MA3D1 Fluid Dynamics Support Class 8 - More Geophysical Flows; Gradient Winds and Rossby Waves

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1 Ekman Pumping

We have from lectures that the vertical velocity $w_I = 1/2(\nu\Omega)^{1/2}\epsilon$. We have seen before that (in the Northern Hemisphere) the flow around a low pressure cell is cyclonic so $\epsilon > 0$ and $w_I > 0$, therefore the ageostrophic flow points in at the bottom, dragging pressure lines in and steepening pressure gradients.

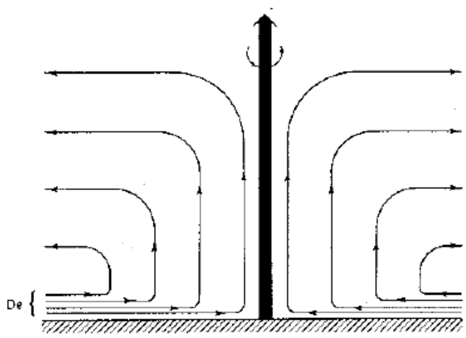


Figure 1: Ekman pumping.

Flow around a high pressure cell is anti-cyclonic so $\epsilon < 0$ and $w_I < 0$, this causes subsidence, and pushes pressure lines out decreasing pressure gradients.

2 Gradient Wind

This is the correction to the geostrophic wind due to the centrifugal force u_θ^2/r. We assume stationarity, inviscid, axisymmetric flow. The geostrophic wind is

$$\boldsymbol{G} = \frac{1}{f\rho}(-\partial_y p, \partial_x p) = \frac{1}{f\rho}\frac{\partial p}{\partial r}. \tag{2.1}$$

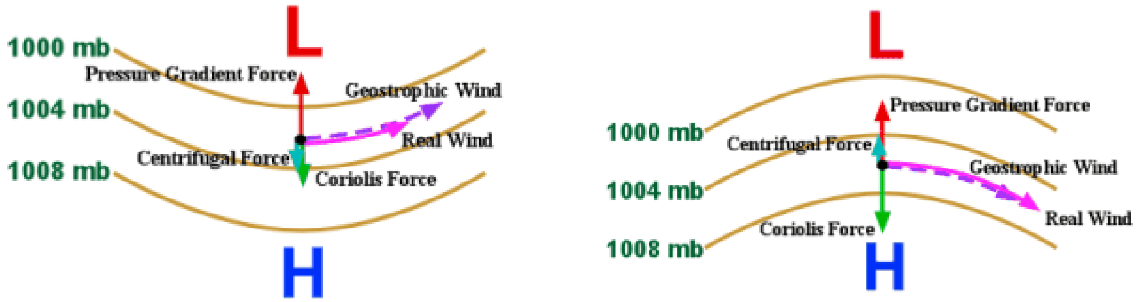
We use the rotating Euler equations in polar coordinates:

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} - fu_\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{2.2}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\theta u_r}{r} + fu_r = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}. \tag{2.3}$$

Axisymmetry means we have $\partial_\theta u_\theta = 0$ and we also assume $u_r \ll u_\theta$. Therefore the equations reduce to

$$\frac{u_\theta^2}{r} + fu_\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r} = 0. \tag{2.4}$$



(a) (b)

Figure 2: Gradient wind around (a) low pressure cell and (b) high pressure cell.

Solving for the gradient wind u_θ we get

$$u_\theta(r) - \frac{fr}{2}\left(1 \pm \sqrt{1 + \frac{4G}{fr}}\right). \tag{2.5}$$

For **cyclonic flow around a low pressure cell**: we have

$$|u_\theta| = \frac{fr}{2}\left[-1 + \sqrt{1 + \frac{4|G|}{fr}}\right]. \tag{2.6}$$

Then using a Taylor expansion up to second order terms we have

$$|u_\theta| \approx \frac{fr}{2}\left[-1 + 1 + \frac{2G}{fr} - \frac{2G^2}{f^2r^2}\right] = G - \frac{G^2}{fr} < G.$$

So the flow is slower than the geostrophic wind, called *sub-geostrophic*. Also $|u_\theta| \to G$ as $r \to \infty$, and $|u_\theta| \to 0$ as $r \to 0$. There are solutions for all $r > 0$. There are strong pressure gradients and strong geostrophic winds as observed in storms around low pressure cells.

