Mini-Course on Mathematical Modelling: Cloaking
Report

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Contents

1 Introduction

We study the following one-dimensional version of the cloaking problem:

$$\phi_{xx}(x) + (k(x))^2\phi(x) = 0, \qquad x \in (0,1), \tag{1}$$
$$\phi(x) = 1, \qquad x = 0, \tag{2}$$
$$\phi'(x) = ik_0, \qquad x = 0, \tag{3}$$
$$\phi(x) = 0, \qquad x = 1. \tag{4}$$

where $k(x)$ is the refractivity index of the cloaking medium in $(0,1)$. For the right-hand side boundary condition to be satisfied we expect $k(1) = \infty$.

Some remarks are in order here.

1. Condition 1 is the Helmholtz equation for waves travelling in a medium with inhomogeneous refractive index $k(x)$.
2. Condition 2 is a continuity condition which we require for $k(x)$, for a wave $\phi^-(x) = e^{ik_0x}$ coming from $-\infty$ we want, $\phi^-(0) = \phi^-(0)$. Similarly, we want $\phi^-_x(0) = \phi_x(0)$ which is Condition 3.
3. Condition 4 is the requirement that no waves reach the target.

We begin with the case where k is constant.

2 The solution for $k(x) = k$ constant

Suppose we neglect equation 4, and consider the corresponding initial-value problem; then the solutions will be of the form

$$\phi(x) = A\cos(kx) + B\sin(kx),$$

where A and B are chosen to satisfy

$$1 = \phi(0) = A,$$

and

$$ik_0 = kB.$$

It follows that $\phi(1) = \cos(k) + \frac{ik_0}{k}\sin(k)$, which is zero if and only if $k \in (n + \frac{1}{2})\pi$ and $k_0 = 0$.

3 Solutions for $k(x) = (1-x)^{-n}$

We have some intuition about $k(x)$.

1. $k(x) = n(x)M$, where $n(x)$ is the index of refractivity of the medium in $(0,1)$, which describes the rate at which the velocity of the wavelength is decreased and M is the angular frequency.
2. Thus expect $n(1) = \infty$.

Without loss of generality assume that $M = 1$ (we can rescale the harmonic components of the solution later). As a guess it would make sense to try something of the form $k(x) = M(1-x)^{-n}$, for some $n \in \mathbb{N}$.

3.1 For $k = M(1-x)^{-1}$

We want to solve

$$\phi_{xx} + \frac{M^2}{(1-x)^2}\phi = 0.$$

Trying $\phi = (1-x)^\lambda$ in the equation gives $\lambda^2 - \lambda + M^2 = 0$. So

$$\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{1-4M^2}}{2}.$$

3.1.1 Two distinct real roots ($1 - 4M^2 > 0$)

General solution

$$\phi = A(1-x)^{\frac{1}{2}+\frac{\sqrt{1-4M^2}}{2}} + B(1-x)^{\frac{1}{2}-\frac{\sqrt{1-4M^2}}{2}}$$

Clearly $\phi(1) = 0$ is satisfied and $\phi(0) = 1$ implies

$$1 = A + B.$$

The condition on the derivative requires

$$\phi'(x) = -A\lambda_1(1-x)^{\lambda_1 - 1} - B\lambda_2(1-x)^{\lambda_2-1} \tag{5}$$

so $\phi'(0) = ik_0 = -\lambda_1 A - \lambda_2(1-A)$ implies

$$A = \frac{\lambda_2 + ik_0}{\lambda_2 - \lambda_1} \qquad \text{and} \qquad B = -\frac{\lambda_1 + ik_0}{\lambda_2 - \lambda_1}.$$

The final solution then is

$$\phi(x) = \frac{\lambda_2 + ik_0}{\lambda_2 - \lambda_1}(1-x)^{\lambda_1} - \frac{\lambda_1 + ik_0}{\lambda_2 - \lambda_1}(1-x)^{\lambda_2}.$$

3.1.2 Two distinct complex roots ($1 - 4M^2 < 0$)

The roots are

$$\lambda = \frac{1}{2} \pm i\frac{\sqrt{4M^2-1}}{2}$$

So

$$\begin{aligned}(1-x)^\lambda = e^{\lambda \ln(1-x)} &= e^{\frac{1}{2}\ln(1-x) \pm i\frac{\sqrt{4M^2-1}}{2}\ln(1-x)} = e^{\frac{1}{2}\ln(1-x)}e^{\pm i\frac{\sqrt{4M^2-1}}{2}\ln(1-x)} \\ &= \sqrt{1-x}\left[\cos\left(\pm\frac{\sqrt{4M^2-1}}{2}\ln(1-x)\right) + i\sin\left(\pm\frac{\sqrt{4M^2-1}}{2}\ln(1-x)\right)\right] \\ &= \sqrt{1-x}\left[\cos\left(\pm C\ln(1-x)\right) + i\sin\left(\pm C\ln(1-x)\right)\right]\end{aligned}$$

where $C = \frac{\sqrt{4M^2-1}}{2}$. So the general solution is

$$\phi(x) = \sqrt{1-x}A\cos\left(C\ln(1-x)\right) + \sqrt{1-x}B\sin\left(C\ln(1-x)\right).$$

We apply the boundary conditions to find the constants A and B. From $\phi(0) = 1$ we immediately obtain that $A = 1$. From the condition $ik_0 = \phi'(0)$ we see that $B = -\frac{2ik_0+1}{2C}$. Our solution therefore is

$$\phi(x) = \sqrt{1-x}\left(\cos\left(C\ln(1-x)\right) - \frac{2ik_0+1}{2C}\sin\left(C\ln(1-x)\right)\right).$$

