabla \boldsymbol{u}\|_2 \left(\|\boldsymbol{B}\|_\infty + c_p \|\nabla \boldsymbol{B}\|_\infty \right), \\
\frac{\nu}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \boldsymbol{u}\|_2^2 &\leq \|\boldsymbol{B}\|_\infty \left(\left\| \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \right\|_1 + \|\nabla \boldsymbol{u}\|_2 \left\| \frac{\partial \boldsymbol{B}}{\partial t} \right\|_2 \right) \\
&\quad + c_p \|\nabla \boldsymbol{B}\|_\infty \left\| \frac{\partial \boldsymbol{B}}{\partial t} \right\|_2 \|\nabla \boldsymbol{u}\|_2 + c_p \|\boldsymbol{f}\|_2 \|\nabla \boldsymbol{u}\|_2.
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Sketch of proof

The proof uses the following estimates:

$$\begin{aligned}
\nu\|\nabla \boldsymbol{u}\|_2 &\le \|\boldsymbol{B}\|_\infty \|\boldsymbol{B}\|_2 + c_p\|\boldsymbol{f}\|_2, \\
\left\|\frac{\partial \boldsymbol{B}}{\partial t}\right\|_2 &\le \|\nabla \boldsymbol{u}\|_2 \left(\|\boldsymbol{B}\|_\infty + c_p\|\nabla \boldsymbol{B}\|_\infty\right), \\
\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|_2^2 &\le \|\boldsymbol{B}\|_\infty \left(\left\|\frac{\partial^2 \boldsymbol{B}}{\partial t^2}\right\|_1 + \|\nabla \boldsymbol{u}\|_2 \left\|\frac{\partial \boldsymbol{B}}{\partial t}\right\|_2\right) \\
&\quad + c_p\|\nabla \boldsymbol{B}\|_\infty \left\|\frac{\partial \boldsymbol{B}}{\partial t}\right\|_2 \|\nabla \boldsymbol{u}\|_2 + c_p\|\boldsymbol{f}\|_2\|\nabla \boldsymbol{u}\|_2.
\end{aligned}$$

Then, since $\nabla \boldsymbol{u} \in L^2((0,\infty), L^2(\Omega))$ and $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|_2^2$ is uniformly bounded, $\|\nabla \boldsymbol{u}(t)\|_2^2 \to 0$ as $t \to \infty$.

What happens in two dimensions?

- So far, we have focussed on three dimensions, where we have no regularity theory.
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- So far, we have focussed on three dimensions, where we have no regularity theory.
- Natural question: what happens in two dimensions?
- It is known that global solutions exist for the diffusive MHD equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p_* = (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} + \boldsymbol{f}, \tag{9a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} - \mu \Delta \boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}, \tag{9b}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{9c}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{9d}$$

What happens in two dimensions?

- In the case $\mu = 0$, we are only guaranteed local existence of solutions. Fan and Ozawa (2009) and Zhou and Fan (2011) proved two conditional regularity results, which say that

$$\nabla \boldsymbol{u} \in L^1(0,T;L^\infty(\Omega)) \qquad \text{or} \qquad \nabla \boldsymbol{B} \in L^1(0,T;\mathrm{BMO}(\Omega))$$

  are both sufficient conditions to guarantee the existence of a solution on time $[0,T]$.

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  are both sufficient conditions to guarantee the existence of a solution on time $[0,T]$.
- However, when $\mu > 0$ but $\nu = 0$, Kozono (1989) proved the existence of weak solutions for all time; a natural question is whether these techniques can be adapted to the case $\nu > 0$ but $\mu = 0$ to prove global existence of (weak) solutions.

Conclusions

While the idea of magnetic relaxation as a means of studying stationary solutions of the Euler equations is important, there are a number of unresolved issues:

- Can we prove that $\frac{\partial \boldsymbol{u}}{\partial t}$ or $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \to 0$ as $t \to \infty$ — i.e., does the limit state actually solve the stationary Euler equations?
- Does an example of a magnetic field with no weak limit as $t \to \infty$ exist in the absence of a (decaying) forcing?
- Can we reduce the hypotheses needed to ensure decay of the kinetic energy in the “Stokes” model?
- Can it all be made rigorous in two dimensions?

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