Hierarchical Bayesian Level Set Inversion

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Outline

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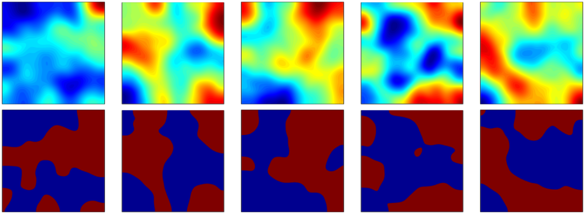
Level Set Representation

S.Osher and J.Sethian
Fronts propagating with curvature-dependent speed . . . .
J. Comp. Phys. **79**(1988), 12–49.

Piecewise constant function v defined through thresholding a continuous level set function u. Let

$$-\infty = c_0 < c_1 < \cdots < c_{K-1} < c_K = \infty.$$

$$v(x) = \sum_{k=1}^{K} v_k \, \mathbb{I}_{\{c_{k-1} < u \le c_k\}}(x); \quad v = F(u).$$



$F : X \to Z$ is the level-set map, X cts fns. Z piecewise cts fns.

Example 1: Image Reconstruction

H.W. Engl, M. Hanke and A. Neubauer
Regularization of Inverse Problems
Kluwer(1994)

Forward Problem

Define $K : Z \to \mathbb{R}^J$ by $(Kv)_j = v(x_j), x_j \in D \subset \mathbb{R}^d$. Given $v \in Z$

$$y = Kv.$$

Let $\eta \in \mathbb{R}^J$ be a realization of an observational noise.

Inverse Problem

Given prior information $v = F(u)$, $u \in X$ and $y \in \mathbb{R}^J$, find v :

$$y = Kv + \eta.$$

Example 2: Groundwater Flow

M. Dashti and A.M. Stuart
The Bayesian approach to inverse problems.
Handbook of Uncertainty Quantification
Editors: R. Ghanem, D.Higdon and H. Owhadi, Springer, 2017.
arXiv:1302.6989

Forward Problem: Darcy Flow

Let $X^+ := \{v \in Z : \mathrm{essinf}_{x\in D} v > 0\}$. Given $\kappa \in X^+$, find $y := \mathcal{G}(\kappa) \in \mathbb{R}^J$ where $y_j = \ell_j(p)$, $V := H_0^1(D), \ell_j \in V^*$, $j = 1, \dots, J$ and

$$\begin{cases} -\nabla \cdot (\kappa \nabla p) = f & \text{in } D \\ p = 0 & \text{on } \partial D. \end{cases}$$

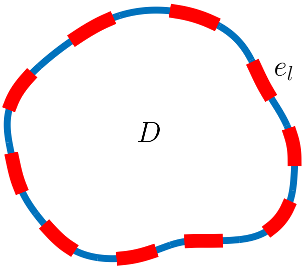
Let $\eta \in \mathbb{R}^J$ be a realization of an observational noise.

Inverse Problem

Given prior information $\kappa = F(u)$, $u \in X$ and $y \in \mathbb{R}^J$, find κ :

$$y = \mathcal{G}(\kappa) + \eta.$$

Example 3: Electrical Impedance Tomography



- Apply currents I_ℓ on e_ℓ, $\ell = 1, \ldots, L$.
- Induces voltages Θ_ℓ on e_ℓ, $\ell = 1, \ldots, L$.
- Input is (σ, I), output is (θ, Θ).
- We have an Ohm's law $\Theta = R(\sigma)I$.

$$\begin{cases} -\nabla \cdot (\sigma(x)\nabla\theta(x)) = 0 & x \in D \\ \displaystyle\int_{e_\ell} \sigma \frac{\partial\theta}{\partial\nu}\,\mathrm{d}S = I_\ell & \ell = 1, \ldots, L \\ \sigma(x)\dfrac{\partial\theta}{\partial\nu}(x) = 0 & x \in \partial D \setminus \bigcup_{\ell=1}^{L} e_\ell \\ \theta(x) + z_\ell\sigma(x)\dfrac{\partial\theta}{\partial\nu}(x) = \Theta_\ell & x \in e_\ell, \ell = 1, \ldots, L \end{cases} \tag{PDE}$$

Example 3: Electrical Impedance Tomography

M. M. Dunlop and A. M. Stuart
The Bayesian formulation of EIT.
arXiv:1509.03136
Inverse Problems and Imaging, Submitted, 2015.

Forward Problem

- Let $X \subseteq L^\infty(D)$, and denote $X^+ := \{u \in X : \operatorname{essinf}_{x \in D} u > 0\}$.
- Given $u \in X, F : X \to X^+$ and $\sigma = F(u)$, find $(\theta, \Theta) \in H^1(D) \times \mathbb{R}^L$ solving (PDE).
- This gives $\Theta = R(F(u))I$.