3.2 Solutions for $k(x) = M(1-x)^{-2}$

The case when $n = 2$ seems equally simple to solve. The resulting boundary value problem is

$$\begin{aligned} \phi_{xx}(x) + \frac{M^2}{(1-x)^4}\phi(x) &= 0, && x \in (0,1), \\ \phi(x) &= 1, && x = 0, \\ \phi'(x) &= ik_0, && x = 0, \\ \phi(x) &= 0, && x = 1. \end{aligned}$$

The fundamental solutions are given by

$$(x-1)\sin\left(\frac{Mx}{x-1}\right) \text{ and } (x-1)\cos\left(\frac{Mx}{x-1}\right).$$

This follows since

$$\phi_1'(x) = \sin\left(\frac{Mx}{x-1}\right) + (x-1)\left(\frac{M}{x-1} - \frac{Mx}{(1-x)^2}\right)\cos\left(\frac{Mx}{x-1}\right),$$

and

$$\phi_1''(x) = -M^2\frac{\sin\left(\frac{Mx}{x-1}\right)}{(x-1)^3},$$

so that

$$\phi_1''(x) + M^2(1-x)^{-4}\phi_1(x) = 0,$$

and similarly

$$\phi_2''(x) + M^2(1-x)^{-4}\phi_2(x) = 0.$$

The Wronskian of these solutions is non-zero, so these solutions are indeed linearly independent. Writing

$$\phi(x) = A(x-1)\sin\left(\frac{Mx}{x-1}\right) + B(x-1)\cos\left(\frac{Mx}{x-1}\right),$$

at $x = 0$, we have

$$1 = \phi(0) = -B,$$

and

$$ik_0 = M.A + B,$$

so that $A = (1 + ik_0)/M$ and $B = -1$. This gives us the solution

$$\phi(x) = \frac{(1+ik_0)}{M}(x-1)\sin\left(\frac{x}{x-1}\right) - (x-1)\cos\left(\frac{x}{x-1}\right)$$

To check whether $\phi(x)$ satisfies the right boundary condition $\phi(1) = 0$, we note that

$$\lim_{x\to 1}\left((x-1)\sin\left(\frac{x}{x-1}\right)\right) = 0 \text{ and } \lim_{x\to 1}\left((x-1)\cos\left(\frac{x}{x-1}\right)\right) = 0,$$

so that the right boundary condition is satisfied also.

