The Bayesian Formulation of EIT

Matt Dunlop

Mathematics Institute, University of Warwick, U.K.

Joint work with Andrew Stuart (Warwick)

SIAM UQ 2016,
Lausanne, Switzerland,
April 5th.

Outline

Introduction

- Electrical Impedance Tomography (EIT) is an imaging technique in which the conductivity of a body is inferred from electrode measurements on its surface.
- Applications range from non-invasive medical imaging to monitoring oil flow in pipelines.
- Abstract formulation of the problem given by Calderón: can the coefficient of a divergence form elliptic PDE be recovered from knowledge of its Neumann-to-Dirichlet operator?

  Specifically, if $g \in H^{-1/2}(\partial D)$ is given and $u \in H^1(D)$ solves

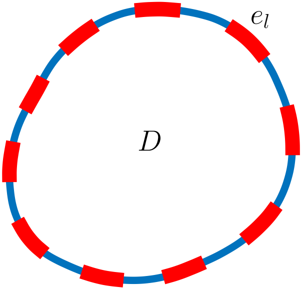
  $$\nabla \cdot (\sigma \nabla u) = 0 \text{ in } D, \quad \sigma \frac{\partial u}{\partial \nu} = g \text{ on } \partial D$$

  does the mapping $\Lambda_\sigma : g \mapsto u|_{\partial D}$ determine σ?

Introduction

- The problem has received much study. Some significant results concern, e.g.,
  - **Uniqueness** (Sylvester, Uhlmann, ‘87)
  - **Reconstruction** (Nachman, ‘88)
  - **Stability** (Alessandrini, ‘88)
  - **Partial data** (Kenig, Sjöstrand, Uhlmann, ‘03)
- We work with a more physically appropriate model of the problem above, introduced in (Somersalo, Cheney, Isaacson, ‘92).
- The problem is ill posed, so we take a probabilistic (Bayesian) approach.
- I'll discuss choices of prior distribution, existence and well-posedness of the resulting posterior distribution, and numerical experiments.

Forward Model: Definition



- Apply currents I_ℓ on e_ℓ, $\ell = 1, \dots, L$.
- Induces voltages Θ_ℓ on e_ℓ, $\ell = 1, \dots, L$.
- Input is (σ, I), output is (θ, Θ).
- We have an Ohm's law $\Theta = R(\sigma)I$.

$$
\begin{cases}
-\nabla \cdot (\sigma(x)\nabla\theta(x)) = 0 & x \in D \\
\displaystyle\int_{e_\ell} \sigma \frac{\partial\theta}{\partial\nu}\, \mathrm{d}S = I_\ell & \ell = 1, \dots, L \\
\sigma(x)\dfrac{\partial\theta}{\partial\nu}(x) = 0 & x \in \partial D \setminus \bigcup_{\ell=1}^{L} e_\ell \\
\theta(x) + z_\ell \sigma(x)\dfrac{\partial\theta}{\partial\nu}(x) = \Theta_\ell & x \in e_\ell, \ell = 1, \dots, L
\end{cases}
\tag{PDE}
$$

Forward Model: Assumptions

Definition

A conductivity field $\sigma : D \to \mathbb{R}$ is said to be admissible if

(i) There exists $N \in \mathbb{N}$, $\{D_n\}_{n=1}^N$ open disjoint subsets of D for which $\overline{D} = \bigcup_{n=1}^N \overline{D}_n$

(ii) $\sigma\big|_{D_n} \in C(\overline{D}_n)$

(iii) There exist $\sigma^-, \sigma^+ \in \mathbb{R}$ such that $0 < \sigma^- \leq \sigma(x) \leq \sigma^+ < \infty$ for all $x \in \overline{D}$.

The set of all such conductivities will be denoted $\mathcal{A}(D)$.

Throughout we will assume that $\{z_\ell\} \in \mathbb{R}^L$ and $\{I_\ell\} \in \mathbb{R}^L$ satisfy

(i) $0 < z_- \leq z_\ell \leq z_+ < \infty, \quad \ell = 1, \dots, L,$

(ii) $\displaystyle\sum_{\ell=1}^{L} I_\ell = 0.$

Forward Model: Existence and Uniqueness

Denote $\mathbb{H} = H^1(D) \times \mathbb{R}^L$.

