A generalised Ladyzhenskaya inequality and a coupled parabolic-elliptic problem

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Summary

- Introduction
- A priori estimates
- Weak L^p spaces and elliptic regularity in L^1
- Interpolation spaces and a generalised Ladyzhenskaya inequality
- Global existence and uniqueness of weak solutions

Ladyzhenskaya's inequality

In proving existence and uniqueness of weak solutions to the 2D Navier–Stokes equations, one uses:

Ladyzhenskaya's inequality

$$\|u\|_{L^4} \leq c\|u\|_{L^2}^{1/2}\|Du\|_{L^2}^{1/2}.$$

Ladyzhenskaya's inequality yields a priori bounds on the nonlinear term $(u \cdot \nabla)u$: if $u \in L^\infty(0,T;L^2)$ and $Du \in L^2(0,T;L^2)$, then

$$\left|\int (u \cdot \nabla)u \cdot \phi\right| = \left|-\int (u \cdot \nabla)\phi \cdot u\right| \leq \|u\|_{L^4}^2\|\nabla\phi\|_{L^2},$$

so

$$\|(u \cdot \nabla)u\|_{H^{-1}} \leq \|u\|_{L^4}^2 \leq c\|u\|_{L^2}\|Du\|_{L^2},$$

and thus $(u \cdot \nabla)u \in L^2(0,T;H^{-1})$, and hence $\partial_t u \in L^2(0,T;H^{-1})$.

A coupled parabolic-elliptic MHD system

We consider the following modified system of equations for magnetohydrodynamics on a bounded domain $\Omega \subset \mathbb{R}^2$:

$$-\Delta u + \nabla p = (B \cdot \nabla) B$$
$$\partial_t B - \varepsilon \Delta B + (u \cdot \nabla) B = (B \cdot \nabla) u,$$

with $\nabla \cdot u = \nabla \cdot B = 0$ and Dirichlet boundary conditions. This is like the standard MHD system, but with the terms $\partial_t u + (u \cdot \nabla) u$ removed.

Theorem

Given $u_0, B_0 \in L^2(\Omega)$ with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$, for any $T > 0$ there exists a unique weak solution (u, B) with

$$u \in L^\infty(0, T; L^{2,\infty}) \cap L^2(0, T; H^1)$$

and

$$B \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1).$$

We prove this using both a generalisation of Ladyzhenskaya's inequality, and some elliptic regularity theory for L^1 forcing.

A priori estimates

Take inner product with u in the first equation, with B in the second equation

$$\|\nabla u\|^2 = \langle (B \cdot \nabla)B, u\rangle = -\langle (B \cdot \nabla)u, B\rangle$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|B\|^2 + \varepsilon\|\nabla B\|^2 = \langle (B \cdot \nabla)u, B\rangle$$

and add:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|B\|^2 + \varepsilon\|\nabla B\|^2 + \|\nabla u\|^2 = 0.$$

We get:

$$B \in L^\infty(0,T;L^2), \qquad \nabla B \in L^2(0,T;L^2), \qquad \nabla u \in L^2(0,T;L^2).$$

We still need elliptic regularity for u:

$$-\Delta u + \nabla p = (B \cdot \nabla)B = \nabla \cdot (\underbrace{B \otimes B}_{L^1})$$

$L^{p,\infty}$: weak L^p spaces

For $f\colon \mathbb{R}^n \to \mathbb{R}$ define

$$d_f(\alpha) = \mu\{x : \ |f(x)| > \alpha\}.$$

Note that

$$\|f\|_{L^p}^p = \int_{\mathbb{R}^n} |f(x)|^p \geq \int_{\{x:\ |f(x)|>\alpha\}} |f(x)|^p \geq \alpha^p d_f(\alpha).$$

For $1 \leq p < \infty$ set

$$\|f\|_{L^{p,\infty}} = \inf\left\{C : \ d_f(\alpha) \leq \frac{C^p}{\alpha^p}\right\} = \sup\{\gamma d_f(\gamma)^{1/p} : \ \gamma > 0\}.$$

The space $L^{p,\infty}(\mathbb{R}^n)$ consists of all those f such that $\|f\|_{L^{p,\infty}} < \infty$.

