MA3D1 Fluid Dynamics Support Class 1

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1 Incompressible Navier-Stokes

The majority of this course will focus on the incompressible Navier-Stokes equations.

$$\underbrace{\frac{\partial \boldsymbol{u}}{\partial t}}_{\substack{\text{Time}\\\text{derivative}}} + \underbrace{(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}}_{\text{Advection}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\substack{\text{Pressure}\\\text{gradient}}} + \underbrace{\nu \Delta \boldsymbol{u}}_{\text{Viscosity}} + \underbrace{\boldsymbol{f}}_{\text{Forcing}} \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{(Incompressibility Condition)} \tag{2}$$

- The pressure gradient term will accelerate the flow in the direction from high pressure areas to low pressure.
- The viscosity term arises due to the stress the fluid exerts on itself. This term will dampen motion, a low viscosity will behave like water whereas a high viscosity will cause the fluid to behave like syrup. The condition (2) comes from the fact that density ρ is constant in the conservation of mass equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{3}$$

- The forcing term includes any external forcing such as gravity/buoyancy.
- The advection term describes the bulk movement of the fluid.

Working in 3D with $\boldsymbol{u} = (u_x, u_y, u_z) = (u, v, w)$ and some quantity of interest f (eg. density or a component of velocity) this advection term is written as

$$\begin{aligned}(\boldsymbol{u} \cdot \nabla)\boldsymbol{f} &= [(u_x, u_y, u_z) \cdot (\partial_x, \partial_y, \partial_z)](f_x, f_y, f_z)\\ &= (u_x\partial_x + u_y\partial_y + u_z\partial_z)(f_x, f_y, f_z)\end{aligned}$$

Here is a derivation of the advection term (from Acheson §1.2): $\frac{\partial f}{\partial t}$ is the rate of change of f at a fixed point (x, y, z) in space. Now the time derivative following the fluid (material derivative) is

$$\frac{D}{Dt}f = \frac{d}{dt}f(x(t), y(t), z(t), t). \tag{4}$$

We have

$$\frac{dx}{dt} = u, \frac{dy}{dt} = v, \frac{dz}{dt} = w. \tag{5}$$

Using the chain rule we get

$$\begin{aligned}\frac{Df}{Dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}\\ &= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}\\ &= \frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f\end{aligned}$$

2 Streamlines and Streamfunctions

Find the streamlines of a flow by solving

$$\frac{1}{u}\frac{dx}{ds} = \frac{1}{v}\frac{dy}{ds} = \frac{1}{w}\frac{dz}{ds}, \tag{6}$$

where the streamline is parameterised by s. For and incompressible ($\nabla\cdot\boldsymbol{u} = 0$), 2D ($\boldsymbol{u} = (u, v, 0)$) flow we can find a *streamfunction* ψ such that

$$u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x}. \tag{7}$$

In polar coordinates this is,

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, u_\theta = -\frac{\partial\psi}{\partial r}, \tag{8}$$

where $\boldsymbol{u} = (u_r, u_\theta, u_z)$. *Streamlines* are when the stream function ψ is constant, ie. level set of the streamfunction.

Example 1. *(Acheson Exercise 1.8) Consider the unsteady flow*

$$u = u_0, v = kt, w = 0, \tag{9}$$

where u_0, k are positive constants. Show that the streamlines are straight lines. Also show any fluid particle follows a parabolic path as time proceeds.

We can find the streamlines by integrating

$$\frac{1}{u_0}\frac{dx}{ds} = \frac{1}{kt}\frac{dy}{ds}, 0 = \frac{dz}{ds} \tag{10}$$

to get

$$y = \frac{kt}{u_0}x + const, z = const. \tag{11}$$

Alternatively, since this is a 2D flow, we may use the streamfunction found by solving:

$$u_0 = \frac{\partial\psi}{\partial y}, kt = -\frac{\partial\psi}{\partial x}, \tag{12}$$

to get $\psi = u_0 y - ktx$. Now the streamlines are when the streamfunction is constant ($\psi = const$) giving the streamlines as in equation (11), which are straight lines with gradient $\frac{kt}{u_0}$. The particle paths may be found by solving

$$\left.\frac{\partial x}{\partial t}\right|_{\boldsymbol{X}} = u_0, \left.\frac{\partial y}{\partial t}\right|_{\boldsymbol{X}} = kt, \left.\frac{\partial z}{\partial t}\right|_{\boldsymbol{X}} = 0, \tag{13}$$

where $\boldsymbol{X} = (X, Y, Z)$ are the Lagrangian coordinates. This gives

$$x = u_0 t + F_1(\boldsymbol{X}), y = \frac{1}{2}kt^2 + F_2(\boldsymbol{X}), z = F_3(\boldsymbol{X}), \tag{14}$$

for some functions F_1, F_2, F_3. We then use the fact that the Eulerian (fixed in space) and Lagrangian (follow fluid) coordinates coincide at $t = 0$, ie. $\boldsymbol{x} = \boldsymbol{X}$, to get

$$x = u_0 t + X, y = \frac{1}{2}kt^2 + Y, z = Z. \tag{15}$$

Eliminating t gives,

$$y = \frac{1}{2}k\left(\frac{x - X}{u_0}\right)^2 + Y. \tag{16}$$

Hence the particle paths are parabolic. Notice that equation (15) gives the transformation from Lagrangian coordinates to Eulerian coordinates $\boldsymbol{x} = \varphi(\boldsymbol{X}, t)$.

