The Bayesian Approach to EIT: Analysis and Algorithms

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Outline

Introduction

- Electrical Impedance Tomography (EIT) is an imaging technique in which the conductivity of a body is inferred from electrode measurements on its surface
- Applications range from non-invasive medical imaging to monitoring oil flow in pipelines
- Abstract formulation of the problem given by Calderón: can the coefficient of a divergence form elliptic PDE be recovered from knowledge of its Neumann-to-Dirichlet operator?

  Specifically, if $g \in H^{-1/2}(\partial D)$ is given and $u \in H^1(D)$ solves

  $$\nabla \cdot (\sigma \nabla u) = 0 \text{ in } D, \quad \sigma \frac{\partial u}{\partial \nu} = g \text{ on } \partial D$$

  does the pair $(u|_{\partial D}, g)$ determine σ?

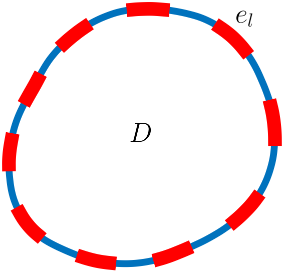
Introduction

- We work with a more physically appropriate version of the problem above
- The problem is ill posed, so we take a probabilistic (Bayesian) approach
- I'll discuss choices of prior distribution, and existence and well-posedness of the resulting posterior distribution
- I'll also present some numerical simulations

Outline

The forward model

$$
\begin{cases}
-\nabla \cdot (\sigma(x)\nabla u(x)) = 0 & x \in D \\
\displaystyle\int_{e_l} \sigma \frac{\partial u}{\partial n}\, \mathrm{d}S = I_l & l = 1, \dots, L \\
\sigma(x)\dfrac{\partial u}{\partial n}(x) = 0 & x \in \partial D \setminus \bigcup_{l=1}^{L} e_l \\
u(x) + z_l \sigma(x)\dfrac{\partial u}{\partial n}(x) = U_l & x \in e_l, l = 1, \dots, L
\end{cases}
\qquad \text{(PDE)}
$$



Input: $\sigma : D \to \mathbb{R}$, $(I_l) \in \mathbb{R}^L$
Output: $u : D \to \mathbb{R}$, $(U_l) \in \mathbb{R}^L$

Existence and uniqueness

Denote $\mathbb{H} = H^1(D) \times \mathbb{R}^L$.

Theorem (Cheney et al [SCI92])

Let $\sigma \in \mathcal{A}(D)$. *Then there exists a unique* $(u, U) \in \mathbb{H}$ *solving the weak form of (PDE), with* $\sum_{l=1}^{L} U_l = 0$

- Fixing a current stimulation pattern and contact impedances $(I_l), (z_l) \in \mathbb{R}^L$, the map $\mathcal{M} : \sigma \mapsto (u, U)$ is hence well-defined.
- It is shown in [KKSV00] that if we equip $\mathcal{A}(D)$ with $\| \cdot \|_\infty$, this map is Fréchet differentiable
- For the conductivities we will be considering, this choice of norm is not appropriate. We establish the following continuity result.

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Continuity of forward map

Proposition

Fix a current stimulation pattern $(I_l) \in \mathbb{R}^L$ *and contact impedances* $(z_l) \in \mathbb{R}^L$*. Define the solution map* $\mathcal{M} : \mathcal{A}(D) \to \mathbb{H}$ *as above. Let* $\sigma \in \mathcal{A}(D)$ *and let* $(\sigma_n)_{n\geq 1} \subseteq \mathcal{A}(D)$ *be such that either*

(i) σ_n *converges to* σ *uniformly; or*

(ii) σ_n *converges to* σ *in measure, and there exist* $\sigma^-, \sigma^+ \in \mathbb{R}$ *such that for all* $n > 0$ *and* $x \in D$*,* $0 < \sigma^- \leq \sigma_n(x) \leq \sigma^+ < \infty$.

Then $\|\mathcal{M}(\sigma_n) - \mathcal{M}(\sigma)\|_{\mathbb{H}} \to 0$.

- Since the projection $\Pi : (u, U) \mapsto U$ is continuous, we also have that $|(\Pi \circ \mathcal{M})(\sigma_n) - (\Pi \circ \mathcal{M})(\sigma)| \to 0$ above.

