# MA3D1 Fluid Dynamics Support Class 4 - More 2D Flows, Gravity Waves, Point Sources/Sinks

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## 1 Power Law Potential Flow

**Example 1.** ***Flow bounded by two intersecting planes***
*Consider a 2D potential flow described by the complex potential*

$$w(z) = Cz^n \tag{1.1}$$

*where $C$ and $n$ are real positive numbers.*

1. *Suppose that $n = 1$. Describe the resulting flow and the meaning of the constant $C$.*

2. *Suppose now that $n = 2$. Describe the resulting flow and the meaning of the constant $C$.*

3. *Now consider the case where $n$ is an arbitrary rational number which is greater or equal to $1/2$. Write down expressions for the velocity potential $\phi$ and the stream function $\psi$ in terms of the polar coordinates $r$ and $\theta$ on the complex plane, $z = re^{i\theta}$. Find the smallest and the second smallest positive angles, $\theta = \theta_1$ and $\theta = \theta_2$, for which the stream function $\psi$ is zero for any $r$.*

4. *Consider the flow in the sector $\theta_1 < \theta < \theta_2$, i.e. between the two intersecting (at $z = 0$) planes $\theta = \theta_1$ and $\theta = \theta_2$. Explain why the flow with the complex potential $w(z) = Cz^n$ satisfies the free-slip boundary conditions at these planes.*

5. *Sketch the flows arising in the cases* $n = 4, n = 4/3, n = 2/3$ *and* $n = 1/2$.

6. *Describe qualitative difference in behaviour of the velocity field near* $r = 0$ *for cases* $n < 1$ *and* $n > 1$.

## 2 Deep Water Waves

We consider surface waves on deep water. Fluid motion will arise from a deformation of the water surface. Denote the free surface $z = \eta(x, t)$.

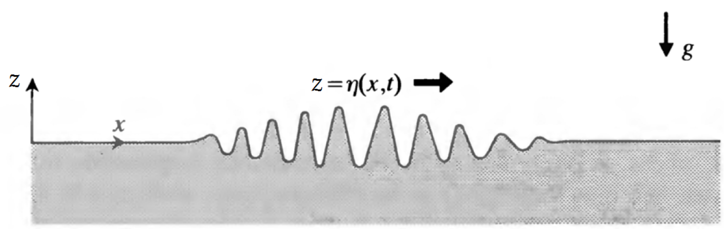


Figure 1: A group of surface waves on deep water.

Fluid particles on the surface remain on the surface. Define $F(x, z, t) = z - \eta(x, t)$ which remains constant (in fact 0) for any fluid particle on the free surface, ie.

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\boldsymbol{u} \cdot \nabla)F = 0 \text{ on } z = \eta(x, t), \tag{2.1}$$

which is equivalent to (using definition of $F$)

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w \text{ on } z = \eta(x, t). \tag{2.2}$$

**Example 2.** *(Small Amplitude Waves) For the surface of deep water let the pressure be* $p = p_0 + \epsilon p_1$*, where* $p_0$ *is atmospheric pressure,* $z = \epsilon\eta + z_0$*. Velocities are small perturbations from rest* $u = \epsilon u_1, w = \epsilon w_1$ *as is the velocity potential* $\phi = \epsilon\phi_1$*, where* $z_0$ *is the average surface level and* $\eta$ *the displacement. We wish to find an exact solution using the deep water equation*

$$\frac{\partial\eta}{\partial t} = \underbrace{u\frac{\partial\eta}{\partial x}}_{O(\epsilon^2)} = w = \frac{\partial\phi}{\partial z}. \tag{2.3}$$

*Now use Bernoulli and pick* $G(t) = \frac{p_0}{\rho} - gz_0$ *with* $\chi = g(\eta + z_0)$ *to get*

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + g\eta = 0, \tag{2.4}$$

*and combine with (2.3) to get*

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0. \tag{2.5}$$

*Assume solution of the form* $\eta = A\cos(kx - \omega t)$ *and* $\phi = f(z)\sin(kx - \omega t)$*, where* $A$ *is the amplitude of the displacement. Use the fact that* $\phi$ *satisfies Laplace's equation*

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

*to find the condition on* $f$*,* $f'' - k^2 f = 0$ *which has general solution* $f = Ce^{kz} + De^{-kz}$*. For a bounded solution as* $z \to -\infty$ *we need* $D = 0$ *so* $\phi = Ce^{kz} sin(kx - \omega t)$*. Substitute into (2.3) and (2.4) to get* $C = A\omega/k$ *and* $\omega^2 = gk$*.*

*To find the particle paths we use* $u = \frac{\partial\phi}{\partial x} = A\omega e^{kz}\cos(kx - \omega t)$ *and* $w = \frac{\partial\phi}{\partial z} = A\omega e^{kz}\sin(kx - \omega t)$*. Assume the particles only move a small amount* $(x', z')$ *from their mean position* $(\bar{x}, \bar{z})$*. Then*

$$x' = -Ae^{k\bar{z}}\sin(k\bar{x} - \omega t), z' = Ae^{k\bar{z}}\cos(k\bar{x} - \omega t).$$

*Therefore the particle paths are circular with radius decreasing with depth.*

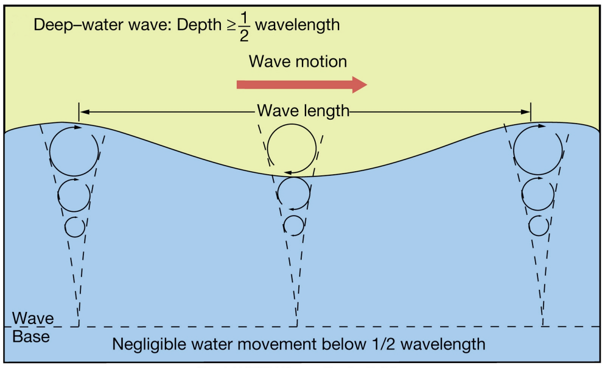



Figure 2: Motion of particles in deep water.