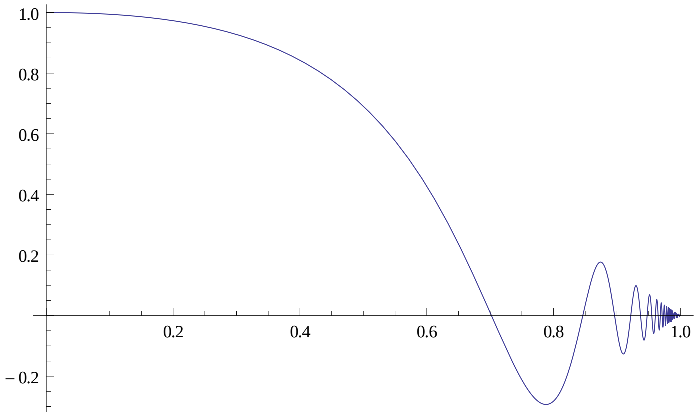

Figure 1: Plot of $Re[\phi(x)]$ for $k_0 = 20$, with $k(x) = (1-x)^{-2}$,

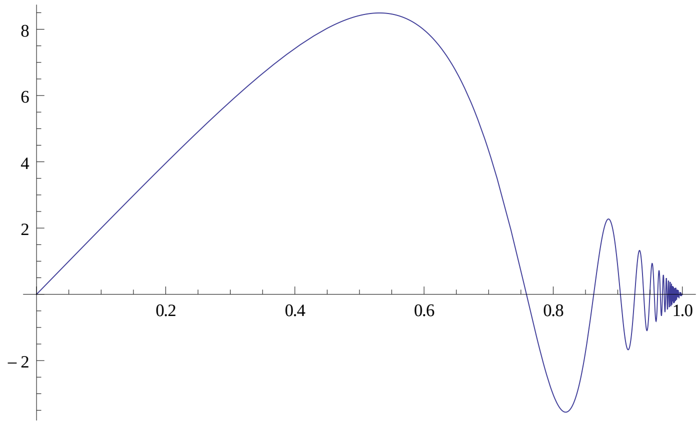

Figure 2: Plot of $Im[\phi(x)]$ for $k_0 = 20$, with $k(x) = (1-x)^{-2}$,

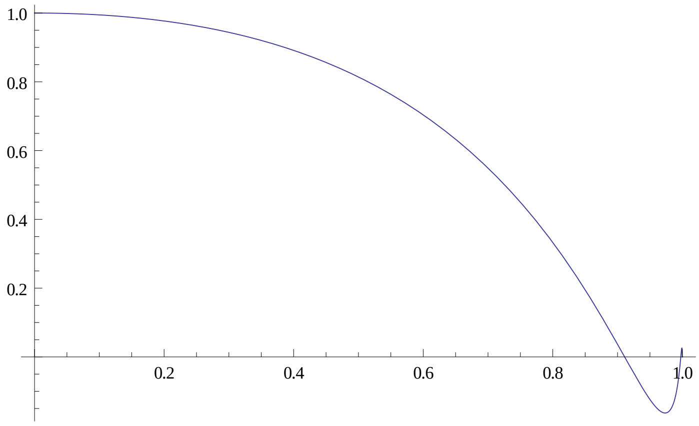

Figure 3: Plot of $Re[\phi(x)]$ for $k_0 = 20$, with $k(x) = (1-x)^{-1}$,

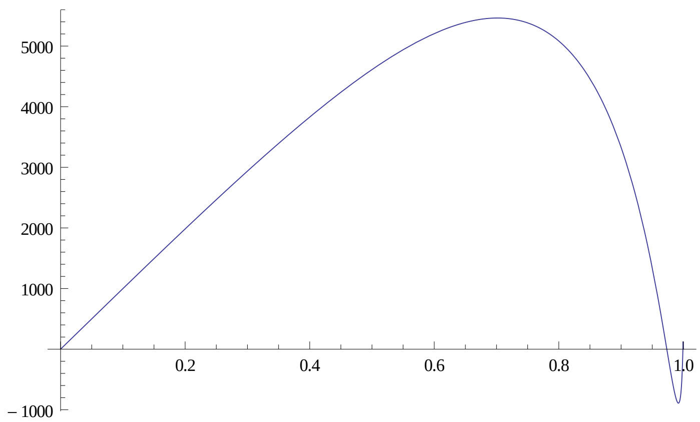

Figure 4: Plot of $Im[\phi(x)]$ for $k_0 = 20$ with $k(x) = (1-x)^{-1}$,

4 Regularisation of $k(x)$.

We now wish to perturb the problem such that $k(x) \to \frac{1}{\epsilon}$ as $x \to 1$, and measure the error introduced in this perturbation. To this end, we consider $k(x) = M(1+\epsilon-x)^{-1}$. We want to solve

$$\phi_{xx} + \frac{M^2}{(1+\epsilon-x)^2}\phi = 0.$$

Trying $\phi = (1+\epsilon-x)^\lambda$ in the equation gives $\lambda^2 - \lambda + M^2 = 0$. So

$$\lambda_\pm = \frac{1 \pm p}{2},$$

where

$$p = \sqrt{1-4M^2}.$$

4.1 Two distinct real roots ($1-4M^2>0$), $n=1$

A general solution will be of the form

$$A(1-x+\epsilon)^{\lambda_+} + B(1-x+\epsilon)^{\lambda_-},$$

for some constants $A, B \in \mathbb{C}$. Applying the boundary conditions at $x=0$:

$$1 = A(1+\epsilon)^{\lambda_+} + B(1+\epsilon)^{\lambda_-} = q_1 A + q_2 B,$$

and

$$ik_0 = (1+\epsilon)^{\lambda_+ - 1}\lambda_+ A + (1+\epsilon)^{\lambda_- - 1}\lambda_- B,$$

which implies that

$$ik_0(1+\epsilon) = q_1\lambda_+ A + q_2\lambda_- B = q_1\lambda_+ A + \lambda_-(1 - Aq_1) = q_1 A(\lambda_+ - \lambda_-) + \lambda_-,$$

so that

$$A = \frac{ik_0(1+\epsilon) - \lambda_-}{q_1(\lambda_+ - \lambda_-)},$$

and

$$B = \frac{\lambda_+ - ik_0(1+\epsilon)}{q_2(\lambda_+ - \lambda_-)},$$

so that for $x=1$:

$$A\epsilon^{\lambda_+} + B\epsilon^{\lambda_-} = \frac{ik_0(1+\epsilon) - \lambda_-}{q_1(\lambda_+ - \lambda_-)}\epsilon^{\lambda_+} + \frac{\lambda_+ - ik_0(1+\epsilon)}{q_2(\lambda_+ - \lambda_-)}\epsilon^{\lambda_-}. \tag{6}$$

If we assume that $p \in \mathbb{R}$ (thus we have two real roots λ_+ and λ_-) then we have that $\phi(1) \approx K\epsilon^{\lambda_+}(1+\epsilon)^{-\lambda_+} + L\epsilon^{\lambda_-}(1+\epsilon)^{-\lambda_-} \propto \epsilon^{\lambda_+} + \epsilon^{\lambda_-} \approx \epsilon^{\frac{1-p}{2}}$. In fact, as $\lambda^+ - \lambda^- = p$, we can see that $\phi(1) \approx |k_0|p^{-1}\epsilon^{\frac{1-p}{2}}$, so that to leading order, the error depends linearly on $|k_0|$.

4.2 Two distinct complex roots ($1 - 4M^2 < 0$), $n = 1$

The roots are

$$\lambda = \frac{1}{2} \pm i\frac{\sqrt{4M^2 - 1}}{2}$$

So

$$\begin{aligned}
(1+\epsilon-x)^{\lambda} &= e^{\lambda \ln(1+\epsilon-x)} \\
&= e^{\frac{1}{2}\ln(1+\epsilon-x) \pm i\frac{\sqrt{4M^2-1}}{2}\ln(1+\epsilon-x)} \\
&= e^{\frac{1}{2}\ln(1+\epsilon-x)} e^{\pm i\frac{\sqrt{4M^2-1}}{2}\ln(1+\epsilon-x)} \\
&= \sqrt{1+\epsilon-x}\left[\cos\left(\frac{\sqrt{4M^2-1}}{2}\ln(1+\epsilon-x)\right) \pm i\sin\left(\frac{\sqrt{4M^2-1}}{2}\ln(1+\epsilon-x)\right)\right] \\
&= \sqrt{1+\epsilon-x}\left[\cos\left(C\ln(1+\epsilon-x)\right) \pm i\sin\left(C\ln(1+\epsilon-x)\right)\right]
\end{aligned}$$

where $C = \frac{\sqrt{4M^2-1}}{2}$. So the general solution is

$$\phi(x) = \sqrt{1+\epsilon-x}A\cos\left(C\ln(1+\epsilon-x)\right) + \sqrt{1+\epsilon-x}B\sin\left(C\ln(1+\epsilon-x)\right) \tag{7}$$