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Outline

The inverse problem

- Given a conductivity field $\sigma \in \mathcal{A}(D)$, the boundary voltage measurements $U(\sigma)$ arising from the solution of the forward model are related to the current stimulation pattern I via Ohm's law:

$$U(\sigma) = R(\sigma)I$$

- Assume that J linearly independent current patterns $I^{(j)} \in \mathbb{R}^L$, $j = 1, \ldots, J$, $J \leq L - 1$ are applied, and noisy measurements of $U^{(j)}(\sigma) = R(\sigma)I^{(j)}$ are made:

$$y_j = U^{(j)}(\sigma) + \eta_j, \quad \eta_j \sim N(0, \Gamma_0) \text{ iid}$$

- Concatenating these observations, we write

$$\begin{aligned} y = \mathcal{G}(\sigma) + \eta, \quad & \eta \sim N(0, \Gamma) \\ & \Gamma = \mathrm{diag}(\Gamma_0, \ldots, \Gamma_0) \end{aligned}$$

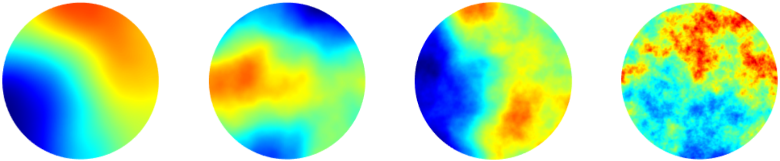
- The inverse problem is then to recover the conductivity field σ from the data y.

Choices of prior

- This inverse problem is highly ill-posed, so we take a Bayesian approach by placing a prior probability distribution on σ
- A solution to the problem is then the posterior distribution $u|y$ of the state u given data y, arising from an application of Bayes' theorem
- We consider three functions $F : X \to \mathcal{A}(D)$ which map draws from prior measures μ_0 on Banach spaces X to the space of conductivities $\mathcal{A}(D)$
- Our prior conductivity distributions will then be $F^*(\mu_0)$, the push forward of the prior measures by these maps F
- Regularity of these maps F will be important for existence of the posterior

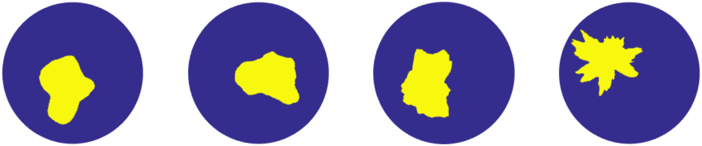
Prior 1: Log-Gaussian

- A simple one to start with: let $F_{\exp} : C^0(\overline{D}) \to \mathcal{A}(D)$ be defined by $F_{\exp}(\varphi) = \exp(\varphi)$
- Typical samples from $F^*_{\exp}(\mu_0)$ when $\mu_0 = N(0, (-\Delta)^{-\alpha})$ for various α:



Prior 2: Star-shaped

- Formally, take a positive periodic function on a line segment/rectangle R, and wrap it around a circle/sphere using polar coordinates to give a closed curve/surface
- Assign one positive value to points within the resulting region, and another to those outside
- Also allow for variation of the centre of the polar coordinate system
- This defines a map $F_{\text{star}} : C^0_{\text{per}}(R) \times D \to \mathcal{A}(D)$
- Typical samples from $F^*_{\text{star}}(\sigma_0 \otimes \tau_0)$ when τ_0 is uniform and $\sigma_0 = \log N(0, (-\Delta)^{-\alpha})$ for various α:

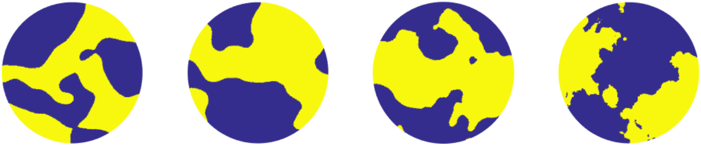


Prior 3: Level set

- These priors are discussed in [ILS]. We give an overview of binary field case
- Let $f_1, f_2 \in C^0(\overline{D})$ be fixed positive functions. Then we can define $F_{\mathrm{lvl}} : C^0(\overline{D}) \to \mathcal{A}(D)$ by

$$F_{\mathrm{lvl}}(\varphi) = f_1 \mathbb{1}_{\varphi \geq 0} + f_2 \mathbb{1}_{\varphi < 0}$$

- Typical samples from $F^*_{\mathrm{lvl}}(\mu_0)$ when $\mu_0 = N(0, (-\Delta)^{-\alpha})$ for various α:



