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Engineering and Physical Sciences Research Council

Hierarchical Bayesian Level Set Inversion

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Overview

- We are interested the determination of the hydraulic permeability of a porous medium given noisy piezometric head measurements, using a Bayesian approach to the inverse problem.
- In particular we are interested in the recovery of interfaces between different media in the subsurface, using a level set approach.
- Treating the length scale of the permeability hierarchically allows for more accurate recovery than non-hierarchical methods.

The forward problem (Darcy model for groundwater flow)

- Piezometric head h.
- Hydraulic permeability κ.
- Given $\kappa \in L^\infty_+$ and $f \in H^{-1}$, plus appropriate boundary conditions, find $h \in H^1_0$ satisfying the PDE

$$-\nabla \cdot (\kappa \nabla h) = f$$

- Define $\mathcal{G}(\kappa) \in \mathbb{R}^J$ to be some measurements of h.

The inverse problem

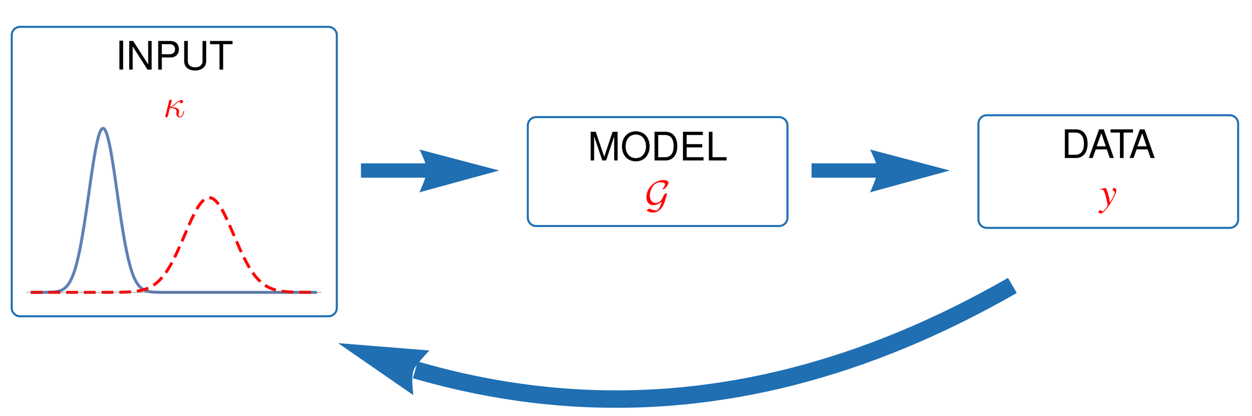
- Let $\eta \sim N(0, \Gamma)$ be some Gaussian noise on $\mathbb{R}^J$. We observe data y,

$$\boxed{y = \mathcal{G}(\kappa) + \eta}$$

- Given y, find the permeability κ.
- Problem is underdetermined: y is finite dimensional, but κ is infinite dimensional.
- Data is noisy: y may not even lie in the image of $\mathcal{G}$ due to the noise term.

Bayesian inversion: the idea

- Probability delivers missing information and accounts for observational noise.



The level set approach

- Often the permeability of interest is approximately piecewise constant. It can then be expressed as a thresholded continuous function, termed the level set function.
- The problem now concerns recovery of the level set function.

