Celestial Mech Dyn Astr (2006) 94:433–449
DOI 10.1007/s10569-006-9006-0

ORIGINAL ARTICLE

Nonplanar second species periodic and chaotic trajectories for the circular restricted three-body problem

S. Bolotin · R. S. MacKay

Received: 3 November 2005 / Revised: 27 January 2006 /
Accepted: 20 February 2006 / Published online: 13 June 2006

Abstract For the circular restricted three-body problem of celestial mechanics with small secondary mass, we prove the existence of uniformly hyperbolic invariant sets of non-planar periodic and chaotic almost collision orbits. Poincaré conjectured existence of periodic ones and gave them the name "second species solutions". We obtain large subshifts of finite type containing solutions of this type.

Keywords Collisions · Regularization · Second species orbits · Singular perturbation · Three-body problem

1 Introduction

In chapter XXXII of Poincaré 1899, he proposed that there are periodic solutions of the three-body problem of celestial mechanics with second and third masses m, μ small compared to the primary mass M, which as $m, \mu \to 0$ converge to sequences of pairs of segments of Kepler orbit joined at collisions. He christened them "second species" orbits. He derived several necessary conditions on the sequences of collision arcs which occur as the limits and sketched an argument that these are sufficient for existence of nearby second species orbits when m, μ are small enough.

It is agreed (Levy 1952), however, that Poincaré did not provide a proof and that the result is not true in the full generality that he claimed. Despite many analyses (e.g., Alexeev 1970; Henrard 1980; Bruno 1981; Perko 1981; Gomez and Olle 1991), it is only recently that any complete proofs have been written, and so far they are only for the "restricted" case where

S. Bolotin (✉)
Department of Mathematics and Moscow Steklov Mathematical Institute,
University of Wisconsin-Madison, VanVleck Hall, 480 Lincoln Drive
Madison, Wisconsin 53706, USA
e-mail: bolotin@math.wisc.edu

R. S. MacKay
Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK
e-mail: mackay@maths.warwick.ac.uk

$\mu = 0$. Marco and Niederman (1995) proved the existence of a periodic second species orbit with two collisions per period for the planar circular restricted case. For the same case, we proved in Bolotin and MacKay (2000) the existence of large sets of second species orbits[1], including aperiodic analogues, to which we proposed to extend the same name. They form uniformly hyperbolic subshifts. A similar result was subsequently obtained by Font et al. (2002) by a different method but it is limited to orbits with small angle changes at collisions. In contrast, the angle changes are large in Bolotin and MacKay (2000) and in the present paper. In recent work (Bolotin 2005, 2006), an extension has been made to the slightly elliptic case.

In the present paper, we extend our analysis of the circular restricted three-body problem to prove existence of uniformly hyperbolic subshifts of *nonplanar* second species orbits. The method is the same as in Bolotin and MacKay (2000). Indeed we already prepared the ground there by allowing for the 3D case in our analysis of the general n-center problem and in remarking that it was likely that existence of one of Poincaré's classes of nonplanar second species orbits could be proved by this method.

Firstly we recall from Bolotin and MacKay (2000) the general mathematical setting for our method. Next we put the spatial circular restricted three-body problem into this setting and enunciate our result. Then we construct the set of collision arcs from which we make our second species orbits and check the conditions of our general setting are satisfied. We close with some comments and open questions. In Appendix A, we prove uniform hyperbolicity of any subshift constructed by the general method of Bolotin and MacKay (2000).

2 Mathematical setting

Let $P = \{p_1, \dots, p_n\}$ be a finite set in a 3D manifold Q. Consider a Lagrangian system (L_ε) with configuration space $Q \setminus P$ and Lagrangian

$$L_\varepsilon(q, \dot{q}) = L_0(q, \dot{q}) - \varepsilon V(q). \tag{2.1}$$

We assume that L_0 is C^4 everywhere in Q and quadratic in the velocity:

$$L_0(q, \dot{q}) = T(q, \dot{q}) + \langle w(q) \cdot \dot{q} \rangle - W(q), \tag{2.2}$$

where the kinetic energy $T(q, \dot{q})$ is a positive definite quadratic form in $\dot{q}$, and $w(q)$ is a covector field on Q. Let V be a C^4 function on $Q \setminus P$ having Newtonian singularities on P. This means that in a neighborhood U_α of any point $p_\alpha \in P$,

$$V(q) = -\frac{f_\alpha(q)}{\operatorname{dist}(q, p_\alpha)}, \quad f_\alpha(p_\alpha) > 0, \tag{2.3}$$

where f_α is a C^4 function on U_α, and the distance is defined by means of the Riemannian metric T. We study system (L_ε) for small $\varepsilon > 0$. Then it is a singular perturbation of system (L_0).

Let

$$H_\varepsilon = H_0 + \varepsilon V, \qquad H_0(q, \dot{q}) = T(q, \dot{q}) + W(q) \tag{2.4}$$

be the energy integral. We fix E such that the domain $D = \{q \in Q \mid W(q) < E\}$ contains the set P and study system (L_ε) on the energy level $\{H_\varepsilon = E\}$.

[1] See also MacKay (2005) for a summary, some minor additions and a correction.

We say that a solution $\gamma\colon [0,\tau] \to D$ of system (L_0) is a *collision arc* if $\gamma(0), \gamma(\tau) \in P$ and has

- **No early collisions**: $\gamma(t) \notin P$ for $0 < t < \tau$.

Let E be the energy of γ. Then γ is a critical point of the Maupertuis–Jacobi functional (see e.g. Arnold et al. 1989) J_E on the set Ω of nonparameterized curves in D with end points in P:

$$J_E(\gamma) = \int_0^\tau g_E(\gamma(t), \dot\gamma(t))\,\mathrm{dt}, \qquad g_E(q,\dot q) = 2\sqrt{(E - W(q))T(q,\dot q)} + \langle w(q)\cdot \dot q\rangle,$$

where g_E is the Jacobi metric. Since $W|_D < E$, the functional J_E is well defined on Ω. We say that the collision arc γ is

- **Nondegenerate** if it is a nondegenerate critical point of J_E.

The functional J_E is independent of the parametrization of γ, so nondegeneracy means nondegeneracy in the set of curves with fixed parametrization, for example, parameterized by time or by Jacobi length.

The definition of nondegeneracy in terms of a variational principle seems the most natural, and it is the one which is actually used in the proof. However, the following definition of nondegeneracy is more suitable for verification in concrete examples. Represent the general solution of system (L_0) as $q(t) = f(q_0, v_0, t)$, where $q_0, v_0 \in \mathbb{R}^3$ are initial position and velocity. Then collision arcs with energy E connecting p_α to p_β correspond to solutions of the system of four equations

$$f(p_\alpha, v, \tau) = p_\beta, \qquad H_0(p_\alpha, v) = E \tag{2.5}$$

in four variables v, τ. The nondegeneracy condition is that the Jacobian at the solution is nonzero. We use a slight variant of this in Section 4.4.