Let's check the boundary conditions. The left-hand boundary condition

$$\begin{aligned}
\phi(0) &= \sqrt{1+\epsilon}A\cos\left(C\ln(1+\epsilon)\right) + \sqrt{1+\epsilon}B\sin\left(C\ln(1+\epsilon)\right) \\
&= 1
\end{aligned} \tag{8}$$

from (7) we can see

$$\begin{aligned}
\phi'(x) = &-\sqrt{1+\epsilon-x}\left(B\cos\left(C\ln(1+\epsilon-x)\right)\frac{C}{1+\epsilon-x} - A\sin\left(C\ln(1+\epsilon-x)\right)\frac{C}{1+\epsilon-x}\right) \\
&- \frac{1}{2\sqrt{1+\epsilon-x}}\left(A\cos(C\ln(1+\epsilon-x)) + B\sin(C\ln(1+\epsilon-x)\right)
\end{aligned}$$

evaluating this at the left-hand end we have

$$\begin{aligned}
\phi'(0) &= -\sqrt{1+\epsilon}\left(B\cos\left(C\ln(1+\epsilon)\right)\frac{C}{1+\epsilon} - A\sin\left(C\ln(1+\epsilon)\right)\frac{C}{1+\epsilon}\right) \\
&\quad - \frac{1}{2\sqrt{1+\epsilon}}\left(A\cos(C\ln(1+\epsilon)) + B\sin(C\ln(1+\epsilon)\right) \\
&= ik_0
\end{aligned}$$

direct substitution of (8) gives

$$-\frac{C}{\sqrt{1+\epsilon}}\left(B\cos\left(C\ln(1+\epsilon)\right) - A\sin\left(C\ln(1+\epsilon)\right)\right) - \frac{1}{2(1+\epsilon)} = ik_0 \tag{9}$$

re-arrangement of (8) gives

$$A = \frac{1}{\sqrt{1+\epsilon}} - B\tan(C(\ln(1+\epsilon)) \tag{10}$$

Substitution into (9) gives

$$ik_0 = -\frac{BC\left(\cos^2(C\ln(1+\epsilon)) + \sin^2(C\ln(1+\epsilon))\right)}{\sqrt{1+\epsilon}\cos(C\ln(1+\epsilon))} + \frac{C\sin(C\ln(1+\epsilon))}{1+\epsilon} - \frac{1}{2(1+\epsilon)}$$

using standard trigonometric identities this is

$$ik_0 = -\frac{BC}{\sqrt{1+\epsilon}\cos(C\ln(1+\epsilon))} + \frac{C\sin(C\ln(1+\epsilon))}{1+\epsilon} - \frac{1}{2(1+\epsilon)}$$

re-arrangement gives

$$B = -\frac{\left(\frac{1}{2} + ik_0(1+\epsilon) - C\sin(C\ln(1+\epsilon))\right)\cos(C\ln(1+\epsilon))}{C\sqrt{1+\epsilon}}$$

using (10) gives

$$A = \frac{1}{\sqrt{1+\epsilon}} - \frac{\left(\frac{1}{2} + ik_0(1+\epsilon) - C\sin(C\ln(1+\epsilon))\right)\sin(C\ln(1+\epsilon))}{C\sqrt{1+\epsilon}}$$

which is just

$$A = \frac{C - \left(\frac{1}{2} + ik_0(1+\epsilon) - C\sin(C\ln(1+\epsilon))\right)\sin(C\ln(1+\epsilon))}{C\sqrt{1+\epsilon}}$$

Thus

$$\begin{aligned}\phi(x) = \sqrt{1+\epsilon-x}&\frac{C - \left(\frac{1}{2} + ik_0(1+\epsilon) - C\sin(C\ln(1+\epsilon))\right)\sin(C\ln(1+\epsilon))}{C\sqrt{1+\epsilon}}\cos\left(C\ln(1+\epsilon-x)\right)\\ &+ \sqrt{1+\epsilon-x}\frac{-\left(\frac{1}{2} + ik_0(1+\epsilon) - C\sin(C\ln(1+\epsilon))\right)\cos(C\ln(1+\epsilon))}{C\sqrt{1+\epsilon}}\sin\left(C\ln(1+\epsilon-x)\right)\end{aligned}$$

We claim

$$\begin{aligned}\phi(1) = \sqrt{\epsilon}&\frac{C - \left(\frac{1}{2} + ik_0(1+\epsilon) - C\sin(C\ln(1+\epsilon))\right)\sin(C\ln(1+\epsilon))}{C\sqrt{1+\epsilon}}\cos\left(C\ln(\epsilon)\right)\\ &+ \sqrt{\epsilon}\frac{-\left(\frac{1}{2} + ik_0(1+\epsilon) - C\sin(C\ln(1+\epsilon))\right)\cos(C\ln(1+\epsilon))}{C\sqrt{1+\epsilon}}\sin\left(C\ln(\epsilon)\right)\end{aligned}$$

$$\begin{aligned}\sqrt{1+\epsilon}^{-1} &= 1 + O(\epsilon)\\ \ln(1+\epsilon) &= \epsilon + O(\epsilon^2)\\ \ln(\epsilon) &= \epsilon - 1 + O(\epsilon-1)\\ \sin(C\ln(1+\epsilon)) &= C\epsilon + O(\epsilon^3)\\ \cos(C\ln(1+\epsilon)) &= C + O(\epsilon^2)\\ \sin(C\ln(\epsilon)) &= C(\epsilon-1) + O(\epsilon-1)\\ \cos(C\ln(\epsilon)) &= C + O(\epsilon-1)\end{aligned}$$