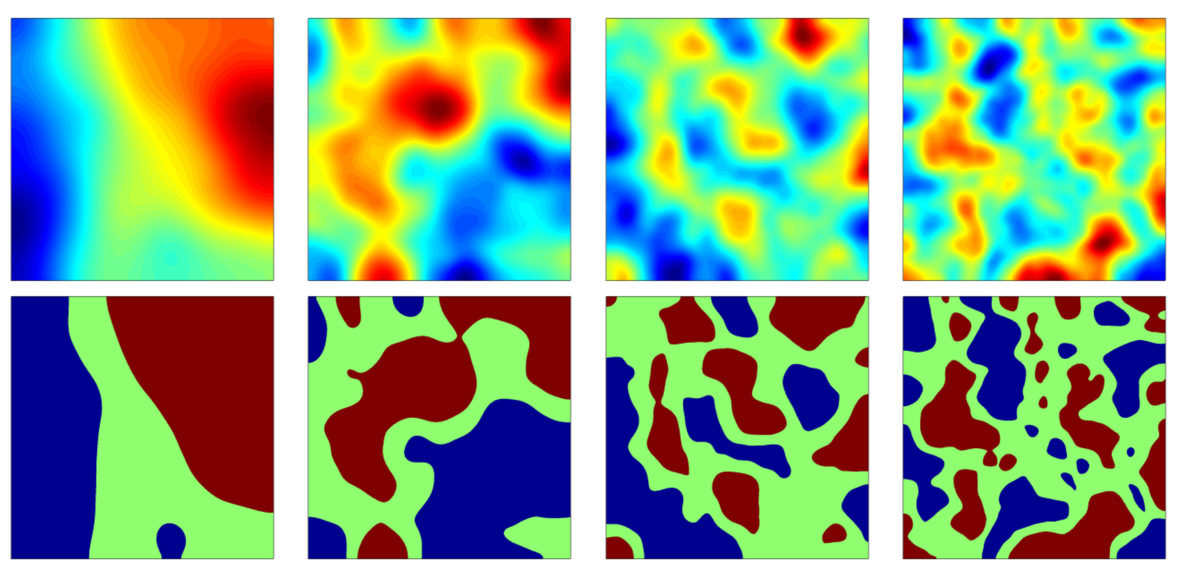


Figure: (Top) Examples of level set functions. (Bottom) The result of thresholding these functions at two levels.

The prior distribution

- We place a probability distribution upon the level set function u, representing our prior beliefs before data is collected.
- This prior distribution may for example be taken to be a Gaussian with Whittle-Matérn covariance function:

$$c(x, y) = \sigma^2 \frac{1}{2^{\nu-1}\Gamma(\nu)}(\tau|x-y|)^\nu K_\nu(\tau|x-y|).$$

- The parameter ν controls the regularity of samples, σ controls the amplitude, and τ controls the (inverse) length scale.
- These parameters can be assumed to be known a priori, though reconstruction of permeabilities may be poor if they are chosen inappropriately.
- To improve reconstruction we treat the parameter τ hierarchically.
- For technical reasons (absolute continuity) the covariance must be rescaled by $\tau^{-\nu}$; the thresholding levels are then given this same scaling to compensate.
- The prior μ_0 is now on both u and τ

$$\boxed{\mu_0(\mathrm{d}u, \mathrm{d}\tau) \propto \mathbb{P}(\mathrm{d}u|\tau)\mathbb{P}(\mathrm{d}\tau)}$$

The likelihood

- Due to the scaling issue above we must pass the length scale parameter τ to the thresholding map.
- We have that $\kappa = \kappa(u, \tau)$ via this map, and so we write $\mathcal{G}(u, \tau)$ in place of $\mathcal{G}(\kappa)$.
- Since $y = \mathcal{G}(u, \tau) + \eta$ and $\eta \sim N(0, \Gamma)$, then $y|(u, \tau) \sim N(\mathcal{G}(u, \tau), \Gamma)$. The model-data misfit Φ is the negative log-likelihood:

$$\boxed{\mathbb{P}(y|u, \tau) \propto \exp(-\Phi(u, \tau; y)), \quad \Phi(u, \tau; y) = \frac{1}{2}|\Gamma^{-1/2}(y - \mathcal{G}(u, \tau))|^2}$$

The posterior distribution

- The posterior distribution μ^y represents information about u and τ after data is collected.
- It can be characterized in terms of Φ and μ_0 using Bayes' theorem:

$$\boxed{\mu^y(\mathrm{d}u, \mathrm{d}\tau) \propto \exp(-\Phi(u, \tau; y))\mu_0(\mathrm{d}u, \mathrm{d}\tau)}$$

- We have the following result concerning well-posedness of the inverse problem:

The map $y \mapsto \mu^y(\mathrm{d}u, \mathrm{d}\tau)$ is Lipschitz in the Hellinger metric. Furthermore, if S is a separable Banach space, and the map $(u, \tau) \mapsto f(u, \tau) \in S$ is square integrable with respect to μ_0, then

$$\|\mathbb{E}^{\mu^{y_1}} f(u, \tau) - \mathbb{E}^{\mu^{y_2}} f(u, \tau)\|_S \le C|y_1 - y_2|.$$

