Cloaking

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Introduction

We study the following one-dimensional version of the cloaking problem.

$$\phi_{xx}(x) + (k(x))^2\phi(x) = 0, \qquad x \in (0,1), \qquad (1)$$

$$\phi(x) = 1, \qquad x = 0, \qquad (2)$$

$$\phi'(x) = ik_0, \qquad x = 0, \qquad (3)$$

$$\phi(x) = 0, \qquad x = 1. \qquad (4)$$

where $k(x)$ is the refractivity index of the cloaking medium in $(0,1)$. For the right-hand side boundary condition to be satisfied we expect $k(1) = \infty$.

First attempt: $k(x) = M(1-x)^{-1}$

Solutions are determined by roots λ_1 and λ_2 of $\lambda^2 - \lambda + M^2 = 0$.

In case there are two distinct real roots, $1 - 4M^2 > 0$, the solution is

$$\phi(x) = \frac{\lambda_2 + ik_0}{\lambda_2 - \lambda_1}(1-x)^{\lambda_1} - \frac{\lambda_1 + ik_0}{\lambda_2 - \lambda_1}(1-x)^{\lambda_2}.$$

In case there are two complex roots, $1 - 4M^2 < 0$, the solution is

$$\phi(x) = \sqrt{1-x}\left(\cos\left(C\ln(1-x)\right) + \frac{2ik_0+1}{2C}\sin\left(C\ln(1-x)\right)\right),$$

where $C = \frac{\sqrt{4M^2-1}}{2}$.

First attempt: $k(x) = M(1-x)^{-1}$

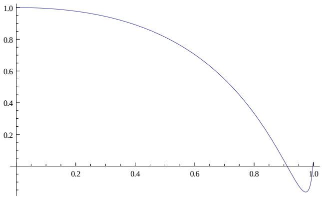


Figure: Plot of $Re[\phi(x)]$ for $k_0 = 20$,

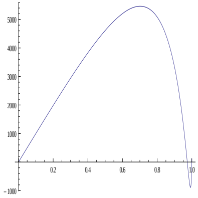


Figure: Plot of $Im[\phi(x)]$ for $k_0 = 20$,

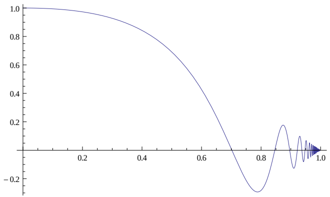
Second attempt: $k(x) = M(1-x)^{-2}$

We are very lucky to have an explicit solutions for the fundamental solutions:

$$\phi_1(x) = (x-1)\sin\left(\frac{x}{x-1}\right) \qquad \phi_2(x) = (x-1)\cos\left(\frac{x}{x-1}\right)$$

Solution given by

$$\phi(x) = (1+ik_0)\phi_1(x) - (x-1)\phi_2(x)$$



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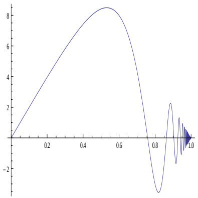

Figure: Plot of $Im[\phi(x)]$ for $k_0 = 20$,

Regularisation of $k(x)$

To avoid the singularity in $k(x)$ we wish to perturb the problem such that $k(x) \to \frac{1}{\epsilon}$ as $x \to 1$, and measure the error introduced in this perturbation. To this end, we consider $k(x) = M(1+\epsilon-x)^{-n}$. We want to solve

$$\phi_{xx} + \frac{M^2}{(1+\epsilon-x)^{2n}}\phi = 0.$$

For $n = 1$ we observe that a general solution will be of the form

$$\phi(x) = A(1-x+\epsilon)^{\lambda_+} + B(1-x+\epsilon)^{\lambda_-},$$

where $\lambda_\pm = \frac{1 \pm p}{2}$.