Using these we conclude that to leading order

$$\phi(1) = |k_0|\sqrt{\epsilon}$$

4.3 For $n = 2$

We notice that to $O(\epsilon)$, the regularised solution $\tilde{\phi}(x)$ can be approximated at 1 by $\phi(1-\epsilon)$, where ϕ is the solution to equation (3.2) with boundary conditions $\phi(0) = 1 + K\epsilon$, $\phi'(0) = ik_0 + L\epsilon$. It follows that to leading order

$$|\tilde{\phi}(1)| \approx |k_0|\epsilon. \tag{11}$$

Note: John Ockendon seems to disagree with this, claiming that the error is independent of k_0. I'd be very surprised if this were the case, but would be interested to see why.

5 The impossibility of having $k(x)$ bounded.

An interesting question for this 1 dimensional model is whether or not it would be possible to achieve the same outcome with an continuous, bounded $k(x)$. Suppose $k(x)$ is bounded and let $K > 0$ be a constant such that $|k(x)| < K$. Suppose the boundary value problem

$$\phi_{xx}(x) + k(x)^2\phi(x) = 0, \qquad x \in (0,1), \tag{12}$$
$$\phi(x) = 1, \qquad x = 0, \tag{13}$$
$$\phi'(x) = ik_0, \qquad x = 0, \tag{14}$$
$$\phi(x) = 0, \qquad x = 1. \tag{15}$$

has a solution ϕ. Note that we can separate ϕ into real and imaginary components ϕ_r and ϕ_i which satisfy the PDE with respective boundary conditions

$$\phi_r(0) = 1, \qquad \phi_r'(0) = 0 \qquad \phi_r(1) = 0 \tag{16}$$
$$\phi_i(0) = 0, \qquad \phi_i'(0) = k_0 \qquad \phi_i(1) = 0. \tag{17}$$

We note that ϕ_r and ϕ_i are linearly independent solutions of the second order PDE. Now consider ϕ_i. Then ϕ_i has at least two roots in $[0,1]$. Suppose now that ϕ_r or ϕ_i had infinitely many roots, then by the Sturm-Picone comparison theorem (see [2] Theorem 5.20) for the following two problems:

$$\phi^{(1)}_{xx}(x) + k(x)^2\phi^{(1)}(x) = 0 \tag{18}$$
$$\phi^{(2)}_{xx}(x) + K^2\phi^{(2)}(x) = 0, \tag{19}$$

it follows that $\phi^{(2)}(x) = A\cos(kx) + B\sin(kx)$ would also have infinitely many roots in $[0,1]$ which is clearly not true.

Let i_0 be the root of ϕ_i immediately before $r_1 = i_1 = 1$. By the Sturm separation theorem (see [2], Section 5.5), then ϕ_r must have a root in (i_0, i_1), say r_0. Similarly, applying the separation theorem again, ϕ_i must have a root $i_2 \in (r_0, r_1)$. Continuing this argument, we note that ϕ_r and ϕ_i must necessarily have infinitely many roots in $[0,1]$, which gives rise to a contradiction.

6 Frobenius solutions

6.1 The case when $k(x) = (1-x)^{-\frac{1}{2}}$

We use the Frobenius method to identify two linearly independent solutions to the boundary value problem, and then show how the solutions cannot satisfy all the boundary conditions.

First, we will make the change of variables $x \to (1-x)$ to get the following BVP

$$\phi_{xx}(x) + x^{-1}\phi(x) = 0 \tag{20}$$
$$\phi(0) = 0, \tag{21}$$
$$\phi(1) = 1, \tag{22}$$
$$\phi'(1) = -ik_0. \tag{23}$$

We look for series solutions of the form $\phi(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$, for some $r \in \mathbb{C}$. Substituting in (20) and using linear independence of monomials we get a series of relationships between the coefficients and r. The indicial equation is

$$r(r-1) = 0,$$

so that $r = 0$ or $r = 1$. The second equation gives us

$$(r+1)ra_1 + a_0 = 0,$$

so that for $r = 1$ we have $a_0 = -\frac{1}{2}a_0$ and for $r = 0$ we have that $a_0 = 0$.

For $r = 1$ we obtain the following recursive relationship for the coefficients a_k:

$$a_k = -\frac{a_{k-1}}{(k+1)k}, \qquad \text{for } k \geq 2, \tag{24}$$

from which it follows that one solution is given by

$$y_1(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k!(k+1)!} \tag{25}$$

Since the indicial roots differ by an integer, to obtain the second independent solution we look for a solution of the form

$$y_2(x) = \alpha y_1(x) \ln(x) + x^0(1 + \sum_{k=1}^{\infty} b_k x^k).$$

Substituting this into equation (20) and writing $b_0 = 1$ we obtain

$$\alpha \left(y_1''(x) + x^{-1}y_1(x)\right) \ln(x) + \frac{\alpha}{x}\left(2y_1'(x) - \frac{y_1(x)}{x}\right) + \sum_{k=0}^{\infty} \left[k(k+1)b_{k+1} + b_k\right] x^{k-1} = 0.$$

Since $y_1(x)$ is a solution, the first term above vanishes:

$$\alpha\left(2y_1'(x) - \frac{y_1(x)}{x}\right) + \sum_{k=0}^{\infty} \left[k(k+1)b_{k+1} + b_k\right] x^k = 0$$

