MA3D1 Fluid Dynamics Support Class 7 - Geophysical Flows

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1   Geostrophic Wind

Suppose we have a rotating fluid, like Earth's atmosphere with rotation $\mathbf{\Omega} = (0, 0, \Omega)$. Consider small Rossby number $Ro = U/\Omega L$ and Ekman number $Ek = \nu/\Omega L^2$, that is Coriolis forces are dominant over inertial and viscous forces, $|\boldsymbol{u} \cdot \nabla \boldsymbol{u}| \leq |\mathbf{\Omega} \times \boldsymbol{u}|$ and $|\nu \Delta \boldsymbol{u}| \leq |\mathbf{\Omega} \times \boldsymbol{u}|$. Then the Euler equation becomes a balance between Coriolis and pressure forces,

$$2\mathbf{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p. \tag{1.1}$$

This is called a *geostrophic flow*. Note that the Coriolis force $2\mathbf{\Omega} \times \boldsymbol{u}$ is always perpendicular to the flow $\boldsymbol{u}$. Assuming this geostrophic balance we have the *geostrophic wind*

$$(u_g, v_g) = \frac{1}{2\Omega\rho}\left(-\frac{\partial p}{\partial y}, \frac{\partial p}{\partial x}\right), \tag{1.2}$$

which is perpendicular to the pressure gradient, therefore blows parallel to isobars (lines of constant pressure). In the Northern Hemisphere the geostrophic wind blows anticlockwise (cyclonic - same rotation direction as the Earth) around low pressure.
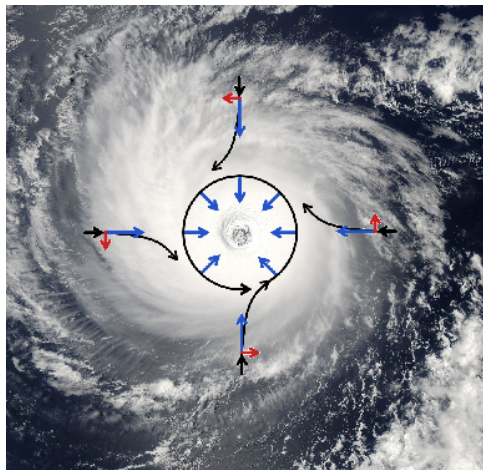

Figure 1: Deflection effects of the Coriolis force on flow towards a low pressure cell causing a cyclone. The aegeostrophic velocity component $\boldsymbol{u}_a$ follows the pressure gradient, shown in blue arrows. The Coriolis effect is shown in red arrows, the geostrophic velocity component $\boldsymbol{u}_g$ is in this direction. The resulting deflected flow is shown in black arrows.

2   Taylor Proudman Theorem

Theorem 1. *Assume we have a rapidly rotating flow, then all the components of velocity are independent of z, that is $\partial_z \boldsymbol{u} = 0$.*

Proof. Use (1.1) and $\nabla \times \nabla p = 0$ to get $\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = 0$. Expanding this we get

$$\boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{\Omega} + \boldsymbol{u}(\nabla \cdot \boldsymbol{\Omega}) - \boldsymbol{\Omega}(\nabla \cdot \boldsymbol{u}) = 0$$

Assuming that $\boldsymbol{\Omega}$ is constant and using incompressibility we get $\boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} = 0 \Rightarrow \partial_z \boldsymbol{u} = 0$. □

3   Taylor Columns

Consider an object in a rotating flow, then due to the Taylor Proudman Theorem the fluid above the object will act as a single rotating body.
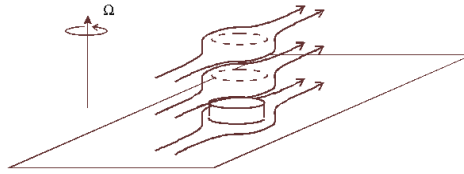

Figure 2: Flow is deflected around a column of fluid above the object.

Any flow through the Taylor column would require a change in the flow with height hence violating $\partial_z \boldsymbol{u} = 0$.