For **anticyclonic flow around a high pressure cell**: we have

$$|u_\theta| = \frac{fr}{2}\left[1 - \sqrt{1 - \frac{4|G|}{fr}}\right]. \tag{2.7}$$

Then using a Taylor expansion up to second order terms we have

$$|u_\theta| \approx \frac{fr}{2}\left[1 - 1 + \frac{2G}{fr} + \frac{2G^2}{f^2r^2}\right] = G + \frac{G^2}{fr} > G.$$

So the flow is faster than the geostrophic wind, called *super-geostrophic*. Also $|u_\theta| \to G$ as $r \to \infty$, however there are no solutions for small $r < 4G/f$. In nature this is resolved by pressure gradients and G tending to zero as $r \to 0$. There are low geostrophic winds.

3 Rotating Bucket Problem

In this problem we want to find the shape of the free surface of an ideal fluid in the presence of a Rankine vortex, which has uniform vorticity $\omega = (0, 0, \Omega)$. This is a rotating fluid with constant angular velocity $u_\theta = \Omega r$. The governing equations in rotating coordinates are

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} - f u_\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{3.1}$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\theta u_r}{r} + f u_r = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{3.2}$$

$$\frac{\partial w}{\partial t} + u_r \frac{\partial w}{\partial r} + \frac{u_\theta}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} + f u_r = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial z} - g. \tag{3.3}$$

To find the free surface we find surfaces of constant pressure. We are not in a rotating frame so $f = 0$, we are considering a steady state so we can drop the time derivatives, also we assume $u_r = 0$. The equations then reduce to

$$-r\Omega^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \text{(Centrifugal balance)} \tag{3.4}$$

$$0 = \frac{1}{\rho}\frac{\partial p}{\partial z} + g \ \text{(Hydrostatic balance)} \tag{3.5}$$

Note that we cannot use Bernoulii's law here ($p/\rho + u^2/2 + gz$ is constant) since the flow is not irrotational. The correct approach is to integrate (3.4) and (3.5) to get the equation for z. Integrating (3.4) with respect to r gives

$$\frac{p}{\rho} = \frac{r^2\Omega^2}{2} + C_1(z),$$

and integrating (3.5) with respect to z gives

$$\frac{p}{\rho} = -gz + C_2(r) + gC.$$

Combining these we get

$$\frac{p}{\rho} = \frac{r^2\Omega^2}{2} - gz + gC.$$

Rearranging this and using the fact that the pressure at the free surface is equal to the atmospheric pressure p_0 gives the equation for z as

$$z = \frac{r^2\Omega^2}{2g} + C - \frac{p_0}{\rho g}, \tag{3.6}$$

where C is a constant depending on the depth of the fluid / choice of where you place the coordinate axes.

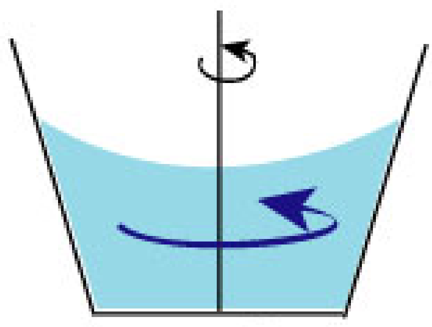


Figure 3: Shape of the free surface in a rotating bucket.

4 Rossby Waves

Atmospheric Rossby waves emerge due to shear in rotating fluids so that the Coriolis force changes along the sheared coordinate. Rossby waves are responsible for the jet stream in Earth's atmosphere. We approximate the Coriolis parameter with Taylor expansion

$$f = 2\Omega \sin(\phi_0) + 2\Omega \frac{y}{a} \cos(\phi_0) + ...$$

where a is the Earth's radius. Take $f_0 = 2\Omega \sin(\phi_0)$ and $\beta_0 = 2\frac{\Omega}{a}\cos(\phi_0)$ then $f = f_0 + \beta_0 y$. The governing equations (from the Shallow Water equations) are

$$\frac{\partial u}{\partial t} - (f_0 + \beta_0 y)v = -g\frac{\partial \eta}{\partial x} \tag{4.1}$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta_0 y)u = -g\frac{\partial \eta}{\partial y} \tag{4.2}$$

$$\frac{\partial \eta}{\partial t} = -b_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.3}$$

where $h = \eta + b_0$ is the depth with b_0 being the mean depth. For planetary waves we assume h is constant, define the *potential vorticity* as

$$q = \frac{\omega + f}{h}, \tag{4.4}$$

this is conserved by the shallow water equations (proved in lectures), that means

$$\frac{Dq}{Dt} = 0 \Rightarrow \frac{D(\omega + f)}{Dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\omega + \beta_0 v = 0. \tag{4.5}$$