Theorem (Somersalo, Cheney, Isaacson, '92)

Let $\sigma \in \mathcal{A}(D)$. Then there exists a unique $(\theta, \Theta) \in \mathbb{H}$ solving the weak form of (PDE), with $\sum_{\ell=1}^{L} \Theta_\ell = 0$.

- Fixing a current stimulation pattern and contact impedances $\{I_\ell\}, \{z_\ell\} \in \mathbb{R}^L$, the map $\mathcal{M} : \sigma \mapsto (\theta, \Theta)$ is hence well-defined.
- It has been shown (Kaipio et al, '00) that if we equip $\mathcal{A}(D)$ with $\|\cdot\|_\infty$, this map is Fréchet differentiable.
- For the conductivities we will be considering, this choice of norm is not appropriate. We establish the following continuity result.

Forward Map: Continuity

Proposition (D, Stuart, '15)

Fix a current stimulation pattern $\{I_\ell\} \in \mathbb{R}^L$ *and contact impedances* $\{z_\ell\} \in \mathbb{R}^L$. *Define the solution map* $\mathcal{M} : \mathcal{A}(D) \to \mathbb{H}$ *as above. Let* $\sigma \in \mathcal{A}(D)$ *and let* $\{\sigma_n\}_{n\geq 1} \subseteq \mathcal{A}(D)$ *be such that either*

- **(i)** σ_n *converges to* σ *uniformly; or*
- **(ii)** σ_n *converges to* σ *in measure, and there exist* $\sigma^-, \sigma^+ \in \mathbb{R}$ *such that for all* $n > 0$ *and* $x \in D$, $0 < \sigma^- \leq \sigma_n(x) \leq \sigma^+ < \infty$.

Then $\|\mathcal{M}(\sigma_n) - \mathcal{M}(\sigma)\|_{\mathbb{H}} \to 0$.

- Since the projection $\Pi : (\theta, \Theta) \mapsto \Theta$ is continuous, we also have that $|(\Pi \circ \mathcal{M})(\sigma_n) - (\Pi \circ \mathcal{M})(\sigma)| \to 0$ above.

The Inverse Problem

- Given a conductivity field $\sigma \in \mathcal{A}(D)$, the boundary voltage measurements $\Theta(\sigma)$ arising from the solution of the forward model are related to the current stimulation pattern I via Ohm's law:

$$\Theta(\sigma) = R(\sigma)I$$

- Assume that J linearly independent current patterns $I^{(j)} \in \mathbb{R}^L$, $j = 1, \dots, J$, $J \leq L - 1$ are applied, and noisy measurements of $\Theta^{(j)}(\sigma) = R(\sigma)I^{(j)}$ are made:

$$y_j = \Theta^{(j)}(\sigma) + \eta_j, \quad \eta_j \sim N(0, \Gamma_0) \text{ iid}$$

- Concatenating these observations, we write

$$y = \mathcal{G}(\sigma) + \eta, \quad \eta \sim N(0, \Gamma)$$
$$\Gamma = \text{diag}(\Gamma_0, \dots, \Gamma_0)$$

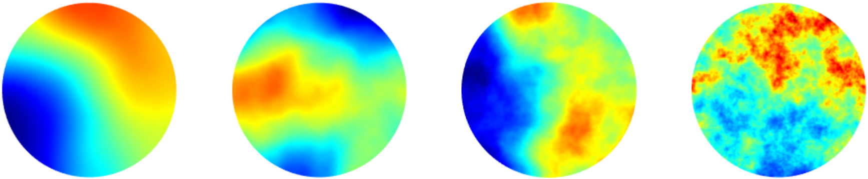
- The inverse problem is to recover the conductivity field σ from the data y.

Choice of Prior Distribution

- This inverse problem is highly ill-posed, so we take a Bayesian approach by placing a prior probability distribution on σ.
- A solution to the problem is then the posterior distribution $\sigma|y$ of the conductivity σ given data y, arising from an application of Bayes' theorem
- We consider three functions $F : X \to \mathcal{A}(D)$ which map draws from prior measures μ_0 on Banach spaces X to the space of conductivities $\mathcal{A}(D)$
- Our prior conductivity distributions will then be $F^*(\mu_0)$, the push forward of the prior measures by these maps F
- Regularity of these maps F will be important for existence of the posterior

