MA3D1 Fluid Dynamics Support Class 4 - Compressible Navier-Stokes and Waves

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1 Compressible Navier Stokes Equations

So far we have been assuming incompressibility, however this is insufficient for sound/pressure waves. The compressible Navier-Stokes equations are,

$$\text{Conservation of mass: } \partial_t \rho + \nabla \cdot (\rho v) = 0 \tag{1}$$

$$\text{Momentum: } \rho[\partial_t v + (v \cdot \nabla) v] = \mu \Delta v + (\lambda + \frac{1}{3}\mu)\nabla(\nabla \cdot v) - \nabla p + f \tag{2}$$

$$\text{Heat transform: } \rho c_v [\partial_t \theta + (v \cdot \nabla)\theta] = -p \nabla \cdot v + \kappa \Delta \theta + \Phi \tag{3}$$

2 Polytropic Waves

A polytropic process is a thermodynamic process that obeys $pv^n = \text{const}$. We have $n = \gamma = \frac{c_p}{c_v}$ the ratio of specific heats, (≈ 1.4 for air at normal temperature and pressure). Also $p = (\gamma - 1)\rho\theta$ from the ideal gas law.

3 Entropy

The entropy (for our purposes) is given by $\eta = c_v \log\left(\frac{p}{\rho^\gamma}\right)$ and is a measure of the number of ways in which a thermodynamic system can be arranged, or a measure of disorder.

If we have an ideal gas and heat conductivity is negligible ($\kappa = 0$) then entropy remains constant, this can be written as,

$$\frac{D}{Dt}(p\rho^{-\gamma}) = 0. \tag{4}$$

This is then called an *isentropic fluid*, note that this means entropy remains constant on each fluid element, however may differ between elements. If the entropy is uniformly constant throughout the fluid then the fluid is *homentropic*.

4 Sound Waves

Sound propagates as waves of pressure causing local regions of compression. We aim to find the equation describing sound waves. Let the undisturbed state be one of rest, with constant pressure p_0 and density ρ_0. We have a slight perturbation from this state of rest,

$$\boldsymbol{u} = \epsilon \boldsymbol{u}_1, p = p_0 + \epsilon p_1, \rho = \rho_0 + \epsilon \rho_1. \tag{5}$$

We aim to linearise the equations ignoring $O(\epsilon^2)$ or higher order terms of the perturbation variables. We know entropy is conserved so $p\rho^{-\gamma} = p_0\rho_0^{-\gamma}$ everywhere. Thus

$$(p_0 + \epsilon p_1)(\rho_0 + \epsilon \rho_1)^{-\gamma} = p_0 \rho_0^{-\gamma},$$

$$\Rightarrow \left(1 + \frac{\epsilon p_1}{p_0}\right)\left(1 + \frac{\epsilon \rho_1}{\rho_0}\right)^{-\gamma} = 1$$

$$\Rightarrow \left(1 + \frac{\epsilon p_1}{p_0}\right)\left(1 - \frac{\gamma \epsilon \rho_1}{\rho_0} + O(\epsilon^2)\right)^{-\gamma} = 1.$$

Ignore $O(\epsilon^2)$ terms,

$$\Rightarrow \frac{p_1}{p_0} = \frac{\gamma \rho_1}{\rho_0}$$
$$\Rightarrow p_1 = c_s^2 \rho_1 \tag{6}$$

where

$$c_s = \sqrt{\frac{\gamma \rho_1}{\rho_0}}. \tag{7}$$

The linearised equations for $\boldsymbol{u}_1$ and ρ_1 are

$$\rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} = -\nabla p_1 \tag{8}$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{u}_1 = 0. \tag{9}$$

Take the divergence of (8) to get,

$$\rho_0 \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{u}_1 = -\Delta p_1. \tag{10}$$

Finally we use (9) to remove the $\nabla \cdot \boldsymbol{u}_1$ term and (6) to get the wave equation

$$\frac{\partial^2 p_1}{\partial t^2} = c_s^2 \Delta p_1. \tag{11}$$

In one dimension this is

$$\frac{\partial^2 p_1}{\partial t^2} = c_s^2 \frac{\partial^2 p_1}{\partial x^2}, \tag{12}$$

which has general solution

$$p_1 = f(x - c_s t) + g(x + c_s t), \tag{13}$$

the first/second term corresponding to right/left propagation of a wave with speed c_s without change of shape. We identify c_s as the *speed of sound.*

The *wavenumber* k is a measure of the number of times a wave has the same phase per unit of space.

