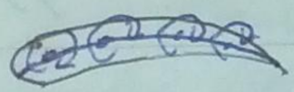
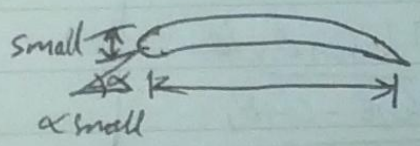
We have seen lift calculated around a cylinder and then transformed to an aerofoil using a Zhukovsky transform, however this is not practical for real aerofoils (doesn't work in 3D). It does tell us that lift is proportional to circulation and a specific value of Γ is needed to resolve singularities and generate lift, that is velocity at trailing edge needs to be finite (Kutta condition)

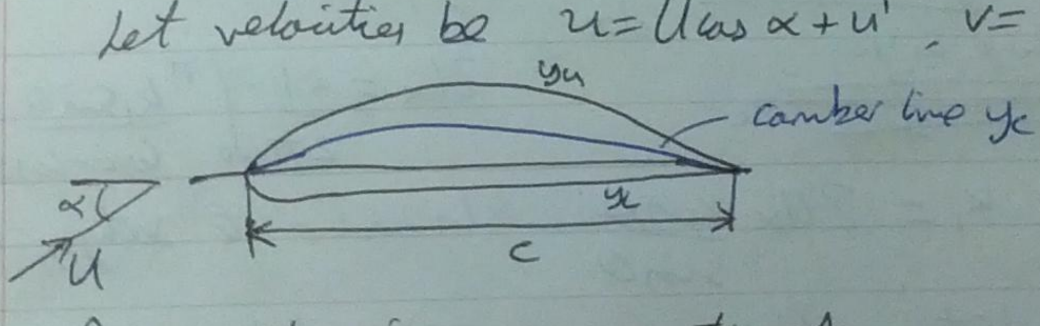
Alternative approach: concentrate a vortex distribution on a camber line



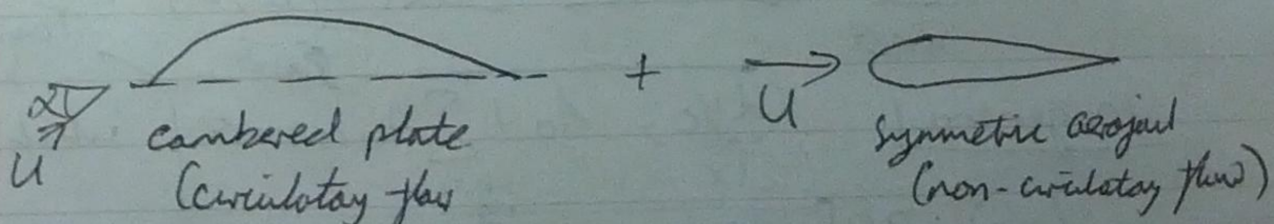
Thin Aerofoil Theory – assumes thickness small compared to chord length
– assume low angle of attack



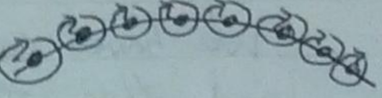
Let velocities be $u = U\cos\alpha + u'$, $v = U\sin\alpha + v'$ (perturbations: u', v')



Assume flow is a superposition of



Circulation about chord is sum of vortex elements $\Gamma = \int_0^c k\,ds$



where k is the distribution of vorticity over the camber line.

Kutta condition $k = 0$ at $x = c$ (trailing edge)

The induced velocity from the circulation of all vortex elements along the chord is

$$v' = \frac{1}{2\pi}\int_0^c \frac{k\,dx}{x - x_1}$$

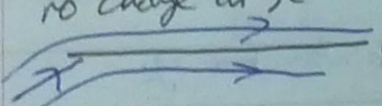
It can be shown that v' must satisfy the boundary condition $v' = U\left[\frac{dy_c}{dx} - \alpha\right]$

Using $x = \frac{c}{2}(1-\cos\theta)$ makes things easier:

$$U\left[\frac{dy_c}{dx} - \alpha\right] = -\frac{1}{2\pi}\int_0^{\pi}\frac{k\sin\theta\,d\theta}{\cos\theta - \cos\theta_1}$$

For a ~~plate~~ plate $\frac{dy_c}{dx} = 0$, so

no change in y_c



$$U\alpha = +\frac{1}{2\pi}\int_0^{\pi}\frac{k_1\sin\theta\,d\theta}{\cos\theta - \cos\theta_1}$$

$\Rightarrow k_1 = 2U\alpha\frac{\cos\theta}{\sin\theta}$. In order to satisfy Kutta condition ($k=0$ at $\theta = \pi$) set $k = k_1 + k_2$ with $k_2 = \frac{2U\alpha}{\sin\theta}$ so $k = 2U\alpha\frac{(1+\cos\theta)}{\sin\theta}$

Fourier-cosine series

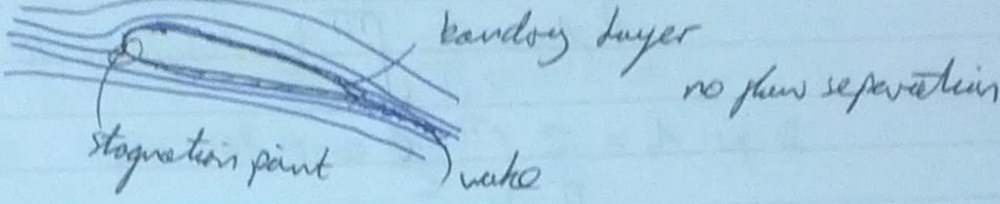
In general $\frac{dy_c}{dx} = A_0 + \sum_{n=1}^{\infty} A_n\cos n\theta$ which gives

$$k(\theta) = 2U\left[(\alpha - A_0)\frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right]$$ where $A_n = \frac{2}{\pi}\int_0^{\pi}\frac{dy_c}{dx}\cos(n\theta)\,d\theta$.