Prior Model #1: Log-Gaussian

- A simple one to start with: let $F_{\exp} : C^0(\overline{D}) \to \mathcal{A}(D)$ be defined by $F_{\exp}(u) = \exp(u)$.
- If $u_n \to u$ in $C^0(\overline{D})$, then we have $\|F_{\exp}(u_n) - F_{\exp}(u)\|_\infty \to 0$.
- Take μ_0 to be Gaussian.
- Typical samples from $F^*_{\exp}(\mu_0)$ when $\mu_0 = N(0, (-\Delta)^{-\alpha})$ for various α:

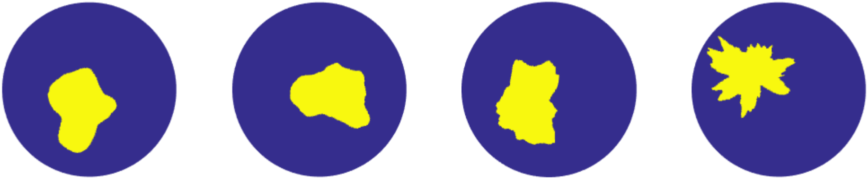


Prior Model #2: Star-shaped

- Formally, take a positive periodic function on a line segment/rectangle R, and wrap it around a circle/sphere using polar coordinates to give a closed curve/surface.
- Assign one positive value to points within the resulting region, and another to those outside.
- Also allow for variation of the centre of the polar coordinate system.
- This defines a map $F_{\text{star}} : C^0_{\text{per}}(R) \times D \to \mathcal{A}(D)$.
- It can be shown that if $x_0 \in D$ and $r \in C^0_{\text{per}}$ is Lipschitz continuous, then $r_n \to r$ and $x_0^n \to x_0$ implies that $F_{\text{star}}(r_\varepsilon, x_0^\varepsilon) \to F_{\text{star}}(r, x_0)$ in measure.

Prior Model #2: Star-shaped

- We assume that r and x_0 are independent under the prior so that we may factor $\mu_0 = \sigma_0 \times \tau_0$.
- Assume that both $\sigma_0(B) > 0$ for all balls B, and τ_0 has exponential moments.
- Typical samples from $F_{\text{star}}^*(\mu_0)$ when $\sigma_0 = \log N(0, (-\Delta)^{-\alpha})$ for various α and τ_0 is uniform:



Prior Model #3: Level Set

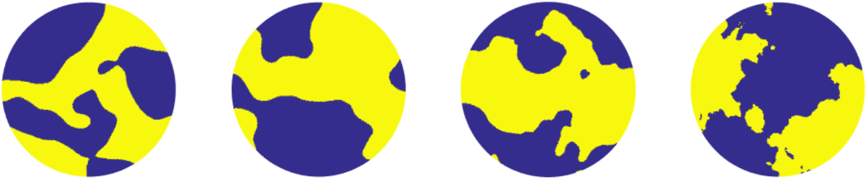
- These priors are discussed in (Iglesias, Lu, Stuart, '16). We give an overview of two-region case
- Let $\sigma_+, \sigma_- > 0$ be fixed positive numbers. Then we can define $F_{\mathrm{lvl}} : C^0(\overline{D}) \to \mathcal{A}(D)$ by

$$F_{\mathrm{lvl}}(u) = \sigma_+ \mathbb{1}_{u \geq 0} + \sigma_- \mathbb{1}_{u < 0}$$

- It can be shown that if $u \in C^0(\overline{D})$ and $|\{u = 0\}| = 0$, then $u_n \to u$ implies that $F_{\mathrm{lvl}}(u_n) \to F_{\mathrm{lvl}}(u)$ in measure
- The assumption that the zero level set has zero measure is in important one, and can be enforced (almost-surely) by choice of prior.

Prior Model #3: Level Set

- Take μ_0 to be Gaussian.
- Typical samples from $F_{\mathrm{lvl}}^*(\mu_0)$ when $\mu_0 = N(0, (-\Delta)^{-\alpha})$ for various α:



Remark

From the regularity result for the forward map $\mathcal{M}$, we now know that $\Pi \circ \mathcal{M} \circ F : X \to \mathbb{R}^L$ is continuous μ_0-a.s. for all three of the choices (F, X) outlined above.

