MA3D1 Fluid Dynamics Support Class 5 - Buoyancy Induced Waves

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1 Deep Water Waves

We consider surface waves on deep water. Fluid motion will arise from a deformation of the water surface. Denote the free surface $z = \eta(x,t)$.

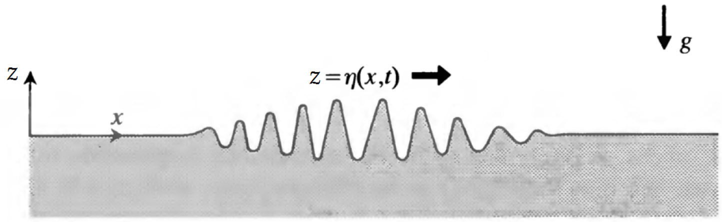

Figure 1: A group of surface waves on deep water.

Fluid particles on the surface remain on the surface. Define $F(x,z,t) = z - \eta(x,t)$ which remains constant (in fact 0) for any fluid particle on the free surface, ie.

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\boldsymbol{u} \cdot \nabla)F = 0 \text{ on } z = \eta(x,t), \tag{1}$$

which is equivalent to (using definition of F)

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w \text{ on } z = \eta(x,t). \tag{2}$$

2 Bernoulli's equation for unsteady flow

Consider Euler's equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + \chi\right). \tag{3}$$

If the flow is irrotational ($\nabla \times \boldsymbol{u} = 0$) so that $\boldsymbol{u} = \nabla\phi$ then

$$\frac{\partial \nabla\phi}{\partial t} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + \chi\right) \text{ where } \chi = gz.$$

Then integrate this to get

$$\partial_t\phi + \frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + \chi = G(t)$$

where $G(t)$ is an arbitrary function of time. Bernoulli's equation can be used to find exact solutions.

Example 1. *(Small Amplitude Waves) For the surface of deep water let the pressure be* $p = p_0 + \epsilon p_1$*, where* p_0 *is atmospheric pressure,* $z = \epsilon\eta + z_0$*. Velocities are small perturbations from* *rest* $u = \epsilon u_1, w = \epsilon w_1$ *as is the velocity potential* $\phi = \epsilon\phi_1$, *where* z_0 *is the average surface level and* η *the displacement. We wish to find an exact solution using the deep water equation*

$$\frac{\partial \eta}{\partial t} = \underbrace{u\frac{\partial \eta}{\partial x}}_{O(\epsilon^2)} = w = \frac{\partial \phi}{\partial z}. \tag{4}$$

Now use Bernoulli and pick $G(t) = \frac{p_0}{\rho} - gz_0$ *with* $\chi = g(\eta + z_0)$ *to get*

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + g\eta = 0, \tag{5}$$

and combine with (4) to get

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0. \tag{6}$$

Assume solution of the form $\eta = A\cos(kx - \omega t)$ *and* $\phi = f(z)\sin(kx - \omega t)$, *where* A *is the amplitude of the displacement. Use the fact that* ϕ *satisfies Laplace's equation*

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

to find the condition on f, $f^{''} - k^2 f = 0$ *which has general solution* $f = Ce^{kz} + De^{-kz}$. *For a bounded solution as* $z \to \infty$ *we need* $D = 0$ *so* $\phi = Ce^{kz} sin(kx - \omega t)$. *Substitute into (4) and (5) to get* $C = A\omega/k$ *and* $\omega^2 = gk$.

To find the particle paths we use $u = \frac{\partial \phi}{\partial x} = A\omega e^{kz}\cos(kx - \omega t)$ *and* $w = \frac{\partial \phi}{\partial z} = A\omega e^{kz}\sin(kx - \omega t)$. *Assume the particles only move a small amount* $(x^{'}, z^{'})$ *from their mean position* $(\bar{x}, \bar{z})$. *Then*

$$x^{'} = -Ae^{k\bar{z}}\sin(k\bar{x} - \omega t), z^{'} = Ae^{k\bar{z}}\cos(k\bar{x} - \omega t).$$

Therefore the particle paths are circular with radius decreasing with depth.

