1.3 Taylor Vortex and Kelvins Circulation Theorem

Consider flow $u = -\frac{x\lambda}{2}$, $v = -\frac{y\lambda}{2}$, $w = z\lambda$

$$\omega(x,y,t) = (0,0,\omega_z(x,y,t)) = \Omega(t)\exp(-a(t)(x^2+y^2))$$

where at $t=0$, $\Omega = \Omega_0$ and $a = a_0$.

(a) Is this incompressible?

Answer $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla\cdot\underline{u} = \frac{-\lambda}{2} - \frac{\lambda}{2} + \lambda = 0$

(b) What is total circulation Γ_T?

$x = r\cos\theta$
$y = r\sin\theta$

Answer $\Gamma_T = \oint_{C_r} \underline{u}\cdot d\underline{x} = \int_S \underbrace{\underline{\omega}\cdot\underline{n}}_{=\omega_z}\, ds$

$$= \int_0^{2\pi}\int_0^{\infty} \omega_z(r,t)\,|J|\,dr\,d\theta = \int_0^{\infty} 2\pi r\,\omega_z(r,t)\,dr$$

$$\left(J = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} \end{bmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r \right.$$

$$\left. \Rightarrow ds = dx\,dy = r\,dr\,d\theta \right)$$

$$= \int_0^{\infty} 2\pi r\,\Omega(t)\exp(-a(t)(r^2))\,dr$$

$$= 2\pi\Omega(t)\int_0^{\infty} r e^{-ar^2}\,dr$$

$$= -\frac{\pi\Omega(t)}{a(t)}\left[e^{-ar^2}\right]_0^{\infty} = +\frac{\pi\Omega(t)}{a(t)}$$

Kelvin's Circulation Theorem say Γ independent of time so $\Gamma_T\big|_{t=0} = \frac{\pi\Omega(0)}{a(0)} = \frac{\pi\Omega_0}{a_0}$

(c) What is $\Omega(t)$?

<u>Answer</u> Recall vorticity equation:

$$\frac{\partial \underline{\omega}}{\partial t} + (\underline{u}\cdot\nabla)\underline{\omega} = (\underline{\omega}\cdot\nabla)\underline{u} + \cancel{\nu\nabla^2\underline{\omega}}$$

assume $\nu = 0$

$\underline{\omega} = (0, 0, \omega_z(x,y,t)) \Rightarrow$

$$\frac{\partial \Omega(t)}{\partial t} e^{-a(x^2+y^2)} - \underbrace{\Omega(t)\,a(t)(x^2+y^2)e^{-a(t)(x^2+y^2)}}_{\substack{\text{non advection} \\ (\underline{u}\cdot\nabla)\omega_z}} = \underbrace{\lambda\,\Omega(t)\,e^{-a(t)(x^2+y^2)}}_{\substack{\text{vortex stretching} \\ (\underline{\omega}\cdot\nabla)\underline{u} \\ = \omega_z \frac{\partial}{\partial z}(\lambda z)}}$$

Set $x=y=0 \Rightarrow \dfrac{\partial \Omega(t)}{\partial t} = \lambda\,\Omega(t)$

$\Rightarrow \Omega(t) = Ae^{\lambda t}$ (solution to first order ODEs)

$\Omega(0) = \Omega_0 \Rightarrow A = \Omega_0 \Rightarrow \underline{\underline{\Omega(t) = \Omega_0 e^{\lambda t}}}.$

(d) What is $a(t)$?

<u>Answer</u> Using conservation of circulation we have

$$\Gamma_T = \frac{\pi\Omega_0}{a_0} = \frac{\pi\Omega_0 e^{\lambda t}}{a(t)}$$

$$\Rightarrow a(t) = a_0 e^{\lambda t}$$

(e) What is azimuthal velocity $u_\theta(r,t)$?

<u>Answer</u> Circulation on circle radius r:

$$\Gamma(r) = \int_0^r 2\pi s\,\omega_z(s,t)\,ds = 2\pi\,\Omega(t)\int_0^r s e^{-a(t)s^2} ds$$

$$= \frac{\pi\,\Omega(t)}{a(t)}\left[e^{-a(t)s^2}\right]_0^r = \frac{\pi\,\Omega(t)}{a(t)}\left(1 - e^{-a(t)r^2}\right)$$

$$= \frac{\pi\,\Omega_0}{a_0}\left(1 - e^{-a(t)r^2}\right)$$

Stokes theorem $\Rightarrow \Gamma(r) = \int_S \underline{\omega}\cdot\underline{n}\,ds = \int_{C_r} \underline{u}\cdot dx$

$$= \cancel{\int_0^{2\pi} u_\theta(r,t)\,d\theta} = \underbrace{2\pi r}_{\substack{\text{length of} \\ \text{circle} \\ \text{“}dx\text{”}}}\ \underbrace{u_\theta(r,t)}_{\substack{\text{velocity} \\ \text{“}\underline{u}\text{”}}}$$

$$\Rightarrow u_\theta(r,t) = \frac{\Gamma(r)}{2\pi r} = \frac{\Omega_0}{2a_0 r}\left(1 - e^{-a(t)r^2}\right)$$

(f) ~~Show~~ $u_\theta(r,t) \to \frac{\Omega_0}{2a_0}$ as $t\to\infty$ for fixed r:

Answer as $t \to \infty$ $e^{-a(t)r^2} \to 0$ since $a(t) = a_0 e^{\lambda t} \to \infty$

$\Rightarrow u_\theta \to \frac{\Omega_0}{2a_0 r}$ for fixed r.