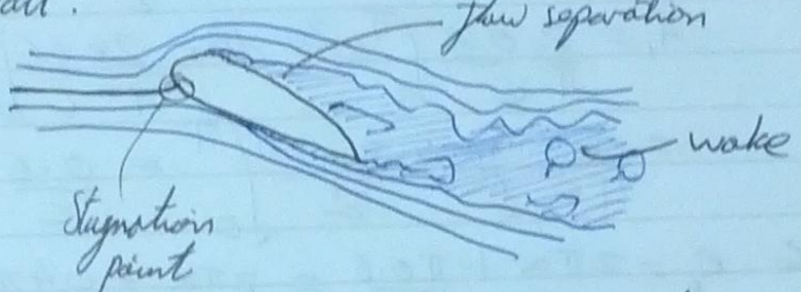
4.4 PD 2009-3

Sketch streamlines, boundary layers and the wake around a wing:

a) In the case of attached flow



(b) In a stall:



(c) Estimate lift force and skin friction acting on the wing of breadth $b = 10\,m$
chord $c = 1.5\,m$

upper aerofoil curve $y_u = 33\,cm \times \frac{x(c-x)}{c^2/4}$

lower aerofoil curve $y_L = -3\,cm\, \frac{x(c-x)}{c^2/4}$

angle of attack $\alpha = 2^\circ = \frac{2}{180}\pi = \frac{\pi}{90}$

for the following background flows:

(i) In air at sea level: $\rho = 1.2\,kg\,m^{-3}$, $\nu = 1.5\times10^{-5}\,m^2 s$
velocity $18\,km\,h^{-1}$

(ii) In water at depth 100m: $\rho = 1000\,kg\,m^{-3}$, velocity $36\,km\,h^{-1}$

Answer The expression for the camber line is

$$y_c = \frac{1}{2}(y_u + y_L) = \frac{x(c-x)}{c^2/4}\, 0.15\,m$$

and $\frac{dy_c}{dx} = \frac{4}{c}\left(1-\frac{2}{c}x\right)0.15$. Using the substitution

$x = \frac{c}{2}(1-\cos\theta)$ we get $\frac{dy_c}{dx} = 0.15\left(\frac{4}{c}\right)\cos\theta$.

For a general thin aerofoil:

$$A_n = \frac{2}{\pi}\int_0^{\pi}\frac{dy_c}{dx}\cos(n\theta)\,d\theta$$

So for $A_0 = \frac{2}{\pi}\int_0^{\pi}\frac{0.6}{c}\cos\theta\,d\theta = 0$

power reduction formula

$$A_1 = \frac{2}{\pi}\frac{0.6}{c}\int_0^{\pi}\cos^2\theta\,d\theta = \frac{2}{\pi}\frac{0.6}{c}\int_0^{\pi}\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{0.6}{\pi c}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{0.6}{c}.$$

So $C_L = 2\pi\alpha + \frac{\pi 0.6}{c} = 2\pi\alpha + 0.4\pi \quad (c=1.5)$

(i) For $\alpha = \frac{\pi}{90} = 2^\circ$; $C_L = 0.4\pi + \frac{\pi^2}{45} \approx 1.47596 \approx 1.5$

Velocity is $18\,\text{km h}^{-1} = 18 \times \frac{1000}{3600} = 5\,\text{ms}^{-1}$

Then the lift force is $\mathcal{L} = C_L \frac{1}{2}\rho U^2 c b$

$$= 1.5 \times \frac{1}{2} \times 1.2 \times 5^2 \times 1.5 \times 10$$

$$= \underline{\underline{337.5\,N}}$$

Skin friction $C_{SF} = 1.3\,Re^{-1/2}$

the Reynolds number is given by

$$Re = \frac{Uc}{\nu} = \frac{5 \times 1.5}{1.5 \times 10^{-5}} = 5 \times 10^5$$

$$\Rightarrow C_{SF} = \frac{1.3}{(5\times10^5)^{1/2}} \approx 1.84 \times 10^{-3}$$

Drag due to skin friction $= \frac{1}{2}\rho U^2 bc\,C_{SF} = 1.2 \times 5^2 \times 10 \times 1.5 \times 1.84 \times 10^{-3}$

$= 0.83\,N$

(ii) Water, $\rho = 1000\,kg\,m^{-3}$, velocity $36\,km\,h^{-1}$

<u>Answer</u> velocity $36\,km\,h^{-1} = 10\,m\,s^{-1}$

Lift $L = \frac{1}{2}\rho u^2 c b C_L$

$= \frac{1}{2} \times 1000 \times 10^2 \times 1.5 \times 10 \times 1.5$

$= 1.125 \times 10^6$