Outline

The forward map and likelihood

- Let X be a separable Banach space and $F : X \to \mathcal{A}(D)$ a map from the state space to the conductivity space.
- Choose a set of current stimulation patterns $I^{(j)} \in \mathbb{R}^L$, $j = 1, \ldots, J$ and let $\mathcal{M}_j : \mathcal{A}(D) \to \mathbb{H}$ denote the solution map when using stimulation pattern $I^{(j)}$.
- Define the projection map $\Pi : \mathbb{H} \to \mathbb{R}^L$ by $\Pi(u, U) = U$.
- Define $\mathcal{G}_j : X \to \mathbb{R}^L$ by $\mathcal{G}_j = \Pi \circ \mathcal{M}_j \circ F$, and let $\mathcal{G} : X \to \mathbb{R}^{JL}$ denote the concatenation of these $\mathcal{G}_j$.
- As before, we assume the data $y \in Y := \mathbb{R}^{JL}$ arises via

$$y = \mathcal{G}(u) + \eta, \quad \eta \sim \mathbb{Q}_0 := N(0, \Gamma)$$

The forward map and likelihood

- Assume that $u \sim \mu_0$, where μ_0 is independent of $\mathbb{Q}_0$. From the above, we see that $y|u \sim \mathbb{Q}_u := N(\mathcal{G}(u), \Gamma)$. This can be used to formally find the distribution of $u|y$.
- First note that

$$\frac{d\mathbb{Q}_u}{d\mathbb{Q}_0}(y) = \exp\left(-\Phi(u; y) + \frac{1}{2}|y|_\Gamma^2\right)$$

where the potential $\Phi : X \times Y \to \mathbb{R}$ is given by

$$\Phi(u; y) = \frac{1}{2}|\mathcal{G}(u) - y|_\Gamma^2 \qquad (1)$$

- Then under suitable regularity conditions, Bayes' theorem tells us that the distribution μ^y of $u|y$ satisfies

$$\mu^y(du) \propto \exp(-\Phi(u; y))\mu_0(du)$$

after absorbing the $\exp(\frac{1}{2}|y|_\Gamma^2)$ term into the normalisation constant.

Existence of posterior

Theorem (Existence)

Let $(X, \mathcal{F}, \mu_0)$ *denote any of the probability spaces associated with the priors introduced previously, and let* $\Phi : X \times Y \to \mathbb{R}$ *be the potential associated with the corresponding forward map. Then the posterior distribution* μ^y *of the state* u *given data* y *is well-defined. Furthermore,* $\mu^y \ll \mu_0$ *with Radon-Nikodym derivative*

$$\frac{\mathrm{d}\mu^y}{\mathrm{d}\mu_0}(u) = \frac{1}{Z}\exp(-\Phi(u;y))$$

where

$$Z := \int_X \exp(-\Phi(u;y))\,\mu_0(\mathrm{d}u) > 0$$

Well-posedness of posterior

Theorem (Well-posedness)

Let $(X, \mathcal{F}, \mu_0)$ *denote any of the probability spaces associated with the priors introduced previously, and let* $\Phi : X \times Y \to \mathbb{R}$ *be the potential associated with the corresponding forward map. Then the posterior measure* μ^y *is locally Lipschitz with respect to* y*, in the Hellinger distance.*

As a consequence, if $y, y' \in B_Y(r)$ *and* $f \in L^2(X, \mu_0)$*, then there is a* $C = C(r) > 0$ *such that*

$$|\mathbb{E}^{\mu^y} f(u) - \mathbb{E}^{\mu^{y'}} f(u)| \leq C|y - y'|_Y$$

Outline

A hierarchical Bayesian approach

- We do not know a priori how clustered together different inclusions will be - there is an intrinsic length scale associated with the conductivity.
- Suppose that our Gaussian prior has stationary covariance function
$$\mathbb{E}u(x)u(y) = c(x,y) = h(|x-y|)$$
- We can then define the family of covariances
$$c_\ell(x,y) = h\left(\frac{|x-y|}{\ell}\right)$$
- We treat this parameter ℓ as an additional unknown and place a hyper-prior on it.

References

- Masoumeh Dashti and Andrew M. Stuart, *The Bayesian Approach to Inverse Problems*, arXiv.
- Marco A. Iglesias, Yulong Lu, and Andrew M. Stuart, *A Bayesian Level Set Method for Geometric Inverse Problems*, arXiv.
- Jari P. Kaipio, Ville Kolehmainen, Erkki Somersalo, and Marko Vauhkonen, *Statistical inversion and Monte Carlo sampling methods in electrical impedance tomography*, Inverse Problems **16** (2000), no. 5, 1487–1522.
- Sari Lasanen, Janne M. J. Huttunen, and Lassi Roininen, *Whittle-Matérn priors for Bayesian statistical inversion with applications in electrical impedance tomography*, Inverse Problems and Imaging **8** (2014), no. 2, 561–586.
- Erkki Somersalo, Margaret Cheney, and David Isaacson, *Existence and Uniqueness for Electrode Models for Electric Current Computed Tomography*, SIAM Journal on Applied Mathematics **52** (1992), no. 4, 1023–1040.