Suppose that system (L_0) has nondegenerate collision arcs $\gamma_k\colon [0,\tau_k] \to D$, $k \in K$ (a finite set) with the same energy E connecting the points p_{α_k} and p_{β_k}. A sequence $(\gamma_{k_i})_{i\in\mathbb{Z}}$ is called a *collision chain* (Poincaré called them “orbites à chocs”) if $\beta_{k_i} = \alpha_{k_{i+1}}$ and satisfies

- **Direction change**: $\dot\gamma_{k_i}(\tau_{k_i}) \neq \pm\dot\gamma_{k_{i+1}}(0)$ for all i.

Collision chains correspond to paths in the graph Γ with the set of vertices K and the set of edges

$$\Gamma = \{(k,k') \in K^2 \mid \beta_k = \alpha_{k'},\ \dot\gamma_k(\tau_k) \neq \pm\dot\gamma_{k'}(0)\}. \tag{2.6}$$

The following result is proved in Bolotin and MacKay (2000).

Theorem 2.1 *Given a finite set K of nondegenerate collision arcs with the same energy E, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and any collision chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$, $k_i \in K$, there exists a unique (up to a time shift) trajectory $\gamma\colon \mathbb{R} \to D\setminus P$ of energy E of system (L_ε), which shadows the chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$ within order ε. More precisely, there exist $c, C > 0$, independent of ε and the collision chain, and a sequence $(t_i)_{i\in\mathbb{Z}}$ such that $|t_{i+1} - t_i - \tau_i| \leqslant C\varepsilon$,* dist $(\gamma(t), \gamma_{k_i}(t - t_i)) \leqslant C\varepsilon$ *for $t_i \leqslant t \leqslant t_{i+1}$, and* dist $(\gamma(t), P) \geqslant c\varepsilon$.

Hence there is an invariant subset Λ_ε in $\{H_\varepsilon = E\}$ on which system (L_ε) is a suspension of a subshift of finite type. It can be proved to be uniformly hyperbolic, and strongly so.

Theorem 2.2 *There exists a cross-section* $N \subset \{H_\varepsilon = E\}$ *such that the corresponding invariant set* $M_\varepsilon = \Lambda_\varepsilon \cap N$ *of the Poincaré map is uniformly hyperbolic with Lyapunov exponents of order* $\log \varepsilon^{-1}$.

Corollary 2.1 *The set* Λ_ε *is uniformly hyperbolic, as a suspension of a hyperbolic invariant set with bounded transition times.*

Theorem 2.2 can be deduced from the proof in Bolotin and MacKay (2000) of Theorem 2.1, but it was not proved there. Thus we prove it here in Appendix A.

The topological entropy of the Poincaré map on M_ε is positive provided the graph Γ has a connected branched sub-graph. In fact the topological entropy is $O(\varepsilon)$-close to that of the topological Markov chain determined by the graph Γ. In the case of a periodic sequence $(k_i)_{i\in\mathbb{Z}}$, local uniqueness of the trajectory γ implies that it is also periodic, closing after one cycle of the sequence.

Remarks One can allow the nonsingular part L_0 of the Lagrangian L_ε also to depend on ε. Then all the results remain true with L_0 replaced by $L_0|_{\varepsilon=0}$.

The result can be extended to some L_ε, which are not quadratic in the velocity; a case like this arises for the reduction of the motion of two charges in a uniform magnetic field with respect to Euclidean symmetry.

3 Application to the spatial circular restricted three-body problem

Consider the spatial circular restricted three-body problem (Sun, Jupiter, and Asteroid, with the Sun and Jupiter moving in circles around their center of mass and the Asteroid of zero mass free to move in 3D) and suppose that the mass of Jupiter is small with respect to the mass of the Sun. We normalize the masses to $1-\varepsilon$ (Sun), ε (Jupiter), and 0 (Asteroid), with the center of mass stationary and the first two masses in circular orbits about it, having separation and angular frequency both normalized to 1.

To apply Theorem 2.1, consider the motion of the Asteroid in the frame $Oxyz$ rotating anti-clockwise about the z-axis through the Sun at angular frequency 1 with Jupiter. Then the Sun is at $O = (0, 0, 0)$, and Jupiter can be chosen at $P = (1, 0, 0)$. The motion of the Asteroid $q = (x, y, z) \in \mathbb{R}^3$ is described by a Lagrangian system (L_ε) of the form (2.1), where

$$L_0(q, \dot q) = \frac{1}{2}|\dot q|^2 + x\dot y - y\dot x + W(q), \qquad W(q) = \frac{1}{2}|q|^2 + \frac{1}{|q|} \tag{3.7}$$

and

$$V(q) = \frac{1}{|q|} - \frac{1}{|q-P|} + x. \tag{3.8}$$

Hence L_0 has the form (2.2), V has the form (2.3), and $Q = \mathbb{R}^3\setminus\{0\}$. The singular set consists of one point P. For the restricted three-body problem, the energy integral (2.4) in the rotating coordinate frame

$$H_\varepsilon = \frac{1}{2}|\dot q|^2 - \frac{1}{2}|q|^2 - \frac{1-\varepsilon}{|q|} - \frac{\varepsilon}{|q-p|} + \varepsilon x$$

is called the Jacobi integral. Its value is traditionally denoted by $-C/2$ and C is called the *Jacobi constant*. Denote the energy of the Asteroid in the fixed coordinate frame by $\mathcal{E}$

and the z-component of its angular momentum about O by G_z. Then the Jacobi constant $C = -2\mathcal{E} + 2G_z$.

For $\varepsilon = 0$, system (L_0) is the Kepler problem of Sun–Asteroid in the rotating coordinate frame. Its orbits with $\mathcal{E} < 0$ are transformations to the rotating frame of ellipses with parameters a, e, ι, where a is the semi-major axis, e is the eccentricity, and ι the inclination of the orbit to the plane of the orbit of the Sun and Jupiter. They have angular frequency $\Omega = a^{-3/2}$ and Jacobi constant $C = a^{-1} + 2\sqrt{a(1-e^2)} \cos \iota$, where ι is taken in such a way that $\cos \iota > 0$ if the projection to the plane of Jupiter's orbit rotates in the same direction as Jupiter, negative if opposite.

Given $C \in \mathbb{R}$ we define the set A_C of *allowed frequencies* of Kepler ellipses to be

- $(0, 1)$ if $C \in [-1, +2]$,
- $(0, (2 + C)^{3/2})$ if $C \in (-2, -1)$,
- $((3 - C)^{3/2}, 1)$ if $C \in (2, 3)$, and
- empty if $C \notin (-2, +3)$

(the motivation will be revealed in Section 4.1).

Theorem 3.1 *For any $C \in (-2, +3)$ there exists a dense subset S of the set A_C of allowed frequencies, such that for any finite set $\Lambda \subset S$ there exists $\varepsilon_0 > 0$ such that for any sequence $(\Omega_n)_{n\in\mathbb{Z}}$ in Λ and $\varepsilon \in (0, \varepsilon_0)$ there is a trajectory of the spatial circular restricted three-body problem with Jacobi constant C, which avoids collisions by order ε and in the rotating frame is within order ε of a concatenation of collision orbits formed from arcs of Kepler ellipses of frequencies Ω_n with inclinations ι_n satisfying* $\cos \iota_n = C/2 - \Omega_n^{2/3}$. *The resulting invariant set is uniformly hyperbolic.*

In particular, the Poincaré map for given Jacobi constant has a chaotic invariant set with Lyapunov exponents of order $\log \varepsilon^{-1}$, and it contains infinitely many nonplanar periodic second species orbits.

