MA3D1 Fluid Dynamics Support Class 7 - Internal Waves, Rayleigh-Taylor Instability

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1 Internal Waves

Example 1. ***Internal waves in stratified fluids***
In this problem we will study waves that arise due to stratification of the fluid density: the so-called internal waves. They play important roles in both atmospheric and ocean dynamics. In particular in the oceans they provide the principal mechanism for bringing the global conveyor belt currents from the ocean's bottom to the surface, and as such they are one of the key ingredients in climate dynamics.

Equations for inviscid incompressible fluid in the presence of a density stratification and gravity $\boldsymbol{f} = -g\hat{\boldsymbol{z}}$ *are*

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p - g\hat{\boldsymbol{z}}, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1.2}$$

$$\partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = 0. \tag{1.3}$$

Consider an incompressible fluid subject to gravity, which at rest has an exponential density profile,

$$\rho = \rho_0(z) = \rho_0(0)e^{-z/h}. \tag{1.4}$$

1. *Find the equilibrium pressure profile* $p_0(z)$*.*

2. *Consider perturbations around the equilibrium state,*

$$\rho = \rho_0(z) + \tilde{\rho}, \qquad \boldsymbol{u} = \tilde{\boldsymbol{u}},$$

   and linearise the fluid equations (1.1) *and* (1.3)*.*

3. *Consider perturbations in the form of a harmonic wave*

$$\tilde{\boldsymbol{u}} = Re\left[\frac{\boldsymbol{A}}{\sqrt{\rho_0(z)}}e^{i(kx-\omega t)}\right], \tag{1.5}$$

$$\tilde{\rho} = Re\left[R\sqrt{\rho_0(z)}e^{i(kx-\omega t)}\right], \tag{1.6}$$

$$\tilde{p} = Re\left[P\sqrt{\rho_0(z)}e^{i(kx-\omega t)}\right] \tag{1.7}$$

where $\boldsymbol{k} \in \mathbb{R}^3$ *is the wave vector,* $\omega \in \mathbb{R}$ *is the frequency,* $\boldsymbol{A} \in \mathbb{C}^3$ *is a constant vector and* $R, P \in \mathbb{C}$ *are constant scalars.*

Substituting (1.5), (1.6) *and* (1.7) *into the linearised equations find the wave dispersion relation,* $\omega = \omega(k)$.

4. *Are these waves dispersive or non-dispersive? Isotropic or anisotropic? Explain why. Consider the short-wave limit* $kh \gg 1$. *Find the group velocity for the inertial waves and comment on its relative direction with respect to the wave vector.*

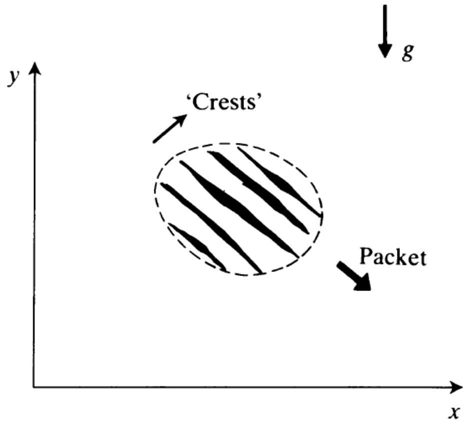

Figure 1: Propagation of a 2D packet of internal gravity waves; the crests denote lines of constant phase $kx + ly - \omega t$.

2 Rayleigh-Taylor Instability

Example 2. ***Rayleigh-Taylor instability***
Consider an ideal incompressible fluid at rest subject to gravity in the negative y-direction. The fluid has density ρ_+ in the upper half space $y \geq 0$ and density ρ_- in the lower half space $y < 0$.

This setup is at a steady state equilibrium which is unstable if $\rho_+ > \rho_-$, i.e. when the upper fluid is heavier than the lower one (e.g. water above oil). We will study such an instability by considering evolution of small irrotational perturbations of the specified configuration.

1. *Use Bernoulli's theorem for time dependent irrotational flow and formulate the pressure boundary conditions at infinity and on the velocity discontinuity surface. Linearise the resulting equation.*

2. *Formulate the kinematic boundary conditions on the discontinuity interface corresponding to the top and the bottom side of this surface. Linearise this condition.*

3. *Write the incompressibility conditions above and below the interface.*

4. Consider a plane wave solution

$$\phi_{\pm}(x, y, t) = Re\left[A_{\pm}(y)e^{i(kx-\omega t)}\right], \qquad h_{\pm}(x, y, t) = Re\left[He^{i(kx-\omega t)}\right], \tag{2.1}$$

where $A_{\pm}(y)$ are real functions, H is complex number, k is the perturbation wavenumber and ω is frequency. Substitute this plane wave solution into the incompressibility conditions and find the shape of the function $A_{\pm}(y)$.

5. Substitute this plane wave solution into the linearised pressure and the kinematic boundary conditions at the interface which you found in the previous parts. Find ω in terms of k from the resolvability condition of the resulting equations. Find the condition under which the values of ω are purely imaginary and one of them has a positive imaginary part, i.e. that the wave perturbations experience an exponential growth (Rayleigh-Taylor instability).