Numerical example

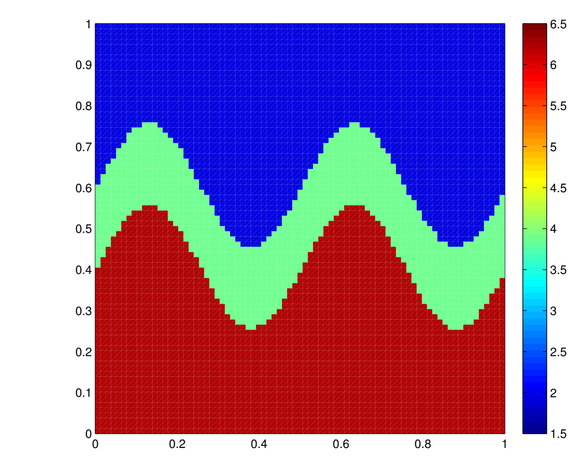


Figure: The true log-permeability used to create the data

- We define a channelized permeability.
- This does not come from the prior, and so there is no 'true' value of the length scale parameter τ.
- Nonetheless there is an intrinsic length scale associated with the field that we aim to recover.
- Data arises from smoothed point observations of the hydraulic head on a uniform grid of 64 points. Noise on the measurements is approximately 2%.
- We perform MCMC simulations to sample from the posterior μ^y arising from both hierarchical and non-hierarchical methods, with τ initialised or fixed at $\tau = 1, 10, 30, 50, 70$ and 90.

Numerics: posterior means

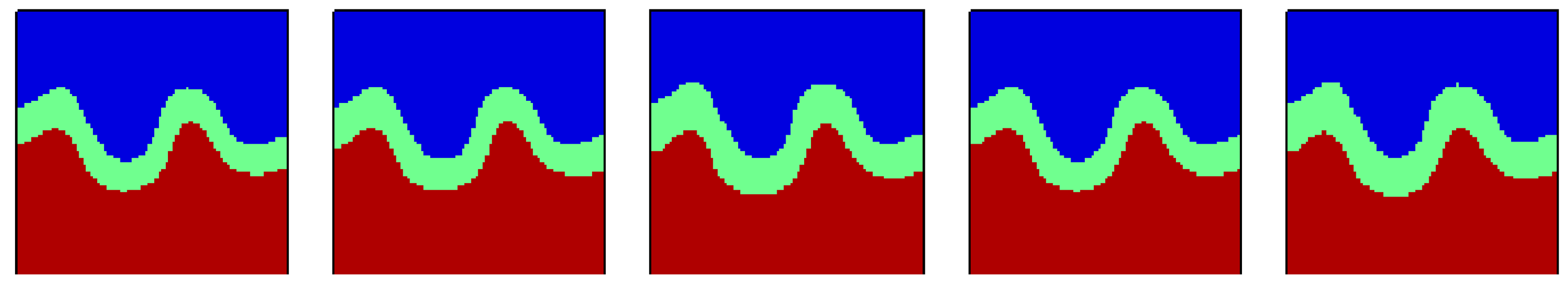


Figure: Approximations of $\kappa(\mathbb{E}(u), \mathbb{E}(\tau))$ under the **hierarchical** posterior, when MCMC is initialized at each value of τ.

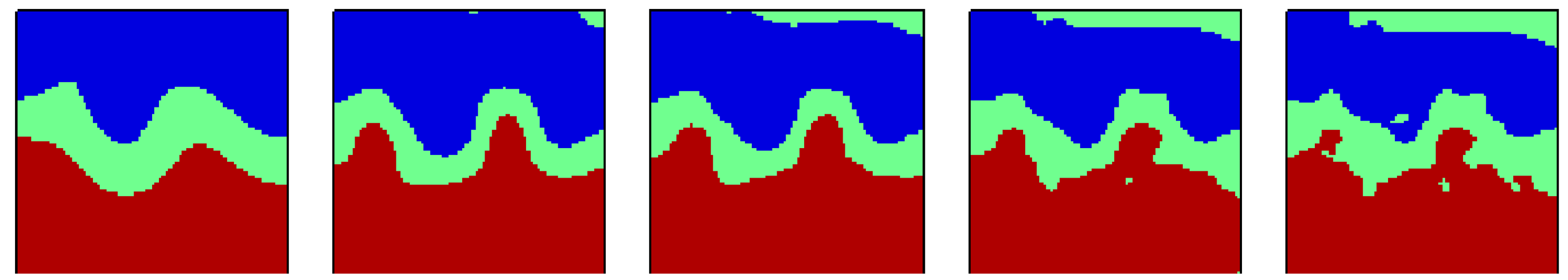


Figure: Approximations of $\kappa(\mathbb{E}(u), \mathbb{E}(\tau))$ under the **non-hierarchical** posteriors, with each fixed value of τ.