Substituting in the equation (25) for $y_1(x)$ we have the relation

$$\alpha \sum_{k=0}^{\infty} \frac{(2k+1)(-1)^k x^k}{(k+1)!k!} = -\sum_{k=0}^{\infty} \left(k(k+1)b_{k+1} + b_k\right) x^k,$$

from which we obtain the following recursive relationship

$$k(k+1)b_{k+1} + b_k = \frac{(-1)^{k+1}\alpha(2k+1)}{(k+1)!k!} \tag{26}$$

so that for $k = 0$,

$$\alpha = -b_0 = -1.$$

For $k = 1$,

$$2b_2 + b_1 = -\frac{3}{2},$$

and for $k = 2$,

$$6b_3 + b_2 = \frac{5}{12}.$$

The constant b_1 is arbitrary, so choose it to be 0, to get

$$y_2(x) = y_1(x)\ln(x) + 1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 + \ldots \tag{27}$$

It's straightforward to see that $\lim_{x \to 0} y_2(x) = 1 \neq 0$. Thus, y_2 does not satisfy the boundary conditions. Thus given a general solution $y(x) = Ay_1(x) + By_2(x)$, to satisfy the boundary conditions we must have that $B = 0$. The constant A must be chosen to satisfy BOTH right-hand side boundary conditions, which we can immediately see is not possible.

6.2 The case when $k(x) = M(1-x)^{-a}$, $0 < a < 2$, $a \in \mathbb{R}$

Let $z = f(y)$. Then the first derivative is

$$\phi'(y) = \phi'(z)\frac{dz}{dy}$$

and similarly

$$\phi''(y) = \phi''(z)\left(\frac{dz}{dy}\right)^2 + \phi'(z)\frac{d^2z}{dy^2} \tag{28}$$

After substituting $x = 1-y$, the original equation is

$$\phi''(y) + M^2y^{-a}\phi(y) = 0 \tag{29}$$

with conditions

$$\begin{aligned}\phi(1) &= 1\\ \phi'(1) &= -ik_0\\ \phi(0) &= 0.\end{aligned}$$

Putting (28) in (29) we have

$$\phi''(z)\left(\frac{dz}{dy}\right)^2 + \phi'(z)\frac{d^2z}{dy^2} + M^2y^{-a}\phi(z) = 0$$

which re-arranges to

$$\phi''(z) + \phi'(z)\frac{\frac{d^2z}{dy^2}}{\left(\frac{dz}{dy}\right)^2} + \frac{M^2y^{-a}}{\left(\frac{dz}{dy}\right)^2}\phi(z) = 0 \tag{30}$$

For the Frobenius method to be amenable we impose the following condition on the coefficient of $\phi(z)$:

$$\frac{M^2y^{-a}}{\left(\frac{dz}{dy}\right)^2} = M^2z^n \qquad n \geq -2\ n \in \mathbb{Z}. \tag{31}$$

This is

$$y^{\frac{-a}{2}} = z^{\frac{n}{2}}\frac{dz}{dy},$$

which integrates to

$$\frac{2y^{\frac{2-a}{2}}}{2-a} = \frac{2z^{\frac{n+2}{2}}}{2+n}.$$

Squaring both sides leads to

$$\frac{y^{2-a}}{(2-a)^2} = \frac{z^{n+2}}{(2+n)^2},$$

which we can re-arrange to give z:

$$z = \left(\frac{n+2}{2-a}\right)^{\frac{2}{n+2}} y^{\frac{2-a}{n+2}}. \tag{32}$$

The first derivative can be calculated as

$$\frac{dz}{dy} = \left(\frac{n+2}{2-a}\right)\frac{z}{y},$$

with the second derivative

$$\frac{d^2z}{dy^2} = \left(\frac{n+2}{2-a}\right)\left(\frac{n+2}{2-a}-1\right)\frac{z}{y^2}.$$

We can now calculate the ratio of the second derivative and the square of the first derivatives which is the coefficient of $\phi'(z)$

$$\frac{\frac{d^2z}{dy^2}}{\frac{dz}{dy}} = \frac{\left(\frac{n+2}{2-a}-1\right)}{\left(\frac{n+2}{2-a}\right)}\frac{1}{z}.$$

Substituting this in (30) and using (31) we have

$$\phi''(z) + \frac{\left(\frac{n+2}{2-a}-1\right)}{\left(\frac{n+2}{2-a}\right)}\frac{\phi'(z)}{z} + M^2 z^n \phi(z) = 0$$

for convenience we set $n = 0$ and define $C_a = \frac{a}{2}$. Thus our equation is

$$\phi''(z) + \frac{C_a \phi'(z)}{z} + M^2\phi(z) = 0. \tag{33}$$

Using (32) we have

$$z = \frac{2}{2-a} y^{\frac{2-a}{2}},$$

and the derivative is given by

$$\frac{dz}{dy} = y^{\frac{2-a}{2}-1}.$$

These allow us to find z and its derivative at the appropriate points:

$$y = 0 \Rightarrow z = 0, \qquad y = 1 \Rightarrow z = \frac{2}{2-a}, \qquad \left.\frac{dz}{dy}\right|_{y=1} = 1$$

which gives us the conditions

$$\begin{aligned} \phi(0) &= 0 \\ \phi\left(\frac{2}{2-a}\right) &= 1 \\ \phi'\left(\frac{2}{2-a}\right) &= -ik_0 \end{aligned}$$