- $L^p \subset L^{p,\infty}$
- $|x|^{-n/p} \in L^{p,\infty}(\mathbb{R}^n)$ but $\notin L^p(\mathbb{R}^n)$.
- if $f \in L^{p,\infty}(\mathbb{R}^n)$ then $d_f(\alpha) \leq \|f\|_{L^{p,\infty}}^p \alpha^{-p}$.

$L^{p,\infty}$: weak L^p spaces

Just as with strong L^p spaces, we can interpolate between weak L^p spaces:

Weak L^p interpolation

Take $p < r < q$. If $f \in L^{p,\infty} \cap L^{q,\infty}$ then $f \in L^r$ and

$$\|f\|_{L^r} \leq c_{p,r,q} \|f\|_{L^{p,\infty}}^{p(q-r)/r(q-p)} \|f\|_{L^{q,\infty}}^{q(r-p)/r(q-p)}.$$

Recall Young's inequality for convolutions: if $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$ then

$$\|E * f\|_{L^p} \leq \|E\|_{L^q} \|f\|_{L^r}.$$

There is also a weak form, which requires stronger conditions on p, q, r:

Weak form of Young's inequality for convolutions

If $1 \leq r < \infty$ and $1 < p, q < \infty$, and $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$ then

$$\|E * f\|_{L^{p,\infty}} \leq \|E\|_{L^{q,\infty}} \|f\|_{L^r}.$$

Elliptic regularity in L^1

Fundamental solution of Stokes operator on $\mathbb{R}^2$ is

$$E_{ij}(x) = -\delta_{ij} \log |x| + \frac{x_i x_j}{|x|^2},$$

i.e. solution of $-\Delta u + \nabla p = f$ is $u = E * f$.
Solution of $-\Delta u + \nabla p = \partial f$ is $u = E * (\partial f) = (\partial E) * f$. Note that

$$\partial_k E_{ij} = \delta_{ij} \frac{x_k}{|x|^2} + \frac{\delta_{ik} x_j + \delta_{jk} x_i}{|x|^2} - \frac{x_i x_j x_k}{|x|^4} \sim \frac{1}{|x|}.$$

Thus $\partial E \in L^{2,\infty}$ and so

$$f \in L^1 \implies u = \partial E * f \in L^{2,\infty}.$$

If we consider the problem in a bounded domain we have the same regularity. We replace the fundamental solution E by the Dirichlet Green's function G satisfying

$$-\Delta G = \delta(x-y) \qquad G|_{\partial\Omega} = 0.$$

Mitrea & Mitrea (2011) showed that in this case we still have $\partial G \in L^{2,\infty}$.
So on our bounded domain, $u \in L^\infty(0,T; L^{2,\infty})$.

Estimates on time derivatives: $\partial_t B \in L^2(0,T;H^{-1})$?

Take $v \in H^1$ with $\|v\|_{H^1} = 1$. Then

$$\begin{aligned} |\langle \partial_t B, v\rangle| &= |\langle \varepsilon \Delta B - (u\cdot\nabla)B + (B\cdot\nabla)u, v\rangle| \\ &\leq \varepsilon\|\nabla B\|\|\nabla v\| + 2\|u\|_{L^4}\|B\|_{L^4}\|\nabla v\|_{L^2}. \end{aligned}$$

so

$$\boxed{\|\partial_t B\|_{H^{-1}} \leq \varepsilon\|\nabla B\| + 2\|u\|_{L^4}\|B\|_{L^4}.}$$

Standard 2D Ladyzhenskaya inequality gives

$$\|B\|_{L^4} \leq c\|B\|^{1/2}\|\nabla B\|^{1/2};$$

but we only have uniform bounds on u in $L^{2,\infty}$.
If $\|f\|_{L^4} \leq c\|f\|_{L^{2,\infty}}^{1/2}\|\nabla f\|^{1/2}$ then

$$\|\partial_t B\|_{H^{-1}} \leq \varepsilon\|\nabla B\| + c\|u\|_{L^{2,\infty}}^{1/2}\|B\|^{1/2}\|\nabla u\|^{1/2}\|\nabla B\|^{1/2}$$

which would yield

$$\partial_t B \in L^2(0,T;H^{-1}).$$

Generalised Ladyzhenskaya inequality and interpolation spaces

For $0 \le \theta \le 1$ one can define an interpolation space $X_\theta := [X^0, X^1]_\theta$ in such a way that $\|f\|_{X_\theta} \le c\|f\|_{X^0}^{1-\theta}\|f\|_{X^1}^{\theta}$. (Note that $\|f\|_{X_1} \le c\|f\|_{X^1}$.)