Let $\eta \in \mathbb{R}^L$ be a realisation of an observational noise.

Inverse Problem

Given prior information $\sigma = F(u)$, $u \in X$, and given currents I and $y \in \mathbb{R}^L$, find σ :

$$y = R(\sigma)I + \eta.$$

Outline

Tikhonov Regularization

$\mathcal{G} : Z \to \mathbb{R}^J$ and $\eta \in \mathbb{R}^J$ a realization of an observational noise.

Inverse Problem

Find $u \in X$, given $y \in \mathbb{R}^J$ satisfying $y = \mathcal{G} \circ F(u) + \eta$.

Model-Data Misfit

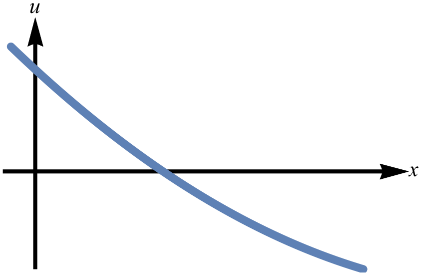
The model-data misfit is $\Phi(u; y) = \frac{1}{2}\left|\Gamma^{-\frac{1}{2}}\big(y - \mathcal{G} \circ F(u)\big)\right|^2$.

Tikhonov Regularization

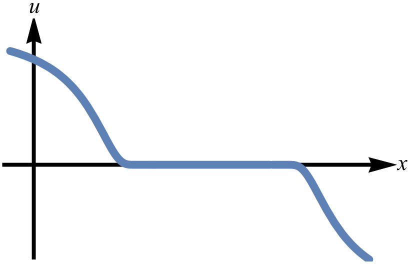
Minimize, for some E compactly embedded into X,

$$I(u; y) := \Phi(u; y) + \frac{1}{2}\|u\|_E^2.$$

Issue 1: Discontinuity of Level Set Map



$F(\cdot)$ is continuous at u

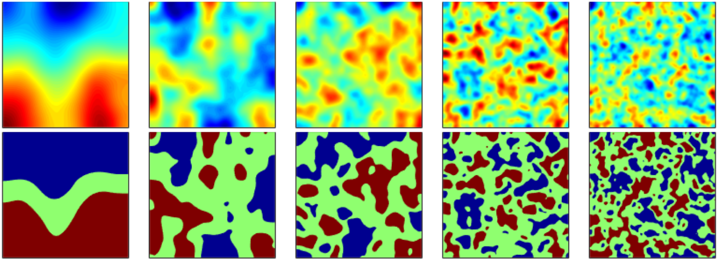


$F(\cdot)$ is discontinuous at u

$$v = F(u) := v^{+}\,\mathbb{I}_{\{u \geq 0\}}(x) + v^{-}\,\mathbb{I}_{\{u < 0\}}(x).$$

Causes problems in classical level set inversion.

Issue 2: Length-Scale Matters



- Figure demonstrates role of length-scale in level-set function.
- Classical Tikhonov-Phillips regularization does not allow for direct control of the length-scale.

New ideas needed.

Issue 3: Amplitude Matters

Y. Lu
Probabilistic Analysis of Interface Problems
PhD Thesis (2016), Warwick University

- Consider case of $K = 2$, $c_1 = 0$.
- For contradiction assume $u^\star$ is a minimizer of $I(u; y)$.
- Now define the sequence $u_\epsilon = \epsilon u^\star$. Then if $0 < \epsilon < 1$,

$$\Phi(u_\epsilon; y) = \Phi(u^\star; y), \|u_\epsilon\|_E = \epsilon\|u^\star\|_E \Rightarrow I(u_\epsilon; y) < I(u^\star; y).$$

- Hence $I(u_\epsilon; y)$ is an infimizing sequence. But $u_\epsilon \to 0$ and $I(0; y) > I(u_\epsilon; y)$ for $\epsilon \ll 1$.
- Thus infimum cannot be attained.
- This issue caused by thresholding at 0. But idea generalizes.

Choice of threshold matters.