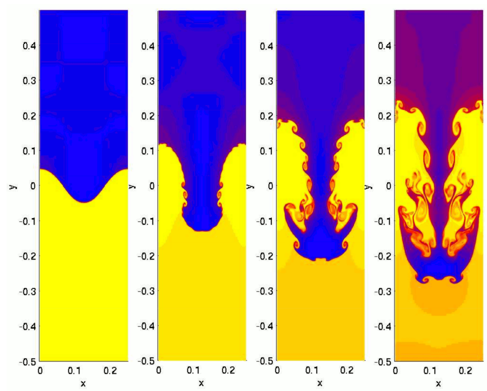

Figure 2: Rayleigh-Taylor instability simulation

3 Sound Waves

Sound propagates as waves of pressure causing local regions of compression. We aim to find the equation describing sound waves. Let the undisturbed state be one of rest, with constant pressure p_0 and density ρ_0. We have a slight perturbation from this state of rest,

$$\boldsymbol{u} = \epsilon \boldsymbol{u}_1, p = p_0 + \epsilon p_1, \rho = \rho_0 + \epsilon \rho_1. \tag{3.1}$$

We aim to linearise the equations ignoring $O(\epsilon^2)$ or higher order terms of the perturbation variables. We know entropy is conserved so $p\rho^{-\gamma} = p_0\rho_0^{-\gamma}$ everywhere. Thus

$$\begin{aligned}
(p_0 + \epsilon p_1)(\rho_0 + \epsilon \rho_1)^{-\gamma} &= p_0 \rho_0^{-\gamma}, \\
\Rightarrow \left(1 + \frac{\epsilon p_1}{p_0}\right)\left(1 + \frac{\epsilon \rho_1}{\rho_0}\right)^{-\gamma} &= 1 \\
\Rightarrow \left(1 + \frac{\epsilon p_1}{p_0}\right)\left(1 - \frac{\gamma \epsilon \rho_1}{\rho_0} + O(\epsilon^2)\right)^{-\gamma} &= 1.
\end{aligned}$$

Ignore $O(\epsilon^2)$ terms,

$$\begin{aligned}
\Rightarrow \frac{p_1}{p_0} &= \frac{\gamma \rho_1}{\rho_0} \\
\Rightarrow p_1 &= c_s^2 \rho_1 \qquad (3.2)
\end{aligned}$$

where

$$c_s = \sqrt{\frac{\gamma \rho_1}{\rho_0}}. \tag{3.3}$$

The linearised equations for $\boldsymbol{u}_1$ and ρ_1 are

$$\rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} = -\nabla p_1 \tag{3.4}$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{u}_1 = 0. \tag{3.5}$$

Take the divergence of (3.4) to get,

$$\rho_0 \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{u}_1 = -\Delta p_1. \tag{3.6}$$

Finally we use (3.5) to remove the $\nabla \cdot \boldsymbol{u}_1$ term and (3.2) to get the wave equation

$$\frac{\partial^2 p_1}{\partial t^2} = c_s^2 \Delta p_1. \tag{3.7}$$

In one dimension this is

$$\frac{\partial^2 p_1}{\partial t^2} = c_s^2 \frac{\partial^2 p_1}{\partial x^2}, \tag{3.8}$$

which has general solution

$$p_1 = f(x - c_s t) + g(x + c_s t), \tag{3.9}$$

the first/second term corresponding to right/left propagation of a wave with speed c_s without change of shape. We identify c_s as the *speed of sound.*

The *wavenumber* k is a measure of the number of times a wave has the same phase per unit of space.

The *wavelength* λ is the distance between repeating units of a propagating wave of a given frequency, it is related to the wavenumber by

$$\lambda = \frac{2\pi}{k}. \tag{3.10}$$

The *phase speed* c_{ph} describes the motion within a wave packet. Velocity at which a phase of any one frequency component of the wave will propagate within the packet,

$$c_{ph} = \frac{\omega}{k}. \tag{3.11}$$

Here ω is the frequency and the *dispersion relation* is the relation $\omega(k)$ between ω and the wavenumber k. In higher dimensions this is

$$c_{ph} = \hat{\boldsymbol{k}}\frac{\omega}{|\boldsymbol{k}|}, \tag{3.12}$$

where $\boldsymbol{k}$ is a vector of the wavenumbers in each direction and $\hat{\boldsymbol{k}} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$.

The *group velocity* c_g describes the motion of the whole wave packet,

$$c_g = \frac{\partial \omega}{\partial k}(= \nabla_{\boldsymbol{k}}\omega \text{ in higher dimensions}). \tag{3.13}$$

If $c_g \neq c_{ph}$ then the waves are *dispersive.*

Example 3. *(Sound Waves are not dispersive)*
Sound waves satisfy (in 1D)

$$\frac{\partial^2 p}{\partial t^2} = c_s^2 \frac{\partial^2 p}{\partial x^2} \tag{3.14}$$

where p is the pressure and c_s the speed of sound. Wave solutions are of the form,

$$p(x,t) = Ae^{i(kx-\omega t)}. \tag{3.15}$$

Put this into (3.14) and we get

$$A\omega^2 e^{i(kx-\omega t)} = Ak^2 c_s^2 e^{i(kx-\omega t)}$$

so $\omega = c_s k$. Then the phase speed is

$$c_{ph} = \frac{\omega}{k} = c_s,$$

and the group speed is

$$c_g = \frac{\partial \omega}{\partial k} = c_s = c_{ph}.$$

Therefore sound waves are not dispersive.