4 Construction of collision arcs

Here we prove Theorem 3.1 by constructing a large set of nonplanar collision arcs with the given value of C for the case $\varepsilon = 0$ (Lemma 4.1), checking their nondegeneracy (Section 4.4), and constructing from them a nontrivial set of collision chains which change direction at each collision (Section 4.5). Theorem 3.1 then follows by applying Theorems 2.1 and 2.2. The first two aspects are most easily done in the nonrotating frame, to which we now revert.

4.1 Nonplanar circle-crossing orbits

First, we summarize the well-known facts (Hénon 1997) about which elliptic orbits of the spatial Kepler problem cross the horizontal unit circle. As Poincaré (1899) remarked, segments of Kepler orbit about the Sun between two intersections with a given one (circle in our case) fall into four classes as follows:

1. a whole number of revolutions of a coplanar orbit;
2. a segment of coplanar orbit between distinct intersection points;
3. a whole number of revolutions of a noncoplanar orbit;
4. a segment of a noncoplanar orbit between points at opposite ends of a straight line through the Sun.

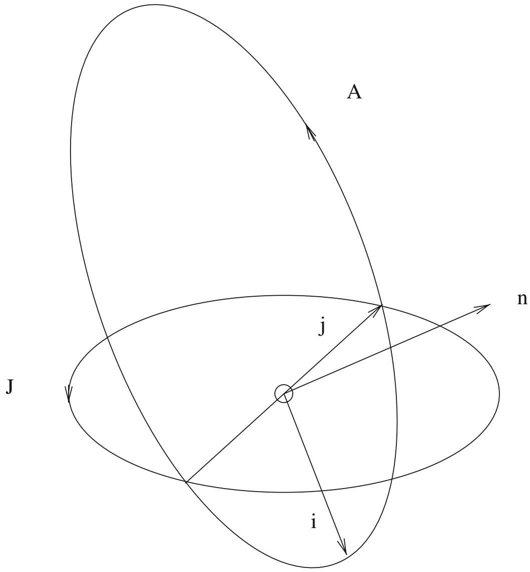

Fig. 1 A nonplanar Kepler ellipse cutting Jupiter's orbit at two points

Here we consider only the last class of orbit, because construction of subshifts from the first class was already done in Bolotin and MacKay (2000), construction from the third class looks problematic to us, and the orbit of Marco and Niederman (1995) was generated from two arcs in the second class so it is not as virgin territory as the fourth class (though still merits treating one day). An interesting question that we will address at the end of the paper is whether one can make subshifts using arcs from a combination of classes.

In this section, we use a nonrotating coordinate system $Oxyz$, centered on the Sun. Jupiter moves anticlockwise along the unit circle in the Oxy-plane, so its position is $(\cos t, \sin t, 0)$. Let $\Pi \subset \mathbb{R}^3$ be the plane containing the elliptic collision arc γ: $[0, \tau] \to \mathbb{R}^3$. We assume that Π is not the plane Oxy of Jupiter's orbit. We orient Π in such a way that the motion of the Asteroid is counter clockwise, and let $\mathbf{n}$ be the corresponding unit normal. Let $\mathbf{i} \in \Pi$ be the unit vector towards the perihelion of the Asteroid's orbit, and let $\mathbf{j} = \mathbf{n} \times \mathbf{i}$ (see Figure 1). Then $\mathbf{j}$ lies in the intersection of Π with the Oxy plane. We will consider chains of elliptic collision arcs γ: $[0, \tau] \to \mathbb{R}^3$ with $\gamma(0) \neq \gamma(\tau)$ (in the fixed coordinate frame). Then there exists $\sigma \in \pm$ such that $\gamma(0) = -\sigma\mathbf{j}$, $\gamma(\tau) = \sigma\mathbf{j}$.

Define the inclination $\iota \in [0, \pi]$ of a Kepler orbit to be the angle of $\mathbf{n}$ to the upward vertical. Then $\cos \iota = \mathbf{n} \cdot \mathbf{e}_z$. We have $\iota \in (0, \pi)$ because we chose to exclude planar orbits.

Denote the semi-major axis of the elliptical orbit of the Asteroid by a and its eccentricity by e. We introduce polar coordinates (r, θ) about O in the plane Π such that $\theta = 0$ corresponds to the perihelion in the $\mathbf{i}$-direction and θ increases in the direction of motion of the Asteroid. Then

$$r = \frac{a(1-e^2)}{1 + e \cos \theta}. \tag{4.9}$$

The angular momentum is $\mathbf{G} = G\mathbf{n}$, with $G = \sqrt{a(1-e^2)}$. The Jacobi constant has the value

$$C = a^{-1} + 2G_z, \qquad G_z = G \cos \iota. \tag{4.10}$$

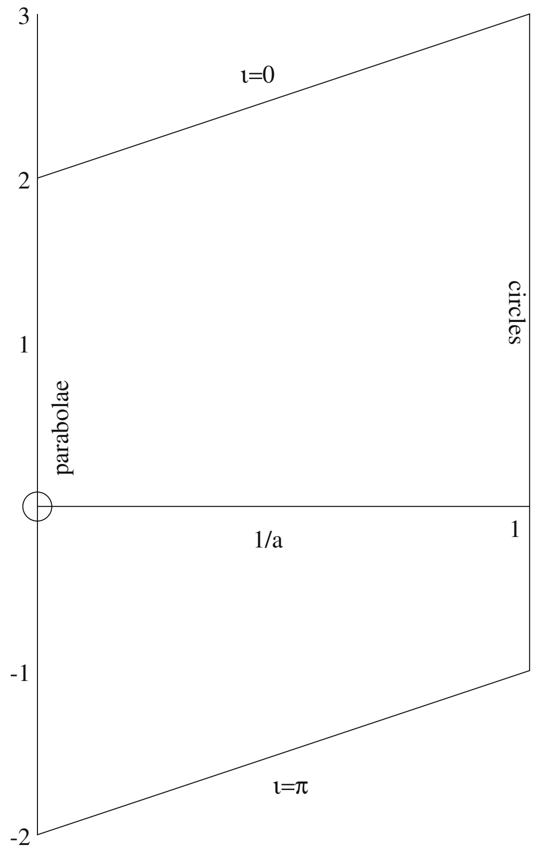

Fig. 2 Region of inverse semi-major axis $1/a$ and Jacobi constant C for the existence of a Kepler ellipse cutting Jupiter's orbit at two points

The points $\pm\mathbf{j}$ at which the intersections of the orbits occur have polar angle $\theta = \pm\pi/2$ in the Π plane. So θ goes from $-\sigma\pi/2$ to $+\sigma\pi/2$ (modulo 2π) as t goes from 0 to τ. Since $r = 1$ at the intersections, we have

$$a(1 - e^2) = 1 \tag{4.11}$$

and $G = 1$. Since Jupiter's period is 2π, the duration of the collision arc is

$$\tau = 2\pi k + \pi \tag{4.12}$$

for some $k \in \mathbb{Z}_+$. For example, one could start at $\mathbf{j}$ in Figure 1 and let the Asteroid perform one half revolution in θ, while Jupiter performs one and a half revolutions (this gives a collision arc with $\sigma = -$).