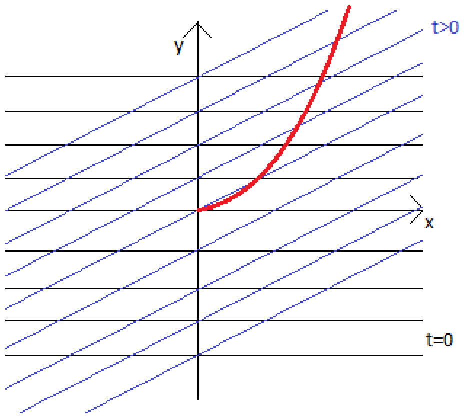

Figure 1: Streamlines are straight lines for this flow. The red line indicates the path of a particle originating from the origin.

Example 2. *Find the streamlines of the 2D flow*

$$u = \frac{y}{x^2+y^2}, v = -\frac{x}{x^2+y^2}. \tag{17}$$

For a 2D flow the streamfunction is found by solving,

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}, \tag{18}$$

which gives $\psi = \frac{1}{2}\log(x^2+y^2)$. *Streamlines are then when this function is constant, that is* $x^2+y^2 = const$, *ie. streamlines are circles.*

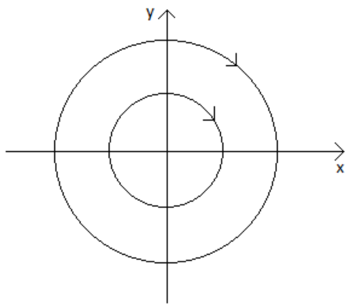

Figure 2: Streamlines are circles (clockwise) for this flow.

3 Vorticity

Vorticity in 3D is defined as

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}, \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}, \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right). \tag{19}$$

In polar coordinates the vorticity is

$$\boldsymbol{\omega} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e_r} & r\boldsymbol{e_\theta} & \boldsymbol{e_z} \\ \partial_r & \partial_\theta & \partial_z \\ u_r & ru_\theta & u_z \end{vmatrix}. \tag{20}$$

If $\boldsymbol{\omega} = 0$ then the flow is *irrotational*.

For a 2D flow $\boldsymbol{u} = (u(x,y,t), v(x,y,t), 0)$ the vorticity is $\boldsymbol{\omega} = (0,0,\omega)$ where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{21}$$

Vorticity is a measure of local rotation of fluid elements.

Example 3. *(Acheson §1.4) Consider the flow* $\boldsymbol{u} = (\beta y, 0, 0)$*. The vorticity is* $\omega = -\beta$*, and as seen in Figure 3 even though there is no global rotation, the fluid elements can be locally rotated.*

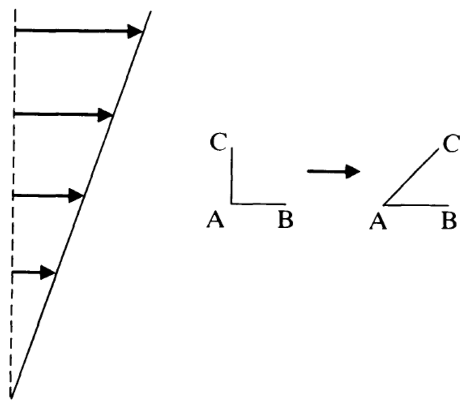

Figure 3: Deformation of two momentarily perpendicular fluid line elements in a shear flow.

4 Velocity Potential

An irrotational flow can be written as the gradient of a potential $\boldsymbol{u} = \nabla\phi$, where ϕ is a scalar function called the *velocity potential*. The gradient operator in polar coordinates is $(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}) = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial}{\partial z}\boldsymbol{e}_z$.

Example 4. *(Point Vortex)*

$$\boldsymbol{u} = \frac{\Gamma}{2\pi r}\boldsymbol{e_\theta} \tag{22}$$

We can find the velocity potential by integrating

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \Rightarrow \phi = \frac{\Gamma\theta}{2\pi}. \tag{23}$$

Similarly the streamfunction is found by integrating

$$u_\theta = -\frac{\partial \psi}{\partial r} \Rightarrow \psi = -\frac{\Gamma}{2\pi}\log(r). \tag{24}$$

5 Bernoulli's equation for unsteady flow

Consider Euler's equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + \chi\right). \tag{25}$$

If the flow is irrotational ($\nabla \times \boldsymbol{u} = 0$) so that $\boldsymbol{u} = \nabla\phi$ then

$$\frac{\partial \nabla\phi}{\partial t} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + \chi\right) \text{ where } \chi = gz.$$

Then integrate this to get

$$\partial_t\phi + \frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + \chi = G(t)$$

where $G(t)$ is an arbitrary function of time. Bernoulli's equation can be used to find exact solutions.