Now we use the streamfunction Ψ with properties $\omega = -\Delta\Psi$ and $u = \partial_y\Psi, v = -\partial_x\Psi$, put this into (4.5) and linearise to get

$$-\Delta\frac{\partial \Psi}{\partial t} - \beta_0\frac{\partial \Psi}{\partial x} = 0 \qquad \text{(Charney-Hasegawa-Miwa equation).} \tag{4.6}$$

To find the dispersion relation put $\Psi = \hat{\Psi}e^{i(kx+ly-\omega t)}$ into the Charney equation to get

$$\omega = -\frac{\beta_0 k}{k^2 + l^2}. \tag{4.7}$$

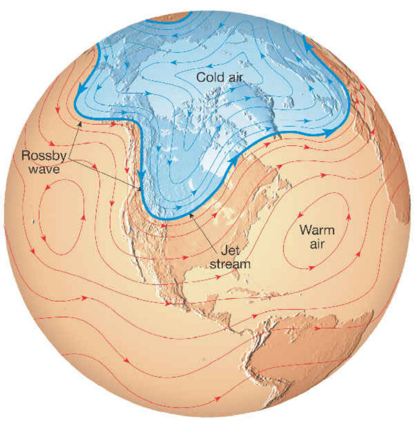


Figure 4: Jet stream in the Northern Hemisphere. Cold air filled troughs can pinch off and form low pressure cyclones. The jet stream transports weather systems around.

From this we get the phase speed

$$c_{ph} = \frac{\omega}{|\boldsymbol{k}|^2}\boldsymbol{k} = -\beta_0 \left(\frac{k^2}{(k^2+l^2)^2}, \frac{kl}{(k^2+l^2)^2} \right), \tag{4.8}$$

notice that the x-component is negative, therefore waves propagate only to the west. The group speed is

$$c_g = (\partial_k, \partial_l)\omega = \beta_0 \left(\frac{k^2-l^2}{(k^2+l^2)^2}, \frac{2kl}{(k^2+l^2)^2} \right). \tag{4.9}$$

This group speed may be in either direction, indeed consider long waves with the x-direction wavelength λ_x larger than the y-direction wavelength λ_y, ie. $1/k > 1/l$ or $l > k$. Then the x-component of the group velocity $c_{g,x} < 0$, so long waves move west. However for short waves with $\lambda_y > \lambda_x$ or $l < k$ then $c_{g,x} > 0$ and so short waves move east. Note that the jet stream blows east with the rest of the atmosphere and carries weather fronts westwards.

4.1 Coastal Rossby Waves

Rossby waves may also occur near coastal regions with sloping boundaries. In this case we have varying depth $h = b + \eta$ with $b = b_0 + \alpha_x x + \alpha_y y$. Again we use the potential vorticity

$$q = \frac{f_0 + \beta_0 y + \omega}{b_0 + \alpha_x x + \alpha_y y + \eta} \approx \frac{1}{b_0}\left(f_0 + \beta_0 y - \frac{\alpha_x f}{b_0}x - \frac{\alpha_y f}{b_0}y + \omega - \frac{f_0}{b_0}\eta \right), \tag{4.10}$$

and using the streamfunction Ψ we get the version of the Charney equation for coastal Rossby waves (assuming $\beta_0 = 0$),

$$-\Delta\Psi - \frac{\alpha_x f_0}{b_0}\frac{\partial \Psi}{\partial y} + \frac{\alpha_y f_0}{b_0}\frac{\partial \Psi}{\partial x} = 0. \tag{4.11}$$

These waves are often found travelling in the opposite direction to Kelvin waves, however they are not restricted to the coast since they have a perpendicular component.