The set of parameters for nonplanar circle-crossing orbits with given Jacobi constant is indicated in the (a^{-1}, C)-plane of Figure 2. In particular, for each $C \in (-2, +3)$ the set of frequencies of Kepler ellipse of Jacobi constant C cutting Jupiter's orbit at opposite ends of a diameter is the set A_C defined in Section 3.

Note that given a, e satisfying (4.11), $\iota \in (0, \pi)$ and a diameter of Jupiter's orbit, there are two Kepler ellipses with these parameters cutting Jupiter's orbit at the ends of the chosen diameter. They are reflections of each other in the horizontal plane (see Figure 3). We

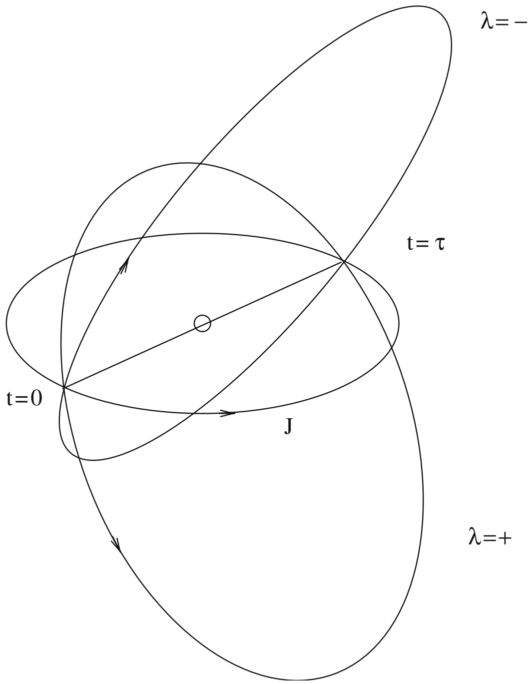

Fig. 3 The two collision arcs with the same starting point, C, a, and σ ($\sigma = -$ in this picture)

distinguish them by a symbol $\lambda \in \pm$, with $\lambda = +$ if the perihelion **i** is above the horizontal plane, $\lambda = -$ if it is below.

4.2 Conditions to start and end on Jupiter

Let $\pm\eta$, $\eta \in [0, \pi/2]$, be the eccentric anomaly of the points $\pm\mathbf{j}$ corresponding to polar angle $\theta = \pm\pi/2$. From the equation

$$r \cos \theta = a(\cos \eta - e)$$

relating θ and eccentric anomaly η, we obtain $e = \cos\eta$. Combining this with (4.11) we obtain $a = \sin^{-2}\eta$. Then the mean angular velocity of the elliptic orbit is $\Omega = a^{-3/2} = \sin^3\eta$. Hence by (4.10),

$$C = \sin^2\eta + 2\cos\iota.$$

There are two cases for the collision arc $\gamma\colon [0, \tau] \to \mathbb{R}^3$:

$\sigma = +$ It starts ($t = 0$) at the point $-\mathbf{j}$, $\theta = -\pi/2$, and ends ($t = \tau$) at the point $\mathbf{j}$, $\theta = \pi/2$. Then for $0 \leqslant t \leqslant \tau$, the eccentric anomaly changes from $-\eta$ to $\eta + 2\pi m$, for some $m \in \mathbb{Z}_+$. Hence for $\sigma = +$, denoting changes by Δ,

$$\Delta\eta = 2\pi m + 2\eta, \qquad \Delta \sin\eta = 2\sin\eta.$$

$\sigma = -$ It starts at $\mathbf{j}$, $\theta = \pi/2$, and ends at $-\mathbf{j}$, $\theta = -\pi/2$. The eccentric anomaly changes from η to $-\eta + 2\pi(m + 1)$, for some $m \in \mathbb{Z}_+$. Hence for $\sigma = -$,

$$\Delta\eta = 2\pi(m + 1) - 2\eta, \qquad \Delta \sin\eta = -2\sin\eta.$$

From Kepler's equation of time

$$\Omega\tau = \Delta\eta - e\Delta \sin \eta$$

and (4.12) we obtain

$$\pi(2k+1)\sin^3\eta - \pi(2m+1) + \sigma g(\eta) = 0, \tag{4.13}$$

where

$$g(\eta) = \pi - 2\eta + \sin 2\eta.$$

Let us analyze solutions of equation (4.13) for $\eta \in [0, \pi/2]$ (see Figure 4). We have $g(0) = \pi$, $g(\pi/2) = 0$ and $0 \leqslant g(\eta) \leqslant \pi$, $g'(\eta) \leqslant 0$ on $[0, \pi/2]$. Hence, for both $\sigma = \pm$, no solutions exist if $m > k$. Write $m = k - l$, $l \in \mathbb{Z}_+$, $0 \leqslant l \leqslant k$. Then (4.13) gives

$$\pi(2k+1)(1 - \sin^3\eta) = 2\pi l + \sigma g(\eta). \tag{4.14}$$

For $0 \leqslant \eta \leqslant \pi/2$, the left-hand side is decreasing from $\pi(2k+1)$ to 0. For $\sigma = -$ the right-hand side is increasing from $2\pi l - \pi$ to $2\pi l$. The derivatives are nonzero on $(0, \pi/2)$. Hence for any $1 \leqslant l \leqslant k$ there exists a unique nondegenerate solution $\eta_-(k, m) \in (0, \pi/2)$. For $l = 0$ there is the unique solution $\eta = \pi/2$ and it is nondegenerate, but it will turn out slightly problematic to make use of this solution, so we will ignore it.

For $\sigma = +$ the right-hand side of (4.14) is decreasing from $\pi(2l+1)$ to $2\pi l$, so both sides are decreasing. Although existence of a solution for each $0 \leqslant l < k$ is clear by the intermediate value theorem, we need to establish their nondegeneracy (we exclude the case $l = k$ from consideration because the obvious solution $\eta = 0$ is degenerate). It will turn out from the calculation that the solutions are also unique. Equating the derivative of (4.13) to zero, we obtain

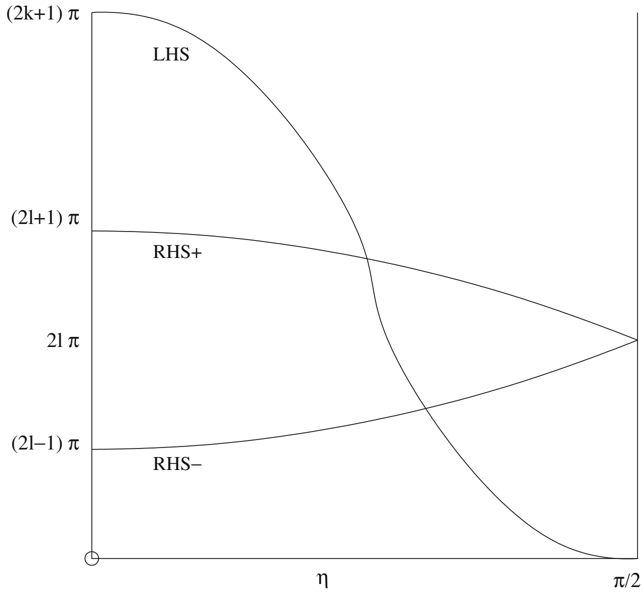

Fig. 4 Sketches of the left (LHS) and right (RHS± for $\sigma = \pm$)-hand sides of (4.14) as functions of the eccentric anomaly η at collision

$$3\pi(2k+1)\sin^2\eta\,\cos\eta - 4\sigma\,\sin^2\eta = 0.$$

Hence, as expected, for $\sigma = -$ the only critical point is $\eta = 0$, and all solutions of (4.14) in $(0, \pi/2)$ are nondegenerate.