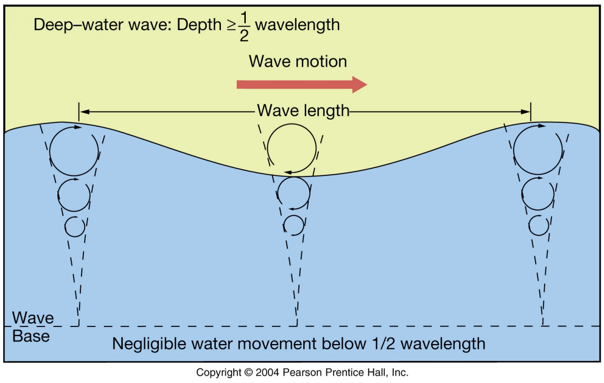

Figure 2: Motion of particles in deep water.

Most of the energy of a surface wave is contained within half a wavelength below the surface.

3 Boussinesq Approximation

Variations of all properties other than density are ignored. Variations of density ignored except when they give rise to a gravitational force (multiplied by g), that is ignored in inertia terms but not buoyancy terms.

4 Anelastic Approximation

Eliminates sound waves by assuming flow has velocities much smaller than the speed of sound. Assumes density as a decreasing function of height.

Example 2. *(Exam Question - Master Equation)*
(a) Explain the difference between the anelastic and Boussinesq approximations.
Answer: *See above.*

(b) Consider gravity waves with no mean shear and the average density is a function of height. What does the master equation reduce to?

$$\hat{w}'' + \frac{\bar{\rho}'}{\bar{\rho}}\hat{w}' + \left[\frac{N^2}{(c-U)^2} + \frac{U''}{c-U} + \frac{\rho_0'}{\rho_0}\frac{U'}{(c-U)} - k^2\right]\hat{w} = 0 \tag{7}$$

where $c = \omega/k$, $U(y)$ is the shear flow and the vertical velocity $w = \hat{w}e^{i(kx-\omega t)}$.
Answer: *No mean shear gives $U = 0$. If mean density is a function of height then $\frac{\bar{\rho}'}{\bar{\rho}} = \frac{\rho_0'}{\rho_0}$ so the master equation (7) reduces to*

$$\hat{w}'' + \frac{\rho_0'}{\rho_0}\hat{w}' + \left(\frac{N^2}{c^2} - k^2\right)\hat{w} = 0. \tag{8}$$

(c) What is N and what does it represent a balance between?

Answer: $N^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}$ *is the Brunt Väisälä frequency. It represents a balance between buoyancy and gravity, and is the frequency at which a vertically displaced air parcel will oscillate in a stable environment.*

Now assume no mean shear and allow for density as a function of height $\rho(z)$.

(d) Assume $\bar{\rho}(z) = \rho_0 e^{-\frac{z}{H}}$. What is the dispersion relation for a wave $w(x,z,t) = w_0 e^{\frac{z}{aH}} e^{i(kx+lz-\omega t)}$ where (k,l) are horizontal and vertical wavenumbers, and ω the frequency.

Answer: *Substitute $w(x,z,t) = w_0 e^{\frac{z}{aH}} e^{i(kx+lz-\omega t)}$ into (8) noting that $\frac{\rho_0'}{\rho_0} = -\frac{1}{H}$ to get,*

$$\left[\left(\frac{1}{aH} + il\right)^2 + \frac{\rho_0'}{\rho_0}\left(\frac{1}{aH} + il\right) + \left(\frac{N^2}{c^2} - k^2\right)\right] w_0 = 0$$

$$\Rightarrow \left[\frac{1}{a^2H^2} + \frac{2il}{aH} - l^2 - \frac{1}{H}\left(\frac{1}{aH} + il\right) + k^2\left(\frac{N^2}{\omega^2} - 1\right)\right] w_0 = 0 \tag{9}$$

(e) Why is $a = 2$ the proper choice?
Answer: *If $a = 2$ the complex term in (9) vanishes, the complex part would cause the waves to decay as they rose.*

(f) Find the dispersion relation $\omega(k,l)$ with $a = 2$.
Answer: *Use $a = 2$ in (9) to get*

$$\omega^2 = \frac{k^2N^2}{k^2 + l^2 + \frac{1}{4H^2}} \tag{10}$$

(g) How would you apply the Boussinesq approximation, what would the dispersion relation be?
Answer: *We assume height variations are small ($\ll H$) and $H \gg \lambda = \frac{2\pi}{(k^2+l^2)^{1/2}}$ so (10) becomes*

$$\omega^2 = \frac{N^2k^2}{k^2 + l^2}.$$