Numerics: posterior samples

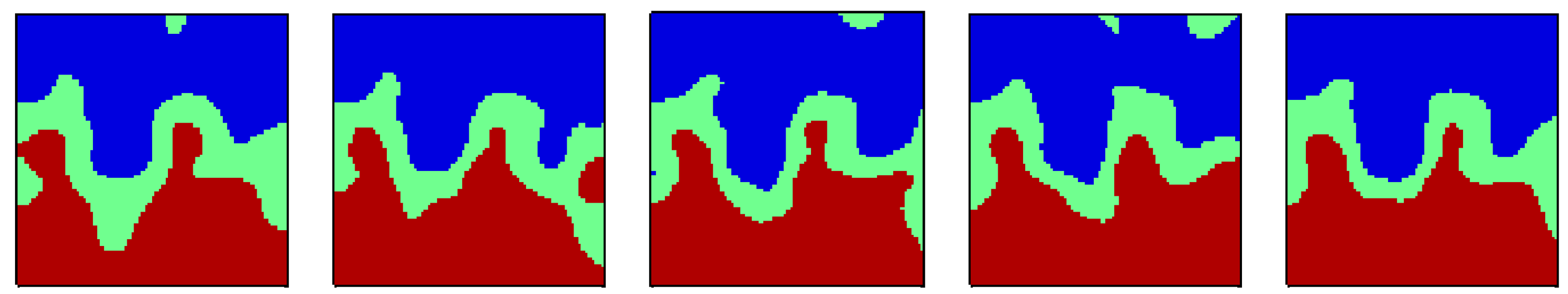


Figure: Typical samples of $\kappa(u, \tau)$ under the **hierarchical** posterior, when MCMC is initialized at each value of τ.

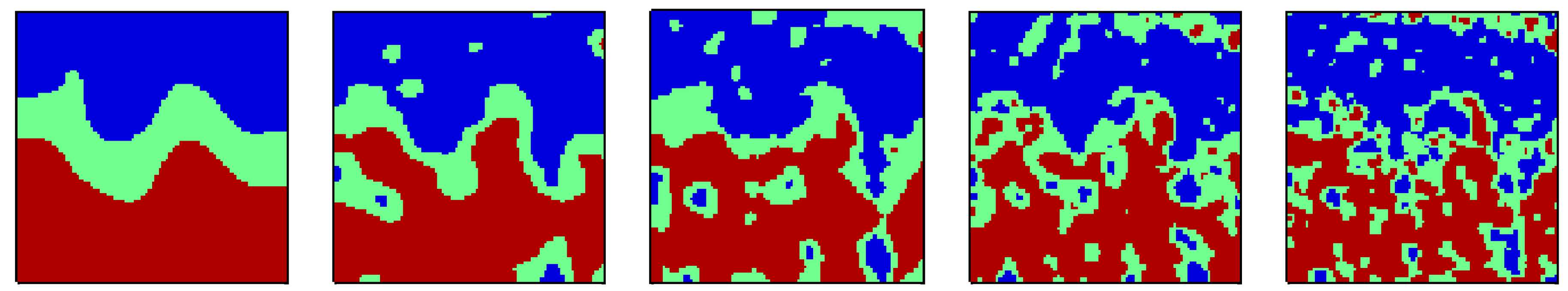
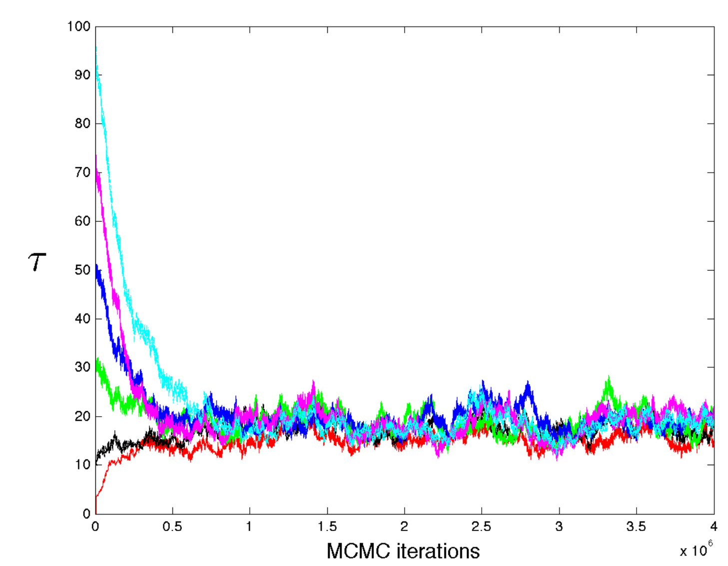


Figure: Typical samples of $\kappa(u, \tau)$ under the **non-hierarchical** posteriors, with each fixed value of τ.

Numerics: trace of length-scale parameter



- The chains for τ all converge within 10^6 samples, to be centred around the value $\tau \approx 18$.
- This can be observed in the hierarchical means, which look essentially identical for each chain. This is in contrast to the non-hierarchical means, wherein the short length scales have allowed for the creation of artifacts towards the top of the domain.
- The effect of length scale is even more stark when comparing the hierarchical and non-hierarchical samples.

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