For $\sigma = +$ there is one other critical point $\eta_* \in (0, \pi/2)$, given by

$$\cos\eta_* = \frac{4}{3\pi(2k+1)}.$$

We will show that this point can not be a solution of (4.13). One can write

$$1 - \sin^3\eta = \cos^2\eta\frac{1+\sin\eta+\sin^2\eta}{1+\sin\eta}.$$

Since

$$g(\eta) = 2(\pi/2-\eta) + 2\sin\eta\cos\eta > 2\cos\eta(1+\sin\eta),$$

for any solution of (4.14) with $\sigma = +$ we obtain

$$\pi(2k+1)\cos\eta\frac{1+\sin\eta+\sin^2\eta}{1+\sin\eta} > 2(1+\sin\eta).$$

This implies

$$\cos\eta > \frac{2(1+\sin\eta)^2}{\pi(2k+1)(1+\sin\eta+\sin^2\eta)} > \frac{2}{\pi(2k+1)} > \cos\eta_*.$$

Hence $0 < \eta_* < \eta$, and no solution is a critical point. Thus for $\sigma = +$ and $0 \leqslant l < k$ there is a unique nondegenerate solution $\eta_+(k, m)$ of (4.13) in $(0, \pi/2)$.

Note that the frequency $\Omega = \sin^3\eta$ satisfies

$$\Omega = \frac{2m+1}{2k+1} - \frac{\sigma g(\eta)}{\pi(2k+1)},$$

and

$$\frac{g(\eta)}{\pi(2k+1)} \in (0, 1).$$

Hence $\Omega \in (2m/(2k+1), (2m+2)/(2k+1))$.

Thus for any inclination $\iota \in (0, \pi)$, the label $\sigma = \pm$ shows the direction of $\gamma(\tau) = \sigma\mathbf{j}$ relative to $\mathbf{j}$. For given σ and any pairs of integers $1 \leqslant m \leqslant k$ (if $\sigma = +$), $0 \leqslant m < k$ (if $\sigma = -$), we obtain an arc $\gamma\colon\ [0, \tau] \to \mathbb{R}^3$ with end points on Jupiter and frequency $\Omega(\sigma, k, m) = \sin^3\eta$ in the interval $(2m/(2k+1), (2m+2)/(2k+1))$. It follows that for given σ the $\Omega(\sigma, k, m)$ form a dense subset of $(0, 1)$.

4.3 Early collisions

Next we have to restrict to arcs $\gamma\colon\ [0, \tau] \to \mathbb{R}^3$ for which there is no early collision, i.e. $t \in (0, \tau)$ for which $\gamma(t)$ coincides with Jupiter, because such an arc should be divided into more than one collision arc.

If an early collision exists, then there are at least two collisions along the arc at the same position of Jupiter in the inertial frame. Hence $\Omega = \sin^3\eta$ must be rational. More importantly, at least one of the arcs produced by subdividing at an early collision has the same start point and same end point as the original arc, so by reducing m and k appropriately, we obtain a replacement collision arc on the same Kepler orbit, with no early collision.

More precisely, suppose for definiteness that $\sigma = +$ and the last early collision in $(0, \tau)$ occurs at $\theta = -\pi/2$ at time $t = 2\pi p$. So $0 < p \leqslant k$ is an integer. Then in the time interval $[0, t]$ the eccentric anomaly increases by $2\pi q$, where $0 < q \leqslant m$ is an integer. Hence $\Omega = q/p$. If we replace the initial time $t = 0$ with $t = 2\pi p$ (and shift time accordingly), then we obtain a collision arc $\tilde{\gamma}\colon [0, \tau - 2\pi p] \to \mathbb{R}^3$, $\tilde{\gamma}(t) = \gamma(t - 2\pi p)$. It has the same start and end points and frequency Ω, and no early collisions. It corresponds to the pair of integers $\tilde{k} = k - p$, $\tilde{m} = m - q$. Alternatively, suppose that $\sigma = +$ and the first early collision in $(0, \tau)$ occurs at $\theta = \pi/2$, at time $t = \tau - 2\pi p$. Then, similarly, $\Omega = q/p$, and we can replace γ by the collision arc $\tilde{\gamma}\colon [0, \tau - 2\pi p] \to \mathbb{R}^3$, $\tilde{\gamma}(t) = \gamma(t)$, corresponding to $\tilde{k} = k - p$, $\tilde{m} = m - q$. Similar observations work for $\sigma = -$.

We obtain

Lemma 4.1 *For $\sigma = \pm$, there exists a dense subset $\Lambda_\sigma \subset (0, 1)$ such that:*

- *For any $\Omega \in \Lambda_\sigma$, $\iota \in (0, \pi)$, $\lambda = \pm$ and starting point u on the horizontal unit circle, there exists a collision arc $\gamma = \gamma(\Omega, \sigma, \lambda)\colon [0, \tau] \to \mathbb{R}^3$ with frequency Ω, inclination ι, $\gamma(0) = u$, and $\gamma(\tau) = -u$.*
- *For $\sigma = +$, at $t = 0$ the orbit moves towards the perihelion and at $t = \tau$ away from it; for $\sigma = -$, at $t = 0$ the orbit moves away from the perihelion and at $t = \tau$ towards it.*
- *For $\lambda = +$ the perihelion is above the horizontal plane; for $\lambda = -$ it is below.*

The Jacobi constant of the arc γ is

$$C = \Omega^{2/3} + 2 \cos \iota.$$

If we fix Jacobi's constant $C \in (-2, +3)$, and Ω in the allowed set A_C of Section 3, then the inclination ι is determined by this equation.

Hence for given $C \in (-2, +3)$ and σ, collision arcs from a given starting point to its opposite point are determined by $\Omega \in \Lambda_\sigma \cap A_C$ and $\lambda = \pm$.

4.4 Nondegeneracy

Given a nonplanar collision arc γ, we evaluate for all nearby trajectories from the same initial point $\mathbf{p}$ the distance D from the origin to the point where it repierces the horizontal plane, the time τ taken to this point, and the Jacobi constant C. By the second criterion of Section 2, the collision arc is nondegenerate if the derivative of (D, τ, C) with respect to initial velocity $\mathbf{v}$ is invertible.