Outline

Bayes' Formula

Prior

Probabilistic information about u before data is collected:

$$\mu_0(du)$$

Likelihood

Since $y = \mathcal{G}(u) + \eta$, if $\eta \sim N(0, \Gamma)$, then $y|u \sim N(\mathcal{G}(u), \Gamma)$. The model-data misfit Φ is the negative log-likelihood:

$$\mathbb{P}(y|u) \propto \exp\big(-\Phi(u;y)\big), \quad \Phi(u;y) = \frac{1}{2}\Big|\Gamma^{-\frac{1}{2}}\big(y - \mathcal{G}(u)\big)\Big|^2.$$

Posterior

Probabilistic information about u after data is collected:

$$\mu^y(du) \propto \exp\big(-\Phi(u;y)\big)\mu_0(du).$$

Rigorous Statement

M. Iglesias, Y. Lu and A. M . Stuart
A level-set approach to Bayesian geometric inverse problems.
arXiv:1504.00313

M. Iglesias
A regularizing iterative ensemble Kalman method for PDE constrained inverse problems.
arXiv:1505.03876

$$L^2_\nu(X;S) = \{f : X \to S : \mathbb{E}^\nu \|f(u)\|_S^2 < \infty\}.$$

Theorem (Iglesias, Lu and S)

Let $\mu_0(du) = \mathbb{P}(du)$ and $\mu^y(du) = \mathbb{P}(du|y)$. Assume that $u \in X$ μ_0-a.s. Then for Examples 1–3 (and more) $\mu^y \ll \mu_0$ and $y \mapsto \mu^y(du)$ is Lipschitz in the Hellinger metric. Furthermore, if S is a separable Banach space and $f \in L^2_{\mu_0}(X;S)$, then

$$\left\|\mathbb{E}^{\mu^{y_1}} f(u) - \mathbb{E}^{\mu^{y_2}} f(u)\right\|_S \le C|y_1 - y_2|.$$

Key idea in proof is that $F : X \to Z$ is continuous μ_0- a.s.

Whittle-Matern Priors

$$c(x,y) = \sigma^2 \frac{1}{2^{\nu-1}\Gamma(\nu)}(\tau|x-y|)^{\nu} K_{\nu}(\tau|x-y|).$$

- ν controls smoothness – draws from Gaussian fields with this covariance have ν fractional Sobolev and Hölder derivatives.
- τ is an inverse length-scale.
- σ is an amplitude scale.

Ignoring boundary conditions, corresponding covariance operator given by

$$C_{\mathrm{WM}(\sigma,\tau)} \propto \sigma^2 \tau^{2\nu} (\tau^2 I - \triangle)^{-\nu - \frac{d}{2}}.$$

Example 2: Groundwater Flow

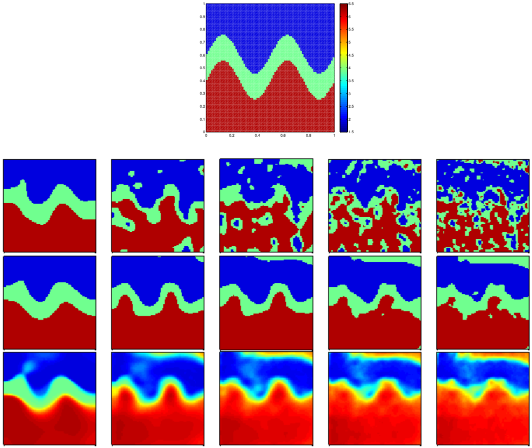


Figure: (Row 1, $\tau = 15$) Logarithm of the true hydraulic conductivity. (Row 2 $\tau = 10, 30, 50, 70, 90$) samples of $F(u)$. (Row 3) $F(\mathbb{E}(u))$. (Row 4) $\mathbb{E}(F(u))$.

Outline

Hierarchical Whittle-Matern Priors (The Problem)

Prior:

$$\mathbb{P}(u, \tau) = \mathbb{P}(u|\tau)\mathbb{P}(\tau).$$

$$\mathbb{P}(u|\tau) = N(0, C_{\mathrm{WM}(\sigma,\tau)}).$$

Recall $C_{\mathrm{WM}(\sigma,\tau)} \propto \sigma^2 \tau^{2\nu} (\tau^2 I - \triangle)^{-\nu - \frac{d}{2}}$.

Theorem

For fixed σ the family of measures $N(0, C_{\mathrm{WM}(\sigma,\cdot)})$ are mutually singular.

Algorithms which attempt to move between singular measures perform badly under mesh refinement (since they fail completely in the fully resolved limit).

Hierarchical Whittle-Matern Priors (The Solution)

M. Dunlop, M. Iglesias and A. M . Stuart
Hierarchical Bayesian Level Set Inversion
arXiv:1601.03605

Hence choose $\sigma = \tau^{-\nu}$ and define: $C_\tau \propto (\tau^2 I - \triangle)^{-\nu - \frac{d}{2}}$. Prior:

$$\mathbb{P}(u, \tau) = \mathbb{P}(u|\tau)\mathbb{P}(\tau).$$

$$\mathbb{P}(u|\tau) = N(0, C_\tau).$$

Theorem

The family of measures $N(0, C_\tau)$ are mutually equivalent.