1.6 Vortex dipole

Streamfunction due to point vortices at $\underline{x}_i = (x_i, y_i)$ is

$$\psi = \frac{-\Gamma_i}{4\pi} \ln\left((x-x_i)^2 + (y-y_i)^2\right) = \frac{-\Gamma_i}{2\pi} \ln\left((x-x_i)^2 + (y-y_i)^2\right)^{1/2}$$

Consider dipole at $y_i = \pm d/2$, $x_i = 0$, $\Gamma_i = \pm\Gamma$.

(a) Write streamfunction

Answer: $\psi = \frac{-\Gamma}{2\pi} \ln |x^2 + (y-\frac{d}{2})^2|^{1/2} + \frac{\Gamma}{2\pi} \ln |x^2 + (y+\frac{d}{2})^2|^{1/2}$

(b) The streamwise velocity due to vortices is $u_x = \frac{\partial}{\partial y}\psi$.

Answer:

$$u_x = \frac{\Gamma}{2\pi}\left(-\frac{y - d/2}{x^2 + (y-d/2)^2} + \frac{y + d/2}{x^2 + (y+d/2)^2}\right)$$

at $y=0$:

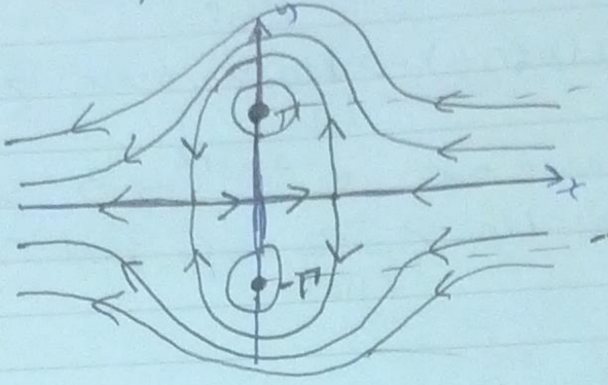
$$u_x = \frac{\Gamma}{2\pi}\left(\frac{d}{x^2 + \frac{d^2}{4}}\right)$$

(c) Assume dipole is in background flow $U = \frac{-\Gamma}{2\pi d}$, where on $y=0$ does $u_x = -U$? These are stagnation points.

Answer: $\frac{\Gamma}{2\pi}\left(\frac{d}{x^2 + \frac{d^2}{4}}\right) = \frac{\Gamma}{2\pi d}$

$$\Rightarrow x^2 = \frac{3}{4} d^2 \Rightarrow x = \pm \frac{\sqrt{3}}{2} d.$$

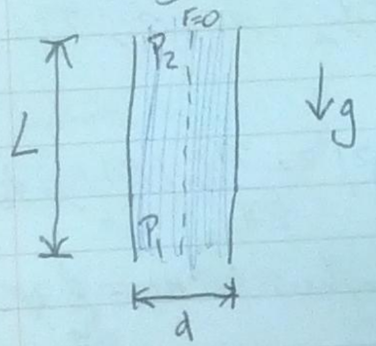
(d) Sketch the flow:



③ 2.1 Flow Through Pipes

Consider vertical pipe, assume cylindrically symmetric, steady, laminar Poiseuille flow.

ρ density, ν viscosity $\}$ constant.



Aim: show flow rate through pipe is

$$Q = \frac{\pi d^4}{128\nu}\left(\frac{P_2 - P_1}{\rho L} + g\right)$$

(a) Simplify Navier-Stokes to a steady Stokes equation.

Answer $\underbrace{\frac{\partial w}{\partial t}}_{\text{steady}}\overset{0}{} + \underbrace{(\underline{u}\cdot\nabla)w}_{\text{ignore advection}}\overset{0}{} = -\frac{1}{\rho}\nabla P + \nu\nabla^2 w + g\underline{z}$

z-component of velocity

Laplacian in polar: $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

So we have $\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = -\frac{r}{\nu}\left(\frac{P_2 - P_1}{\rho L} + g\right)$ (1)

(b) Find velocity profile:

Answer Flow is symmetric at $r=0$ and boundary condition at $r=0$ is $\frac{dw}{dr} = 0$

also note $w=0$ at $r = d/2$

Integrate (1) to get $r\frac{dw}{dr} = -\frac{r^2}{2\nu}\left(\frac{P_2 - P_1}{\rho L} + g\right) + c_1$

use $\frac{\partial w}{\partial r} = 0$ at $r = 0$ to get $c_1 = 0$.

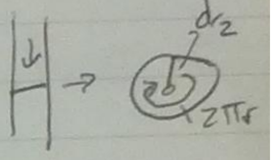
Integrate again to get $w = -\frac{r^2}{4\nu}\left(\frac{P_2 - P_1}{\rho L} + g\right) + c_2$

$w = 0$ at $r = d/2 \Rightarrow c_2 = \frac{d^2}{16\nu}\left(\frac{P_2 - P_1}{\rho L} + g\right)$

so that $w = \frac{1}{4\nu}\left(\frac{P_2 - P_1}{\rho L} + g\right)\left(\frac{d^2}{4} - r^2\right)$

(c) Find flow rate:

<u>Answer</u> Flow rate $Q = 2\pi \int_0^{d/2} w\, r\, dr$



$$= 2\pi \int_0^{d/2} \frac{1}{4\nu}\left(\frac{P_2 - P_1}{\rho L} + g\right)\left(\frac{d^2 r}{4} - r^3\right) dr$$

$$= \left[\frac{2\pi}{4\nu}\left(\frac{P_2 - P_1}{\rho L} + g\right)\left(\frac{d^2 r^2}{8} - \frac{r^4}{4}\right)\right]_0^{d/2}$$

$$= \frac{\pi d^4}{128\nu}\left(\frac{P_2 - P_1}{\rho L} + g\right)$$