As in Bolotin and MacKay (2000),[2] we can replace initial velocity $\mathbf{v}$ by position $\mathbf{F} \in \mathbb{R}^3$ of the second focus. Indeed, for fixed $\mathbf{p}$ the parameters of the elliptic orbit are smooth functions of $\mathbf{v}$. In particular this holds for $\mathbf{F} = -2a\mathbf{L}$, where $\mathbf{L} = \mathbf{v} \times \mathbf{G} - \mathbf{p}$ is the Laplace vector. Conversely, the parameters of the ellipse passing through $\mathbf{p}$ are smooth functions of $\mathbf{F}$. In particular this holds for $\mathbf{L} = e\mathbf{i}$ and $\mathbf{G} = \sqrt{a(1 - e^2)}\mathbf{n}$. Thus $\mathbf{v} = G^{-2}\mathbf{G} \times (\mathbf{L} + \mathbf{p})$ is a smooth function of $\mathbf{F}$.

Moving $\mathbf{F}$ on a circle around O perpendicular to Π makes no change to D or τ but changes C at nonzero rate, because it changes ι at nonzero rate and does not change a, and $C = a^{-1} + 2 \cos \iota$ and $\iota \neq 0, \pi$. Moving $\mathbf{F}$ radially within Π makes no change to D but changes τ at nonzero rate: this is equivalent to the nondegeneracy of the solutions η of the equation for a collision arc, proved in Section 4.2. Moving $\mathbf{F}$ parallel to the intersection of Π with the horizontal plane changes D at nonzero rate, because

[2] We take the opportunity to correct a typographical error in the proof of Lemma 3.2 there: the three references to Equation (1.6) should refer to (1.8).

$$D = \frac{1 + e \cos \alpha}{1 - e \cos \alpha},$$

where α is the angle between the vectors from O to $\mathbf{F}$ and the initial point (measured anticlockwise), so

$$\frac{\partial D}{\partial \alpha} = \frac{-2e \sin \alpha + 2 \cos \alpha \partial e/\partial \alpha}{(1 - e \cos \alpha)^2} = \mp 2e \neq 0$$

at $\alpha = \pm\pi/2$ (since $\cos \alpha = 0$ there was no need to evaluate the derivative of e with respect to α).

Thus the derivative of (D, τ, C) with respect to $\mathbf{F}$ is triangular with nonzero diagonal entries, so invertible.

4.5 Direction change

We make collision chains by connecting collision arcs with the same Jacobi constant, but we must be sure that they satisfy the "direction change" condition. This requires the velocity in the rotating frame just after each collision to be neither parallel nor opposite to the velocity just before the collision.[3]

Suppose the chain contains consecutive collision arcs $\gamma\colon [0, \tau] \to \mathbb{R}^3$, $\gamma'\colon [0, \tau'] \to \mathbb{R}^3$ with given C. Suppose they correspond to σ, Ω, λ and $\sigma', \Omega', \lambda'$. We want to rule out the possibility that the relative (to Jupiter) velocities v of γ at $t = \tau$ and v' of γ' at $t = 0$ satisfy $v' = \pm v$.

Let us represent the relative collision velocity v in the cylindrical coordinates z, ρ, ϕ in the rotating frame:

$$v = v_z \mathbf{e}_z + (v_\phi - 1)\mathbf{e}_\phi + v_\rho \mathbf{e}_\rho.$$

We have $v_\phi = G_z = \cos \iota$. Thus, if the direction change condition does not hold, then

$$\cos \iota = \cos \iota' \quad \text{or} \quad \cos \iota + \cos \iota' = 2.$$

For a nonplanar orbit, the second case is impossible. In the first case, conservation of $C = \frac{1}{a} + 2 \cos \iota$ implies that $\Omega = \Omega'$. Hence if $\Omega \neq \Omega'$, the changing direction condition holds automatically.

Now suppose that $\Omega = \Omega'$. Then $\eta = \eta'$, $\cos \iota = \cos \iota'$, and $v_\phi = v'_\phi$. So there are two choices: to continue on the same ellipse, or to change to the one with the opposite sign of λ. In the first case, the direction change condition fails, but in the second it is always satisfied because the orbits are nonplanar. So by choosing to switch sign of λ we satisfy the changing direction condition.

Now for given C and any sequence Ω_n, σ_n, we can find a sequence λ_n so that the corresponding collision chain satisfies the changing direction condition.

This completes the proof of Theorem 3.1.

5 Comments

For small enough ratio ε of secondary to primary mass in the circular restricted three-body problem and any value of Jacobi constant in the range for which there exist nonplanar Kepler

[3] At the analogous point in Bolotin and MacKay (2000) we mistakenly studied the direction change in the inertial frame; this error was corrected in MacKay (2005).

ellipses crossing the unit circle twice, we have proved existence of arbitrarily large uniformly hyperbolic subshifts of finite type consisting of non-planar second species orbits. This result is a 3D analog of the planar result proved in Bolotin and MacKay (2000).

As in the planar case, the result remains true if the Sun is replaced by an extended mass distribution provided it is constant in the rotating frame, because the only effect is to make a small change to the potential W. Similarly, Jupiter can be replaced by a spherically symmetric mass distribution confined to a sphere of radius $c\varepsilon$, because it produces the same field as a point mass at its center and the constructed orbits avoid collision by at least $c\varepsilon$. In fact, one can also replace Jupiter by any nonspherically symmetric mass distribution provided it is constant in the rotating frame, contained within a radius less than $c\varepsilon$ about its center of mass, and the effect on the gravitational field of deviation from spherical symmetry has decayed to much less than $1/c^2\varepsilon$ at this radius. This allows Jupiter a significant oblateness (J_2 component) for example.

We have also proved in Appendix A a general result which implies that in both the planar and nonplanar cases, the resulting second species orbits are highly unstable, with Lyapunov exponents of order $\log \varepsilon^{-1}$. This strong instability implies strong controllability, a key fact long recognized by the designers of solar system exploration missions using flyby.

Probably we could also make subshifts of finite type using some parabolic and hyperbolic Kepler arcs in addition to the elliptical ones used here.

We could probably make subshifts using infinitely many collision arcs (also in the planar case), by restricting to sequences for which the direction change is bounded away from 0 and π, but this would need more careful control on the nondegeneracy, and uniform hyperbolicity for the flow (though perhaps not the map) would be lost because the durations of the collision arcs would be unbounded. Probably, we could make unbounded orbits too, for any $C \in (-2, +2)$, by taking a sequence with a_n going to infinity [and the analogous result for any $C \in (-2\sqrt{2}, +2\sqrt{2})$ for the planar case]. However, it easier to obtain unbounded orbits by switching from an ellipse to a hyperbola via just one near collision (as e.g. in Alexeev 1970). There are many other mechanisms for the existence of unbounded orbits (Xia 1994).

An interesting question is whether one could construct subshifts based on sequences of both planar and nonplanar collision arcs. Their existence does not follow from our analysis because although the planar arcs used in Bolotin and MacKay (2000) are non-degenerate with respect to variations in the plane, they are degenerate with respect to 3D variations.