The Forward Map & Likelihood

- Let X be a separable Banach space and $F : X \to \mathcal{A}(D)$ a map from the state space to the conductivity space.
- Choose a set of current stimulation patterns $I^{(j)} \in \mathbb{R}^L$, $j = 1, \dots, J$ and let $\mathcal{M}_j : \mathcal{A}(D) \to \mathbb{H}$ denote the solution map when using stimulation pattern $I^{(j)}$.
- Define the projection map $\Pi : \mathbb{H} \to \mathbb{R}^L$ by $\Pi(u, U) = U$.
- Define $\mathcal{G}_j : X \to \mathbb{R}^L$ by $\mathcal{G}_j = \Pi \circ \mathcal{M}_j \circ F$, and let $\mathcal{G} : X \to \mathbb{R}^{JL}$ denote the concatenation of these $\mathcal{G}_j$.
- As before, we assume the data $y \in Y := \mathbb{R}^{JL}$ arises via

$$y = \mathcal{G}(u) + \eta, \quad \eta \sim \mathbb{Q}_0 := N(0, \Gamma)$$

The Forward Map and Likelihood

- Assume that $u \sim \mu_0$, where μ_0 is independent of $\mathbb{Q}_0$. From the above, we see that $y|u \sim \mathbb{Q}_u := N(\mathcal{G}(u), \Gamma)$. This can be used to formally find the distribution of $u|y$.
- First note that

$$\frac{d\mathbb{Q}_u}{d\mathbb{Q}_0}(y) = \exp\left(-\Phi(u; y) + \frac{1}{2}|y|_\Gamma^2\right)$$

where the potential $\Phi : X \times Y \to \mathbb{R}$ is given by

$$\Phi(u; y) = \frac{1}{2}|\mathcal{G}(u) - y|_\Gamma^2.$$

- Then under suitable regularity conditions, Bayes' theorem tells us that the distribution μ^y of $u|y$ satisfies

$$\frac{d\mu^y}{d\mu_0}(u) \propto \exp(-\Phi(u; y))$$

after absorbing the $\exp(\frac{1}{2}|y|_\Gamma^2)$ term into the normalisation constant.

Existence and Well-posedness of the Posterior

Theorem (D, Stuart, '15)

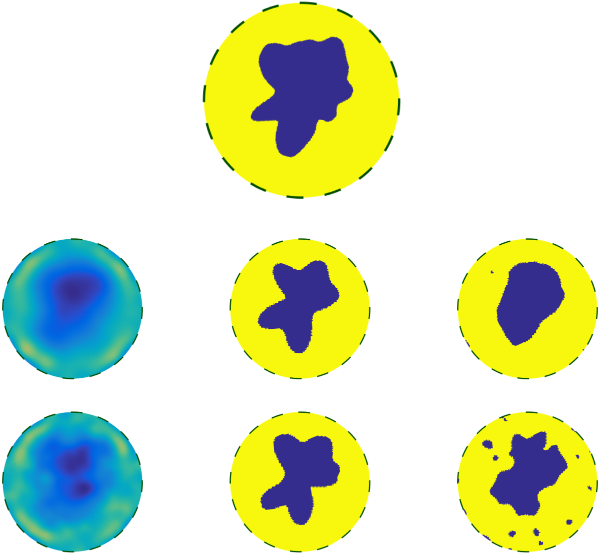
Let $(X, \mathcal{F}, \mu_0)$ denote any of the probability spaces associated with the priors introduced previously, and let $\Phi : X \times Y \to \mathbb{R}$ be the potential associated with the corresponding forward map. Then the posterior distribution μ^y of the state u given data y is well-defined. Furthermore, $\mu^y \ll \mu_0$ with Radon-Nikodym derivative

$$\frac{d\mu^y}{d\mu_0}(u) \propto \exp(-\Phi(u; y)).$$

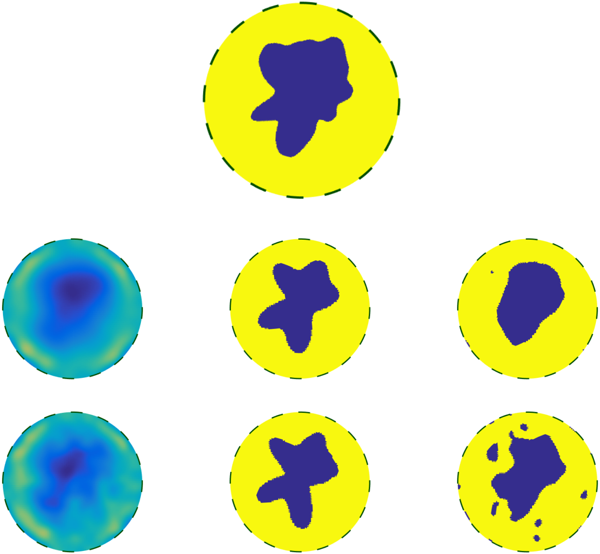
Additionally, μ^y locally Lipschitz with respect to y, in the Hellinger distance, and so if $y, y' \in Y$ and $f \in L^2_{\mu_0}(X; S)$, then there is a $C > 0$ such that