Suggests need to scale thresholds in F by τ. Let

$$-\infty = c_0 < c_1 < \cdots < c_{K-1} < c_K = \infty.$$

$$v(x) = \sum_{k=1}^{K} v_k \, \mathbb{I}_{\{c_{k-1} < u\tau^\nu \le c_k\}}(x); \quad v = F(u, \tau).$$

Hierarchical Posterior

Prior

$$\mu_0(\mathrm{d}u, \mathrm{d}\tau) = \mu_0(\mathrm{d}u \,|\, \tau)\pi_0(\mathrm{d}\tau)$$

where $\mu_0(\,\cdot\,|\tau) = N(0, C_\tau)$, π_0 is hyperprior on τ.

Likelihood

Likelihood is given by $\mathbb{P}(y|u,\tau) \propto \exp\Big(-\Phi(u,\tau;y)\Big)$. where the model-data misfit is now $\Phi(u,\tau;y) = \frac{1}{2}\Big|\Gamma^{-\frac{1}{2}}\big(y - \mathcal{G}\circ F(u,\tau)\big)\Big|^2$.

Posterior

Probabilistic information about u after data is collected:

$$\mu^y(\mathrm{d}u, \mathrm{d}\tau) \propto \exp\big(-\Phi(u,\tau;y)\big)\mu_0(\mathrm{d}u, \mathrm{d}\tau).$$

MCMC

We implement a Metropolis-within-Gibbs algorithm to generate a posterior-invariant Markov chain (u_k, τ_k):

- Propose v_{k+1} from a $\mathbb{P}(u|\tau_k)-$reversible kernel.
- $u_{k+1} = v_{k+1}$ w.p. $1 \wedge \exp(\Phi(u_k, \tau_k) - \Phi(v_{k+1}, \tau_k))$,
  $u_{k+1} = u_k$ otherwise.
- Propose t_{k+1} from a $\mathbb{P}(\tau)-$reversible kernel.
- $\tau_{k+1} = t_{k+1}$ w.p. $1 \wedge \exp(\Phi(u_{k+1}, \tau_k) - \Phi(u_{k+1}, t_{k+1}))w(\tau_k, t_{k+1})$,
  $\tau_{k+1} = \tau_k$ otherwise.

Here $w(\tau, t)$ is density of $N(0, C_t)$ with respect to $N(0, C_\tau)$.

Example 2: Groundwater Flow

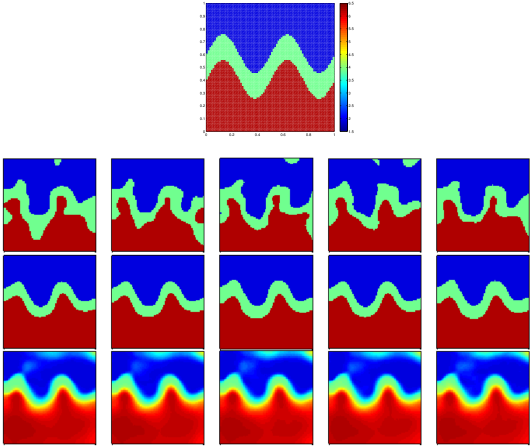


Figure: (Row 1, $\tau = 15$) Logarithm of the true hydraulic conductivity (middle, top). (Row 2 $\tau = 10, 30, 50, 70, 90$) samples of $F(u, \tau)$. (Row 3) $F(\mathbb{E}(u), \mathbb{E}(\tau))$. (Row 4) $\mathbb{E}(F(u, \tau))$.

Outline

Summary

- Last 5 years development of a theoretical and computational framework for infinite dimensional Bayesian inversion with wide applicability:
  1. Framework allows for noisy data and uncertain prior information.
  2. Probabilistic well-posedness.
  3. Theory clearly delineates (and links) analysis and probability.
  4. Theory leads to new algorithms (defined on Banach space).
  5. Grid-independent convergence rates for MCMC.
- The methodology has been extended to solve interface problems. This is achieved via the level set representation.
  1. Level set method becomes well-posed in this setting.
  2. Discontinuity in level set map is a probability zero event.
  3. Hierarchical choice of length-scale improves performance.
  4. Amplitude scale is linked to length-scale via measure equivalence.
  5. Algorithms which reflect this link are mesh-invariant.
- Potential for many future developments for both applications and theory:
  1. Applications: subsurface imaging, medical imaging.
  2. Theory: inhomogenous length-scales, new hierarchies.