Presumably we could extend the result of Bolotin and MacKay (2000) to make planar subshifts using class 2 arcs (in the terminology of Section 4.1 here), like Marco and Niederman's orbit. Presumably we could also combine them with the class 1 arcs used in Bolotin and MacKay (2000), to make even bigger planar subshifts.

We recall from Bolotin and MacKay (2000), however, a problem with using class 3 arcs, namely that they are all degenerate. So more delicate analysis would be required to make trajectories to shadow sequences of them. Existence is not impossible, but might require an analog of the method of Bolotin (2006).

Acknowledgements Our collaboration was partly funded by an INTAS grant. SB was supported by the NSF grant # 0300319 and RFBR grant # 050101119. RSM thanks IMPA (Rio de Janeiro) and the Fields Institute (Toronto) for hospitality during the writing.

Appendix A: Hyperbolicity

In this appendix, we prove Theorem 2.2.

For each singularity $p_\alpha \in P$ let Σ_α be a small sphere (with respect to the metric T) around p_α. The proof of Theorem 2.1 of this paper in Bolotin and MacKay (2000) involved showing uniform nondegeneracy of the critical points for the variational problem for the sequence of points at which orbits cross the spheres Σ_α for small enough ε. This in turn implies hyperbolicity of the shadowing orbit.

Let us recall some notations from the proof of Theorem 2.1 in Bolotin and MacKay (2000). Let $(\gamma_{k_i})_{i\in\mathbb{Z}}$ be a collision chain. The collision arcs γ_k joining the points p_{α_k} and p_{β_k} cross the spheres Σ_{α_k} and Σ_{β_k} at the points u_k^0 and v_k^0, respectively. Then shadowing orbits of system (L_ε) were obtained as critical points of a formal functional

$$F_\varepsilon(u, v) = \sum_{i\in\mathbb{Z}} f_{k_i k_{i+1}}(u_i, v_i, u_{i+1}, \varepsilon),$$

where for each $(k, k') \in \Gamma$ with $\beta_k = \alpha_{k'} = \alpha$,

$$f_{kk'}(u, v, u', \varepsilon) = g_k(u, v, \varepsilon) + \varepsilon s_\alpha(v, u', \varepsilon), \qquad u \in A_k,\ v \in B_k,\ u' \in A_{k'}.$$

Here, A_k, B_k are neighborhoods of u_k^0 and v_k^0 in Σ_{α_k} and Σ_{β_k}, respectively, g_k is the action for given[4] ε from u to v near γ_k plus the actions for $\varepsilon = 0$ from p_{α_k} to u and v to p_{β_k}, and εs_α is the action for given ε from v to u' near p_α minus the actions for $\varepsilon = 0$ from v to p and from p to u'. The function g_k is C^2 on $A_k \times B_k \times [0, \varepsilon_0)$, and for $\varepsilon = 0$, it has a nondegenerate critical point (u_k^0, v_k^0). The function s_α is C^2 in a set in Σ_α^2 containing $B_k \times A_{k'}$. Thus for small $\varepsilon \in (0, \varepsilon_0]$ the functional F_ε has a nondegenerate critical point near (u^0, v^0), which gives the shadowing orbit.

We will show that the resulting orbit is uniformly hyperbolic. To do this, we will reduce F_ε to a functional Φ_ε of the sequence $u = (u_i)_{i\in\mathbb{Z}}$ only,

$$\Phi_\varepsilon(u) = \sum_{i\in\mathbb{Z}} S_{k_i,k_{i+1}}(u_i, u_{i+1}, \varepsilon), \qquad \Phi_0(u) = \sum_{i\in\mathbb{Z}} \phi_{k_i}(u_i),$$

eliminating v by stationarity, where $S_{kk'}$ is defined on $A_k \times A_{k'}$. Then we apply a result of Aubry et al. (1992), where it was proved that if the $S_{kk'}$ have nondegenerate mixed second derivative then nondegeneracy of a stationary sequence u for Φ_ε is equivalent to uniform hyperbolicity of the corresponding orbit of the associated symplectic twist map. Actually, the proof was written for the case that all the $S_{kk'}$ are the same function, but it goes through without change if the $S_{kk'}$ have uniformly nondegenerate mixed second derivative, as in the present case where there are only finitely many of them (and the same number of associated symplectic twist maps). In fact we can replace $S_{kk'}$ with a single function defined on a disjoint union of $A_k \times A_{k'}$.

First we perform the reduction to Φ_ε, next we state a result about a mixed second derivative, then we use this to deduce the uniform hyperbolicity from Aubry et al. (1992), plus bounds on the Lyapunov exponents, and finally we prove the claimed result about the mixed second derivative.

An alternative strategy would have been to write F_ε without adding and subtracting the extra terms, check the nondegeneracy of the mixed second derivatives, and extend the proof

[4] The dependence of g_k on ε was not made explicit in Bolotin and MacKay (2000), but the proofs need virtually no change.

of Proposition 1 of Aubry et al. (1992) to the case of block tridiagonal matrices with weak coupling between only alternate pairs of blocks.

Without loss of generality we assume that

$$\det D_v^2 g_k(u_k^0, v_k^0, 0) \neq 0.$$

Indeed, if this is not true, then u_k^0 is conjugate to p_{β_k} along γ_k. Since conjugate points are isolated, it is enough to change the radii of the spheres Σ_α a little. Then for small $\varepsilon > 0$ we can locally solve the equation

$$D_v f_{kk'}(u, v, u', \varepsilon) = 0, \quad u \in A_k, \qquad v \in B_k, \qquad u' \in A_{k'}$$

for

$$v = w_{kk'}(u, u', \varepsilon) = w_k(u) + O(\varepsilon).$$

Then

$$f_{kk'}(u, v, u', \varepsilon) = S_{kk'}(u, u', \varepsilon) = \phi_k(u) + \varepsilon\psi_{kk'}(u, u') + O(\varepsilon^2),$$

where

$$\phi_k(u) = g_k(u, w_k(u), 0), \qquad \psi_{kk'}(u, u') = s_\alpha(w_k(u), u', 0).$$

The function $S_{kk'}$ is well defined in a small neighborhood $A_k \times A_{k'}$ of $(u_k^0, u_{k'}^0)$. Stationary sequences (u, v) of F_ε correspond to stationary sequences u of Φ_ε. For small ε, Φ_ε has a nondegenerate critical point near u^0.

Since $D_2 g_k(u, w_k(u), 0) \equiv 0$, u_k^0 is a nondegenerate critical point of ϕ_k. We will show later that

$$\det D_{vu'}^2 s_\alpha(v, u', 0) \neq 0 \quad \text{in} \quad Y_\alpha. \tag{5.15}$$

It follows that $D_{uu'}^2 S_{kk'}(u, u', \varepsilon)$ is nondegenerate for small $\varepsilon \in (0, \varepsilon_0]$, and:

$$\|(D_{uu'}^2 S_{kk'}(u, u', \varepsilon))^{-1}\| \leqslant C\varepsilon^{-1} \quad \text{in} \quad W_{kk'}. \tag{5.16}$$

Hence $S_{kk'}$ is a generating function of a symplectic map $T_{kk'}\colon N_k \to N_{k'}$. The cross-section N_k is an open set in T^*A_k, which can be identified with an open set in $T_{A_k}Q \cap \{H_\varepsilon = E\}$ via the Legendre transform. In fact the twist condition (5.16) is equivalent to $\|DT_{kk'}\| \leqslant c\varepsilon^{-1}$ uniformly in N_k.