$$|\mathbb{E}^{\mu^y} f(u) - \mathbb{E}^{\mu^{y'}} f(u)| \leq C|y - y'|_Y.$$

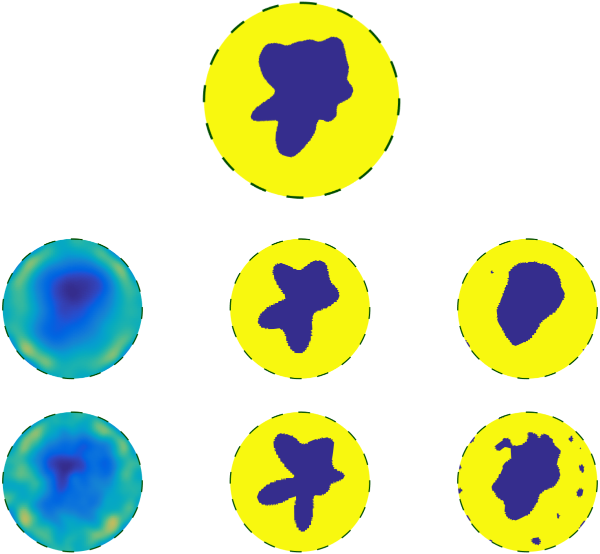
Numerical Experiments: Truth From Star-shaped Prior



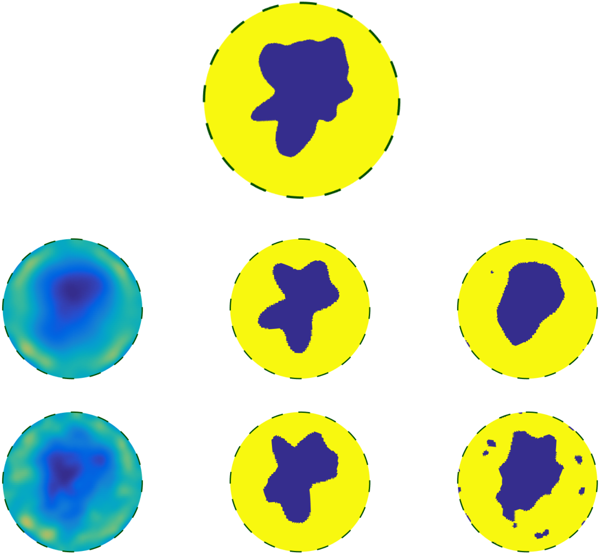
Numerical Experiments: Truth From Star-shaped Prior



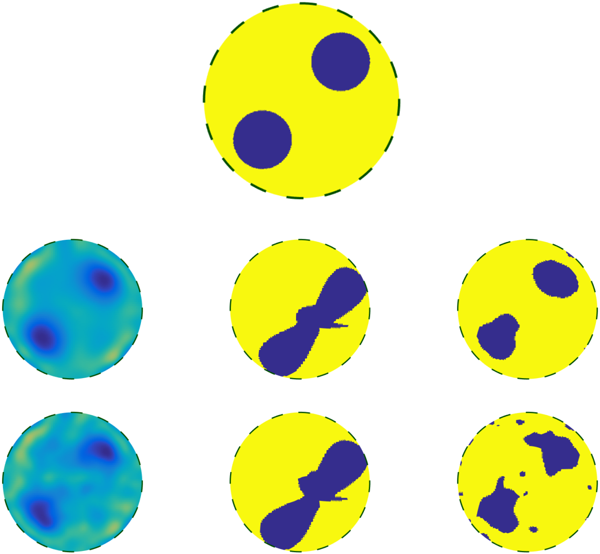
Numerical Experiments: Truth From Star-shaped Prior