4   Ekman Boundary Layers

An *Ekman spiral* is a structure of currents or winds near a horizontal boundary in which the flow rotates as one moves away from the boundary.
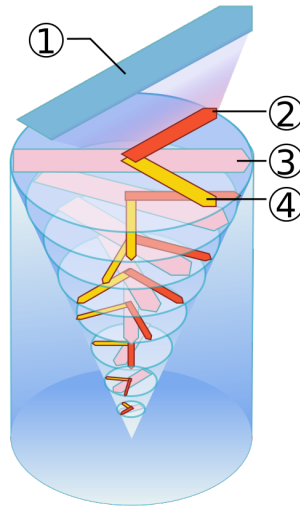

Figure 3: (1) Wind direction. (2) Force from above. (3) Effective direction of flow. (4) Coriolis force. The mass transport of the fluid ends up going at 90° to right (left) of the wind direction in the Northern (Southern) Hemisphere.

Here we find the solution for the flow of an Ekamn spiral with rigid bottom boundary. Assume the main body flow is geostrophic so that the velocities are

$$u_G = -\frac{1}{2\Omega\rho}\frac{\partial p}{\partial y}, \quad v_G = \frac{1}{2\Omega\rho}\frac{\partial p}{\partial x}. \tag{4.1}$$

Near the boundary, within the spiral, there will be an ageostrophic component of velocity $\boldsymbol{u}_a$ varying slowly in (x, y) with $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_a$ and $\nabla \cdot (\boldsymbol{u}_G + \boldsymbol{u}_a) \approx 0$, $w_a = 0$. Then the equations in this boundary layer are (remember viscosity cannot be ignored here)

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \tag{4.2}$$

$$2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} \tag{4.3}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2} \tag{4.4}$$

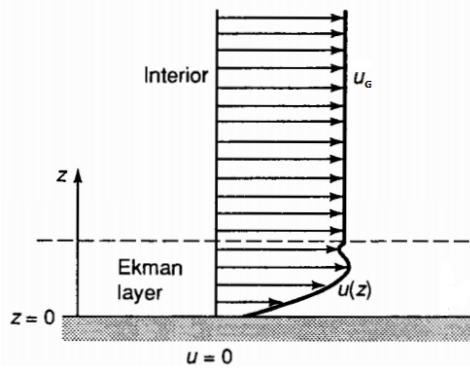$$\nabla \cdot \boldsymbol{u} = 0 \tag{4.5}$$

Figure 4: Ekman layer with rigid bottom boundary. This could be the atmosphere near the surface of the Earth or the bottom of the ocean. Note that the velocity vectors are also rotating within the Ekman layer.

The vertical velocity component w will be much smaller in comparison to u and v near the boundary so from the equations we deduce $\partial_z p \approx 0$. This pressure is a function of x and y so retains its inviscid value from the main body of the fluid throughout the boundary layer, which is given by (4.1). Then noting that $\partial_{zz}\boldsymbol{u}_G = 0$ the boundary layer equations become

$$-2\Omega(v - v_G) = -2\Omega v_a = \nu\frac{\partial^2 u_a}{\partial z^2}$$

$$-2\Omega(u - u_G) = -2\Omega u_a = \nu\frac{\partial^2 v_a}{\partial z^2}$$

Multiply the second equation by i and add them together to get

$$2\Omega i Z = \nu\frac{\partial^2 Z}{\partial z^2}$$

where $Z = u_a + iv_a$. This has general solution

$$Z = Ae^{-(1+i)z_*} + Be^{(1+i)z_*} \tag{4.6}$$

where $z_a = (\Omega/\nu)^{1/2} z$. Let $u_* = u + iv = u_G + Z$ (assume $v_G = 0$ for simplicity). To match with the interior flow we need $u_* \to u_G$ as $z_* \to \infty$ so $B = 0$. We also need $u_* = 0$ at $z_* = 0$ which gives $A = -u_G$ so