We look for a Frobenius solution of the form

$$U(z) = \sum_{k=0}^{\infty} A_k z^{k+r}.$$

Substitution into (33) gives

$$\sum_{k=0}^{\infty} A_k \left[\left((k+r)(k+r-1) + C_a(k+r)\right) z^{k+r-2} + M^2 z^{k+r}\right] = 0,$$

which becomes

$$\sum_{k=0}^{\infty} A_k (k+r)(k+r-1+C_a) z^{k+r-2} + \sum_{k=2}^{\infty} M^2 A_{k-2} z^{k+r-2} = 0.$$

The indicial ($k = 0$) equation is $r(r - 1 + C_a) = 0$, which gives the values for r to be $r = 0$ and $r = 1 - C_a$. If these differ by an integer (i.e. $C_a \in \mathbb{Z}$) we have the same problem as we had in the case $a = 1$. Otherwise, looking at the $k = 1$ equation, we have

$$(1 + r)(C_a + r)A_1 = 0$$

which implies $A_1 = 0$.

For $k \geq 2$, we obtain the recursive relation

$$A_k = \frac{-M^2}{(k + r)(k + r - 1 + C_a)} A_{k-2},$$

which implies either $k = 2p + 1$ $p \in \mathbb{Z}$, in which case as $A_1 = 0$, $A_{2p+1} = 0$, or $k = 2p$, which gives rise to

$$A_{2p} = \prod_{i=0}^{p} \frac{-M^2}{(2i + r)(2i + r - 1 + C_a)} A_0.$$

We can now formulate the two solutions noting first that

$$1 - C_a = 1 - \frac{a}{2} = \frac{2 - a}{2},$$

then if $\phi_0(z)$ denotes the solution at $r = 0$, we have

$$\phi_0(x) = \sum_{k=0}^{\infty} \prod_{i=0}^{k} \frac{-M^2}{(2k)(2k - 1 + C_a)} A_0 x^{2k}.$$

If $\phi_a(z)$ is the solution when $r = \frac{2+a}{2}$, we have

$$\phi_a(x) = \sum_{k=0}^{\infty} \prod_{i=0}^{p} \frac{-M^2}{\left(2i + \frac{2-a}{2}\right) 2i} A_0 x^{2k + \frac{2-a}{2}}. \tag{34}$$

Our true solution is

$$\phi(z) = A\phi_0(z) + B\phi_a(z).$$

Clearly, $\phi_a(0) = 0$ so $\phi(0) = AA_0$. Thus for $\phi(0) = 0$, we require $A = 0$. From (34),

$$\phi_a'(x) = \sum_{k=0}^{\infty} \left(2k + \frac{2 - a}{2}\right) \prod_{i=0}^{p} \frac{-M^2}{\left(2i + \frac{2-a}{2}\right) 2i} A_0 x^{2k + \frac{2-a}{2} - 1},$$

hence

$$\phi_a'(1) = \sum_{k=0}^{\infty} \left(2k + \frac{2 - a}{2}\right) \prod_{i=0}^{p} \frac{-M^2}{\left(2i + \frac{2-a}{2}\right) 2i} A_0.$$

Since M, i, $k \in \mathbb{R}$, we have $\phi'\left(\frac{2}{2-a}\right) = -ik_0$, implying BA_0 is imaginary. However,

$$\phi_a(1) = \sum_{k=0}^{\infty} \prod_{i=0}^{p} \frac{-M^2}{\left(2i + \frac{2-a}{2}\right) 2i} A_0$$

with M, i, $k \in \mathbb{R}$, we see that $\phi\left(\frac{2}{2-a}\right) = 1$, meaning $BA_0 \in \mathbb{R}$ which is a contradiction, **so no solution exists**.

7 Asymptotics when $k(x) = M(1-x)^{-n}$ for $n > 1$

While we don't have a closed formula solution for $n > 2$ one can derive approximations of the solution $\phi(x)$ close to the singularity. For $n > 1$, the point $x = 1$ no longer remains a regular singular point and so we can no longer hope to obtain a solution via the Frobenius method. Instead we make a WKB approximation. First, we make the change of variables $x \to (1-x)$ to get

$$\phi_{xx}(x) + M^2 x^{-2n}\phi(x) = 0 \tag{35}$$
$$\phi(0) = 0, \tag{36}$$
$$\tag{37}$$

We interested in the behaviour of the solution ϕ near the singularity. We assume that near 0, $\phi(x)$ is of the form $\phi(x) = \exp[S(x)]$ for some function $S(x)$. Substituting this into the equation we see that $S(x)$ must satisfy

$$x^{2n}((S'(x))^2 + S''(x)) + M^2 = 0. \tag{38}$$

Suppose that $S'(x) = cx^{\alpha}$ to leading order. Substituting we get that

$$c^2 x^{2\alpha+2n} + c\alpha x^{\alpha+(2n-1)} + M^2 = 0. \tag{39}$$

Balancing the dominant terms we see that $\alpha = -n$ and $c^2 = -M^2$, so that $c = \pm iM$.