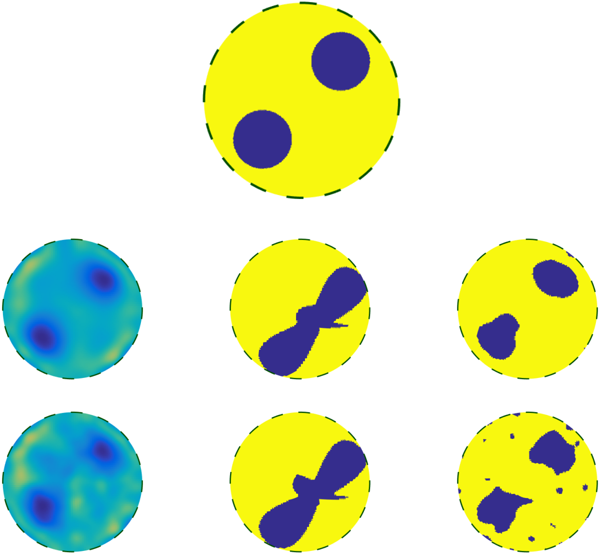
Numerical Experiments: Truth From Star-shaped Prior



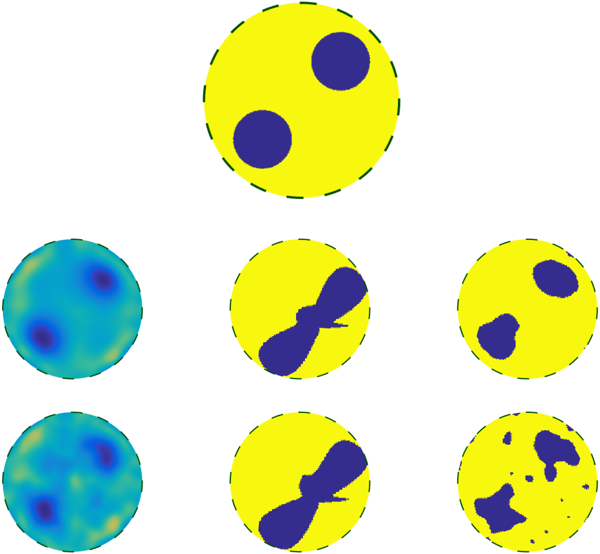
Numerical Experiments: Truth Not From Prior



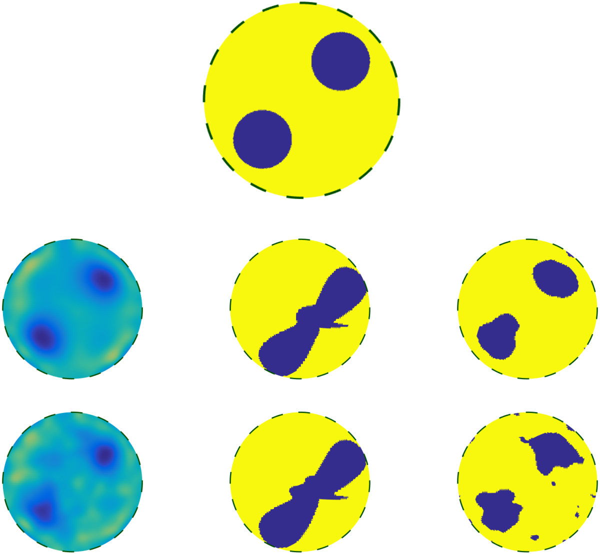
Numerical Experiments: Truth Not From Prior



Numerical Experiments: Truth Not From Prior



Numerical Experiments: Truth Not From Prior



References

Masoumeh Dashti and Andrew M. Stuart, *The Bayesian Approach to Inverse Problems*, arXiv.

Marco A. Iglesias, Yulong Lu, and Andrew M. Stuart, *A Bayesian Level Set Method for Geometric Inverse Problems*, arXiv.

Jari P. Kaipio, Ville Kolehmainen, Erkki Somersalo, and Marko Vauhkonen, *Statistical inversion and Monte Carlo sampling methods in electrical impedance tomography*, Inverse Problems **16** (2000), no. 5, 1487–1522.

Sari Lasanen, Janne M. J. Huttunen, and Lassi Roininen, *Whittle-Matérn priors for Bayesian statistical inversion with applications in electrical impedance tomography*, Inverse Problems and Imaging **8** (2014), no. 2, 561–586.

Erkki Somersalo, Margaret Cheney, and David Isaacson, *Existence and Uniqueness for Electrode Models for Electric Current Computed Tomography*, SIAM Journal on Applied Mathematics **52** (1992), no. 4, 1023–1040.

Thank you!