$$u = u_G(1 - e^{-z_*}\cos(z_*)), \quad v = u_G e^{-z_*}\sin(z_*).$$

For small z_* we have (from Taylor expansions) $e^{-z_*} \approx 1 - z_*$, $\cos(z_*) \approx 1 - \frac{1}{2}z_*^2$ and $\sin(z_*) \approx z_*$. Therefore, very close to the boundary the flow is at $45°$ to the interior flow.
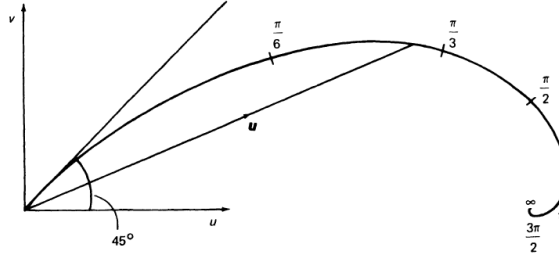
Figure 5: Ekman spiral: polar diagram of velocity vector $\boldsymbol{u}$ in Ekman layer. Numbers along the spiral are values of z_*.

Ekman layers can occur at the bottom of the atmosphere near the surface of the Earth or ocean (like in the example above). They can also occur at the bottom of the ocean (also like the example), or at the top of the ocean near the air-water interface. The thickness of the Ekamn layer is $(\nu/\Omega)^{1/2}$.

5 Kelvin Waves

Kelvin waves are travelling disturbances of that require the support of a lateral boundary. Therefore it most often occurs in the ocean where it travels along coastlines. For these waves we use the linearised shallow water equations

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{5.1}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{5.2}$$

$$\frac{\partial \eta}{\partial t} = -b\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{5.3}$$

where the depth has been linearised $h = b + \eta$ with b the average depth. Now considering a wave with the coast at $x = 0$ and the domain $x > 0$, we assume the velocity in the x-direction to be zero $u = 0$. Then taking a second time derivative of (5.2) and then using (5.3) we get

$$\frac{\partial^2 v}{\partial t^2} = -g\frac{\partial}{\partial y}\frac{\partial \eta}{\partial t} = c^2\frac{\partial^2 v}{\partial y^2} \tag{5.4}$$

where $c = \sqrt{gb} = c_g = c_{ph}$, which is identified as the speed of surface gravity waves in non-rotating shallow water. Equation (5.4) is the one dimension wave equation and has general solution

$$v = V_+(x, y + ct) + V_-(x, y - ct) \tag{5.5}$$

and using (5.2) or (5.3) we get the surface displacement,

$$\eta = -\sqrt{\frac{b}{g}}V_+(x, y + ct) + \sqrt{\frac{b}{g}}V_-(x, y - ct) \tag{5.6}$$

(where the integration constant can be removed my a redefinition of the mean depth). Using (5.1) gives the form of V_+ and V_- as

$$V_+ = V_{0+}(y + ct)e^{-x/R}, \quad V_- = V_{0-}(y - ct)e^{+x/R} \tag{5.7}$$

where $R = c/f$ is the *Rossby radius of deformation* (distance covered by a wave travelling at speed c during one inertial period $2\pi/f$) and V_{0+}, V_{0-} are arbitrary functions determined by

initial conditions. For the Northern Hemisphere (where $f > 0$) the solution V_- explodes as $x \to \infty$, hence we can only use V_+ for a physical solution. We therefore get the general solution

$$u = 0 \tag{5.8}$$

$$v = \sqrt{gb}F(y + ct)e^{-x/R} \tag{5.9}$$

$$\eta = -bF(y + ct)e^{-x/R} \tag{5.10}$$

where F is an arbitrary function. Kelvin waves are *trapped* due to the exponential decay away from the boundary. Observing that writing the arbitrary function $F(y + ct) = e^{ik(y+ct)} = e^{i(ky-\omega t)}$ we see that $-\omega = ck$. Then $c = c_{ph} = c_g = -\omega/k < 0$ (pointing south), Kelvin waves are non-dispersive. The waves propagate without distortion at the speed of surface gravity waves. In the Northern Hemisphere the waves travel with the coast on its right; in the Southern Hemisphere with the coast on its left. Surface Kelvin waves are usually generated by the ocean tides and by local wind effects in coastal areas.

Figure 6: Direction of Kelvin waves depicted by arrows along coastlines. Kelvin waves will travel south on the east coast of Britain, north along the west coast and east along the north coast of France coming in off of the Atlantic ocean. A Kelvin wave travelling south along the east coast of Great Britain could travel around the coasts of the North sea in an anticlockwise direction and reach the west coast of Norway.