*Most of the energy of a surface wave is contained within half a wavelength below the surface.*

**Example 3.** ***Complex potential for the gravity water waves***
*The velocity potential corresponding to the gravity wave on deep water is (c.f. question 4.3.2):*

$$\phi = Ce^{ky}\cos(kx - \omega t) \tag{2.6}$$

*where the water surface is at* $y = 0$*, and* $C, k$ *and* $\omega$ *are real positive constants having the meaning of the wave amplitude, wave number and frequency.*

1. *Find the velocity field under the water surface.*

2. *Find the stream function and the complex potential.*

3. *Find the trajectories of the fluid particles and the streamlines. Comment on the differences of these two types of curves.*

# 3 Point Source / Sink Flows

Another kind of point object arising in the study of ideal fluid flows are point sources or sinks. They have only a radial (with respect to the position of the source/sink) component of velocity, $v(r)$, which is independent of the angular coordinates:

$$\boldsymbol{u} = \hat{\boldsymbol{r}} v(r), \tag{3.1}$$

where $\hat{\boldsymbol{r}}$ is a unit vector pointing away from the source/sink in the radial direction; see Figure 3. From incompressibility we have $r^{d-1}v(r) = \text{const}$, where $d$ is the dimension of the system (which is 2 or 3) and $r$ is the distance from the source/sink. Therefore

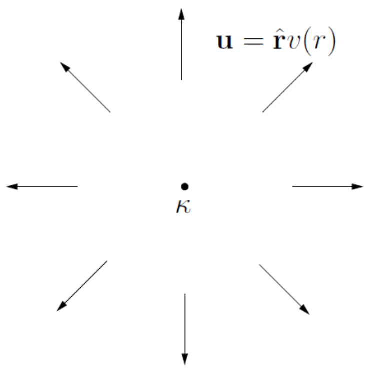


Figure 3: A point source flow.

$$v(r) = \frac{\kappa}{r^{d-1}} \tag{3.2}$$

where constant $\kappa$ is called the source strength when it is positive and the sink strength when it is negative. The constant $\kappa$ is proportional to the mass flux out of the source or into the sink respectively. The source/sink flow is irrotational and its velocity potential is $\phi(r) = \int v\, dr$ i.e.

$$\phi(r) = \begin{cases} \kappa \ln r & \text{if } d = 2 \\ -\frac{\kappa}{r} & \text{if } d = 3 \end{cases}. \tag{3.3}$$

**Example 4.** ***Discharge through a hole***

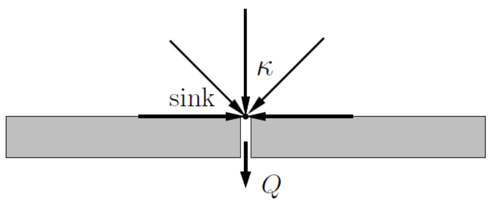


Figure 4: Water discharging through a hole at the bottom.

*A flow of water is discharging through a hole at the at bottom of a large flume, see Figure 4.*

1. *Explain why the resulting water flow is equivalent to the flow produced by a point sink of strength $\kappa < 0$ assuming that the free-slip boundary condition at the wall is satisfied.*

   ***Answer:*** *The velocity field produced by a sink located at the wall will be tangential to this wall, i.e. the free-slip boundary condition at the wall is satisfied.*

2. *Find $\kappa$ in terms of the volume flow $Q$ through the hole.*

***Answer:*** *The radial component of the velocity field produced by the sink of strength* $\kappa$ *in the 3D case is*

$$v(r) = \frac{\kappa}{r^2} \tag{3.4}$$

*and the inward volume flux through a sphere of radius* $r$ *surrounding the sink is* $-4\pi r^2 v(r) = -4\pi\kappa$. *However, in the problem only half of the sink flow is realised in the fluid volume and, therefore, the volume flow through the hole is* $Q = -2\pi\kappa$. *Thus, for* $\kappa$ *in terms of the volume flow through the pump* $Q$ *we have:*

$$\kappa = -\frac{Q}{2\pi} \tag{3.5}$$

3. *Now suppose that the water is pumped into the flume through the same whole rather than being discharged. Explain why would it be less realistic to consider the resulting flow as a point source flow.*

   ***Answer:*** *For the water discharging from the same whole we would most likely encounter the flow separation phenomenon. In the other words, the flow would emerge out of the hole as a jet which is strongly anisotropic and, therefore, very dissimilar to the point sink flow which is isotropic. In a much more viscous fluid such a flow separation would be suppressed, but the free-slip boundary condition would have to be replaced with the no-slip condition, and the isotropic flow would be again impossible.*