- For two distinct real roots ($p > 0$), we observe that

$$|\phi(1)| \approx |k_0|\epsilon^{\frac{1-p}{2}},$$

- For two distinct complex roots ($p < 0$) we observe that

$$|\phi(1)| \approx |k_0|\epsilon^{\frac{1}{2}}.$$

Regularisation of $k(x)$

- Similarly for

$$k(x) = M(1 - x + \epsilon)^{-2},$$

we observe that

$$|\phi(1)| \approx |k_0|\epsilon.$$

What about bounded $k(x)$?

It would very surprising if there was some bounded, continuous $k(x)$ on $[0, 1]$, having a solution satisfying the boundary conditions. Suppose there was such a $k(x)$ with solution

$$\phi(x) = \phi_r(x) + i\phi_i(x)$$

that satisfied the boundary conditions.

- Sturm-Picone separation theorem $\Rightarrow \phi_r$ and ϕ_i have infinitely many roots on $[0, 1]$.
- Let $||k||_\infty < K$, the Sturm Picone comparison theorem would then imply that $\cos(Kx)$ and $\sin(Kx)$ have infinitely many roots on $[0, 1]$ which is clearly a contradition.

Frobenius method: $k(x) = M(1-x)^{-\frac{1}{2}}$

It would be interesting to try weaker singularities. So consider $k(x) = (1-x)^{-\frac{1}{2}}$. We use the Frobenius method to identify two linearly independent solutions to the problem, and then show how the solutions cannot satisfy all the boundary conditions.

Changing variables $x \to (1-x)$ we get

$$\phi_{xx}(x) + M^2 x^{-1}\phi(x) = 0 \tag{5}$$

$$\phi(0) = 0, \tag{6}$$

$$\phi(1) = 1, \tag{7}$$

$$\phi'(1) = ik_0. \tag{8}$$

We look for series solutions of the form $\phi(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$, for some $r \in \mathbb{C}$. Substituting in (5), we get a series of relationships between the coefficients and r.

Frobenius method: $k(x) = (1-x)^{-\frac{1}{2}}$

The indicial equation is $r(r-1) = 0$, so that $r = 0$ or $r = 1$.

- For $r = 1$ we obtain

$$y_1(x) = a_0 \sum_{k=0}^{\infty} \frac{(-M^2)^k x^{k+1}}{k!(k+1)!} \tag{9}$$

- Second independent solution has the form

$$y_2(x) = \alpha y_1(x) \ln(x) + x^0 (1 + \sum_{k=1}^{\infty} b_k x^k).$$

  For $x \approx 0$, $y_1(x) \approx x$ and $\lim_{x\to 0} y_1(x) \ln(x) = 0$ (L'Hopital's rule). Thus $\lim_{x\to 0} y_2(x) = 1 \neq 0$.

- Thus, y_2 does not satisfy the boundary conditions.
- General solution is of the form $y = Ay_1$, but this cannot satisfy BOTH of the remaining boundary conditions.

Faster singularities: $k(x) = M(1-x)^{-n}$, $n \geq 2$

We do a WKB approximation around the the irregular singular point.

- We get that for x close to 1

$$\phi(x) = A(1-x)^{\frac{n}{2}} e^{\frac{iM(1-x)^{1-n}}{1-n}} + B(1-x)^{\frac{n}{2}} e^{-\frac{iM(1-x)^{1-n}}{1-n}}$$

- If we consider the corresponding regularised problem where $k(x) = M(1-x+\epsilon)^{-n}$, then we can see that

$$|\phi(1)| = O(\epsilon^{\frac{n}{2}}),$$

- Note we don't know how the constant depends on k_0, but we expect the dependence to be linear.

Conclusions

- We considered a very basic 1-D model.
- Our choice of $k(x)$ is quite arbitrary. Other choices of $k(x)$ have other interesting properties $k(x) = M(1 - x^2)^{-2}$ for example
- Would be interesting to see how the above could be applied to $2D$, $3D$.

Thank you for listening!