The *wavelength* λ is the distance between repeating units of a propagating wave of a given frequency, it is related to the wavenumber by

$$\lambda = \frac{2\pi}{k}. \tag{14}$$

The *phase speed* c_{ph} describes the motion within a wave packet. Velocity at which a phase of any one frequency component of the wave will propagate within the packet,

$$c_{ph} = \frac{\omega}{k}. \tag{15}$$

Here ω is the frequency and the *dispersion relation* is the relation $\omega(k)$ between ω and the wavenumber k. In higher dimensions this is

$$c_{ph} = \hat{\boldsymbol{k}} \frac{\omega}{|\boldsymbol{k}|}, \tag{16}$$

where $\boldsymbol{k}$ is a vector of the wavenumbers in each direction and $\hat{\boldsymbol{k}} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$.

The *group velocity* c_g describes the motion of the whole wave packet,

$$c_g = \frac{\partial \omega}{\partial k} (= \nabla_{\boldsymbol{k}} \omega \text{ in higher dimensions}). \tag{17}$$

If $c_g \neq c_{ph}$ then the waves are *dispersive.*

Example 1. *(Sound Waves are not dispersive) Sound waves satisfy (in 1D)*

$$\frac{\partial^2 p}{\partial t^2} = c_s^2 \frac{\partial^2 p}{\partial x^2} \tag{18}$$

where p is the pressure and c_s the speed of sound. Wave solutions are of the form,

$$p(x,t) = Ae^{i(kx-\omega t)}. \tag{19}$$

Put this into (18) and we get

$$A\omega^2 e^{i(kx-\omega t)} = Ak^2 c_s^2 e^{i(kx-\omega t)}$$

so $\omega = c_s k$. Then the phase speed is

$$c_{ph} = \frac{\omega}{k} = c_s,$$

and the group speed is

$$c_g = \frac{\partial \omega}{\partial k} = c_s = c_{ph}.$$

Therefore sound waves are not dispersive.

Example 2. *(Internal Gravity Waves) These are waves generated by buoyancy forces. We use the following linearised equations for a stratified fluid with small 2D motion,*

$$\rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial p_1}{\partial x}, \qquad \rho_0 \frac{\partial v_1}{\partial t} = -\frac{\partial p_1}{\partial y} - \rho_1 g,$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \qquad \frac{\partial \rho_1}{\partial t} + v_1 \frac{\partial \rho_0}{\partial y} = 0.$$

Find the dispersion relation $\omega(k)$, we assume solutions of the form

$$u_1 = \hat{u_1} e^{i(kx+ly-\omega t)}, \quad v_1 = \hat{v_1} e^{i(kx+ly-\omega t)},$$
$$p_1 = \hat{p_1} e^{i(kx+ly-\omega t)}, \quad \rho_1 = \hat{\rho_1} e^{i(kx+ly-\omega t)}.$$

Put these into the linearised equations and eliminate $\hat{u_1}, \hat{v_1}, \hat{p_1}, \hat{\rho_1}$ to get

$$\omega^2 = \frac{k^2 N^2}{(l^2+k^2)} \text{ where } N^2 = \frac{-g}{\rho_0} \frac{\partial \rho_0}{\partial y}.$$

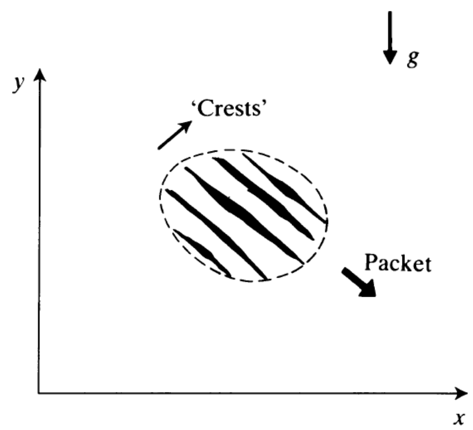

Figure 1: Propagation of a 2D packet of internal gravity waves; the crests denote lines of constant phase $kx + ly - \omega t$.

The phase velocity is then (where $\boldsymbol{k} = (k, l)$*)*

$$c_{ph} = \frac{\omega \boldsymbol{k}}{|\boldsymbol{k}|^2} = \frac{\omega}{(k^2 + l^2)}(k, l),$$

and the group velocity is then

$$c_g = \left(\frac{\partial}{\partial k}, \frac{\partial}{\partial l}\right)\omega = \frac{\omega l}{k(k^2 + l^2)}(l, -k).$$

So we can see that $c_{ph} \propto (k, l)$ *is perpendicular to* $c_g \propto (l, -k)$*. As time proceeds the crests move in direction* (k, l) *and the packet moves in the perpendicular direction* $(l, -k)$*.*