Further expanding $S'(x)$ as
$$S'(x) = \pm iMx^{-n} + A(x),$$
for $A(x) = o(x^{-n})$, then writing $c_1 x^{\beta}$ and substituting we get

$$\left(c^2 + 2c_1 c x^{n+\beta} + c_1^2 x^{2\beta+2n} - ncx^{n-1} + c_1\beta x^{2n+\beta-1}\right) + M^2 = 0. \tag{40}$$

Balancing dominant terms we have that $\beta = -1$,and $c_1 = n/2$, and so

$$S'(x) = \pm iMx^{-n} + \frac{n}{2}x^{-1}. \tag{41}$$

Integrating we get

$$S(x) = \pm iM\frac{x^{1-n}}{1-n} + \frac{n}{2}\ln|x|, \tag{42}$$

and so for $0 \le x \ll 1$, $\phi(x)$ has the general form

$$\phi(x) = Ax^{\frac{n}{2}}e^{\frac{iMx^{1-n}}{1-n}} + Bx^{\frac{n}{2}}e^{-\frac{iMx^{1-n}}{1-n}} \tag{43}$$

8 Regularised cloaking for $n > 1$

If we consider instead $k(x) = (1-x+\epsilon)^{-n}$, then in this case we see that, we can approximate the corresponding solution at zero, by evaluating (43) at ϵ. Doing so gives us

$$\phi(x) = A\epsilon^{\frac{n}{2}}e^{\frac{iM\epsilon^{1-n}}{1-n}} + B\epsilon^{\frac{n}{2}}e^{-\frac{iM\epsilon^{1-n}}{1-n}} = O(\epsilon^{\frac{n}{2}}). \tag{44}$$

Calculating how the constants behave isn't straightforward as the solution ϕ is only valid around 0. To extend it to the whole domain we'd need a boundary layer approximation matching this solution to an outer solution. However, we expect the constants to behave linearly with respect to k_0.

9 A piecewise constant $k(x)$

Another possibility would be to consider whether "staircase" of piecewise constant functions would serve as a good candidate for this one-dimensional cloaking problem. The appeal of this approach is that it reduces the problem to an algebraic problem.

Let $(l_n)_{n\in\mathbb{N}}$ be a sequence of lengths partitioning $[0,1]$, so that $\sum_n l_n = 1$. Choose $k_n \in \mathbb{R}$ such that $k_n.l_n = \frac{\pi}{2}$. Then suppose we consider the following problem

$$\begin{aligned} \phi_n''(x) + k_n^2\phi_n(x) &= 0 \qquad\qquad x \in (0, l_n) \\ \phi_n(0) &= A_n \\ \phi_n'(0) &= B_n \end{aligned}$$

Then one can see that $\phi_n(l_n) = A_n \cos(k_n.l_n) + \frac{B_n}{k_n}\sin(k_n.l_n) = \frac{B_n}{k_n}$ since $k_n.l_n = \frac{\pi}{2}$. Similarly $\phi'(l_n) = -k_n.A_n$. Let's introduce a scheme where the values $\phi_n(l_n)$ and $\phi_n'(l_n)$ become the left-hand side for the $(n+1)^{th}$ problem. Indeed, starting from $A_1 = 1$, $B_1 = ik_0$ we obtain the following recursive relationship

$$A_{n+1} = \frac{B_n}{k_n}, \qquad n > 0$$

and

$$B_{n+1} = -A_n k_n \qquad n > 0.$$

So that

$$A_n = (-1)^n \frac{ik_0}{k_1}\frac{k_2}{k_3}\cdots\frac{k_{n-2}}{k_{n-1}},$$

and

$$B_n = (-1)^{n-1} ik_0 \frac{k_1}{k_2}\cdots\frac{k_{n-1}}{k_{n-2}},$$

so that as $n \to \infty$, A_n converges to 0

$$k_n < c.k_{n+1}, \qquad \text{for all } n \tag{45}$$

for some $0 < c < 1$ or equivalently

$$l_{n+1} < c.l_n, \qquad \text{for all } n \tag{46}$$

This is the growth condition we require for the piecewise-constant function to be a suitable candidate for perfect cloaking. Note that $B_n \to \infty$ as $n \to \infty$, which is how we expect the derivative to behave, since we need a shock at $x = 1$.

Now consider the ODE on $[0,1]$ given by

$$\begin{aligned} \phi''(x) + k_n^2\phi(x) &= 0 \qquad\qquad x \in \Big(\sum_{k=1}^{n-1} l_k, \sum_{k=1}^{n} l_k\Big) \\ \phi(x) &= 1 \qquad\qquad x = 0 \\ \phi'(x) &= ik_0 \qquad\qquad x = 0 \end{aligned}$$

Then it's clear that

$$\lim_{x\to 1}\phi(x) = \lim_{n\to\infty}\phi(\sum_{k=1}^{n} l_n) = \lim_{n\to\infty}\phi_n(l_n) = \lim_{n\to\infty}A_{n+1} = 0.$$

It would be interesting to consider the error introduced in truncated part of the "staircase" in a neighbourhood of $x = 1$ where it attains ∞, but we don't have time to consider this.

10 Conclusion

We have studied a very basic one-dimensional model of cloaking and considered a possible candidate for the refractive index of the form $k(x) = M(1-x)^n$ for $0 < n < \infty$. We exhaustively studied the range of possible values of n and identified the values for which $k(x)$ is a suitable candidate. Wanting to avoid the singularity at $x = 1$ which seems necessary for perfect cloaking, we then considered a regularisation of the problem and provided bounds for the error in the solution for general n, thus allowing partial cloaking with an error term that can be controlled. Finally we considered a possibly piecewise constant candidate for the refractive index and showed that it satisfies our requirements to be a perfect cloaking medium.

Possible future work would be extending the above results to a $2D$ and $3D$ medium (a brief study suggests this is certainly not as straightforward as initially supposed, at least following the approach of [1]).

References

[1] S.A. Cummer, B.I. Popa, D. Schurig, D.R. Smith, J. Pendry, M. Rahm, and A. Starr. Scattering theory derivation of a 3d acoustic cloaking shell. *Physical review letters*, 100(2):24301, 2008.

[2] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*, volume 140. American Mathematical Society, 2012.