Theorem (Bennett & Sharpley, 1988)

$L^{p,\infty} = [L^1, \mathrm{BMO}]_{1-(1/p)}$ *for* $1 < p < \infty$*; so* $L^{2,\infty} = [L^1, \mathrm{BMO}]_{1/2}$.

Reiteration Theorem

If $A_0 = [X_0, X_1]_{\theta_0}$, $A_1 = [X_0, X_1]_{\theta_1}$ then $[A_0, A_1]_\theta = [X_0, X_1]_{(1-\theta)\theta_0 + \theta\theta_1}$ provided that $\theta \in (0, 1)$.

Write $\mathfrak{B} = [L^1, \mathrm{BMO}]_1$ and note that $\|f\|_{\mathfrak{B}} \le c\|f\|_{\mathrm{BMO}}$. Then $L^{3,\infty} = [L^{2,\infty}, \mathfrak{B}]_{1/3}$ and $L^{6,\infty} = [L^{2,\infty}, \mathfrak{B}]_{2/3}$, and hence

$$\begin{aligned}\|f\|_{L^4} &\le c\|f\|_{L^{3,\infty}}^{1/2}\|f\|_{L^{6,\infty}}^{1/2} \\ &\le c[c\|f\|_{L^{2,\infty}}^{2/3}\|f\|_{\mathfrak{B}}^{1/3}]^{1/2}[c\|f\|_{L^{2,\infty}}^{1/3}\|f\|_{\mathfrak{B}}^{2/3}]^{1/2} \\ &= c\|f\|_{L^{2,\infty}}^{1/2}\|f\|_{\mathfrak{B}}^{1/2} \le c\|f\|_{L^{2,\infty}}^{1/2}\|f\|_{\mathrm{BMO}}^{1/2}.\end{aligned}$$

Since $\dot H^1 \subset \mathrm{BMO}$ (see Evans, for example) this yields

$$\boxed{\|f\|_{L^4} \le c\|f\|_{L^{2,\infty}}^{1/2}\|f\|_{\dot H^1}^{1/2}.}$$

Estimates on time derivatives 2: $\partial_t u \in L^1(0,T;H^{-1})$

We have now obtained $\partial_t B \in L^2(0,T;H^{-1})$.
Now from $-\Delta u + \nabla p = (B\cdot\nabla)B$ we have

$$-\Delta u_t + \nabla p_t = (B_t\cdot\nabla)B + (B\cdot\nabla)B_t.$$

Take $v \in H^1$ let ϕ satisfy

$$-\Delta\phi + \nabla p = v \qquad \Longrightarrow \qquad \|\phi\|_{H^3} \le c\|v\|_{H^1}.$$

$$\begin{aligned}
|\langle u_t, v\rangle| &= |\langle u_t, -\Delta\phi + \nabla p\rangle| \\
&= |\langle \Delta u_t, \phi\rangle| \\
&\le |\langle (B_t\cdot\nabla)B, \phi\rangle| + |\langle (B\cdot\nabla)B_t, \phi\rangle| \\
&\le c\|B_t\|_{H^{-1}}\|B\nabla\phi\|_{H^1} \\
&\le c\|B_t\|_{H^{-1}}\|B\|_{H^1}\|\phi\|_{H^3} \\
&\le c\|B_t\|_{H^{-1}}\|B\|_{H^1}\|v\|_{H^1},
\end{aligned}$$

so

$$u_t \in L^1(0,T;H^{-1}).$$

Conclusion

By using Galerkin approximations, we can make the previous a priori estimates rigorous, and using a variant of the Aubin–Lions compactness lemma (Temam, 1979; Simon, 1987) we obtain a weak solution (u, B) of the equations; similar arguments to the a priori estimates show uniqueness of weak solutions, and so:

Theorem

Given $u_0, B_0 \in L^2(\Omega)$ with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$, for any $T > 0$ there exists a unique weak solution (u, B) with

$$u \in L^\infty(0, T; L^{2,\infty}) \cap L^2(0, T; H^1)$$

and

$$B \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1).$$

What about $\varepsilon = 0$?

- Try looking at more regular solutions and taking the limit $\varepsilon \to 0$ to get local existence
- Assume regularity and show that $u(t) \to 0$ as $t \to \infty$ (Moffatt)?