Critical points of Φ_ε correspond to orbits of a sequence of symplectic twist maps $T_{k_i k_{i+1}}$ with the generating functions $S_{k_i k_{i+1}}(u_i, u_{i+1}, \varepsilon)$. As proved by Aubry et al. (1992), hyperbolicity of an orbit of this sequence is equivalent to nondegeneracy of the critical point of Φ_ε.

Moreover the Lyapunov exponents of the corresponding Poincaré map are of order $\log \varepsilon^{-1}$. An upper bound of this order comes from $\|DT_{kk'}\| \leqslant c\varepsilon^{-1}$. A lower bound of this order comes from the proof of Proposition 1 of Aubry et al. (1992).

Thus to complete the proof of Theorem 2.2 it is enough to prove (5.15). We recall how the function s_α was defined in Lemma 4.1 of Bolotin and MacKay (2000). For any $a, b \in \Sigma_\alpha$ let $v_+(a)$ be the collision velocity of the collision arc γ_a^+ of system (L_0) joining a with p. Similarly, let $v_-(b)$ be the collision velocity of the collision arc γ_b^- of system (L_0) joining p with b. Both have energy E:

$$\|v_+(a)\| = \|v_-(b)\| = \sqrt{2(E - W(p_\alpha))}.$$

Fix small $\delta > 0$ and let

$$Y_\alpha = \{(a, b) \in \Sigma_\alpha^2 : \|v_+(a) \times v_-(b)\| \geqslant \delta\}.$$

The cross-product is taken with respect to the Riemannian metric T.

It was proved in Bolotin and MacKay (2000) that for any $\varepsilon \in (0, \varepsilon_0)$ and any $(a, b) \in Y_\alpha$ there exists a unique orbit γ_{ab}^ε: $[0, \tau] \to U_\alpha$ of system (L_ε) of energy E joining a, b. Its Maupertuis action has the form

$$J_E(\gamma_{ab}^\varepsilon) = S_\alpha(a, b, \varepsilon) = S_+(a) + S_-(b) + \varepsilon s_\alpha(a, b, \varepsilon) - c_\alpha \varepsilon \log \varepsilon,$$

where $c_\alpha = f_\alpha(p_\alpha)$ in (2.3) and $S_+(a)$ and $S_-(b)$ are Maupertuis actions of the collision orbits γ_a^+ and γ_b^-. This defines a function s_α on $Y_\alpha \times (0, \varepsilon_0)$. One can show s_α can be C^2 extended to $\varepsilon = 0$ and

$$s_\alpha(a, b, 0) = c_\alpha \log \|v_+(a) \times v_-(b)\|. \tag{5.17}$$

Computing the derivative and using that $Dv_+(a)\delta a \perp v_+(a)$, $Dv_-(b)\delta b \perp v_-(b)$, we obtain

$$D_{ab}^2 s_\alpha(a, b, 0)(\delta a, \delta b) = -\frac{c_\alpha \langle Dv_+(a)\,\delta a, Dv_-(b)\,\delta b\rangle}{\|v_+(a) \times v_-(b)\|}.$$

Hence $D_{ab}^2 s_\alpha(a, b, 0)$ is nondegenerate as a bilinear form on $T_a\Sigma_\alpha \times T_b\Sigma_\alpha$. Indeed, the maps $Dv_+(a)$: $T_a\Sigma_\alpha \to T_{p_\alpha}Q$ and $Dv_-(b)$: $T_b\Sigma_\alpha \to T_{p_\alpha}Q$ have rank 2.

Equation (5.17) was not proved in Bolotin and MacKay (2000), although all the ingredients were there. However, we can also verify nondegeneracy of s_α without performing the computation. Indeed, the twist condition for the generating function S_α means that the corresponding Poincaré map P_ε is well defined. Take some $(a, b) \in Y_\alpha$ and the corresponding connecting orbit $\gamma = \gamma_{a,b}^\varepsilon$: $[0, \tau] \to U_\alpha$. Let $v = \dot\gamma(0)$, $w = \dot\gamma(\tau)$. Since γ crosses Σ_α transversely at b, by the implicit function theorem (b, w) is locally a C^2 function of (a, v). The map $(a, v) \to (b, w)$ is the Poincaré map P_ε. It is well defined and C^2 in the set

$$N = \{(a, v) : v = \dot\gamma(0),\ (a, b) \in Y_\alpha\} \subset \{H_\varepsilon = E\} \cap T_{\Sigma_\alpha}Q.$$

Uniform twist condition for the function $S_\alpha(a, b, \varepsilon)$ on Y_α is equivalent to boundedness of the derivative of P_ε:

$$\det D_{ab}^2 s_\alpha(a, b, \varepsilon) \geqslant c > 0 \Leftrightarrow \|DP_\varepsilon(a, v)\| \leqslant C\varepsilon^{-1}.$$

To estimate DP_ε we recall that Lemma 4.1 was proved in Bolotin and MacKay (2000) by KS regularization. Let h: $\mathbb{R}^4 \to \mathbb{R}^3$ be the quadratic Hopf map such that $h(0) = p_\alpha$. There exists a C^{3+} Hamiltonian

$$\mathcal{H}(x, y) = \frac{1}{2}(|y|^2 - |x|^2) + O_4(x, y)$$

in a neighborhood of 0 in $\mathbb{R}^8$, invariant under the transformation group $(x, y) \to (e^{tJ}x, e^{-tJ}y)$, such that if $(x(t), y(t))$ is an orbit of the regularized system $(\mathcal{H})$ on the level set

$$Z_\varepsilon = \{(x, y) : \langle Jx, y\rangle = 0,\ \mathcal{H}(x, y) = \varepsilon\}$$

of the first integrals of $(\mathcal{H})$, then $h(x(t))$ is an orbit of system (L_ε) with $H_\varepsilon = E$.

The Poincaré map P_ε of system (L_ε) is a quotient of the Poincaré map of system $(\mathcal{H})$ in Z_ε. Orbits of system (L_ε) connecting $(a, b) \in Y_\alpha$ correspond to orbits of system $(\mathcal{H})$ in Z_ε connecting a pair of points in $X = h^{-1}(\Sigma_\alpha)$. It is shown in Bolotin and MacKay (2000) that connecting orbits have time intervals of order log ε^{-1}. Since the regularized system $(\mathcal{H})$ is nonsingular, for τ of order $\log \varepsilon^{-1}$ the time-τ map of system $(\mathcal{H})$ is uniformly C^2-bounded by $C\varepsilon^{-1}$. Corresponding solutions cross X transversely with speed bounded away from zero independent of ε. By the implicit function theorem, the Poincaré map is uniformly C^1 bounded by $C\varepsilon^{-1}$. This proves nondegeneracy of the mixed second derivative of s_α.

The proof of Theorem 2.2 is finished.

References

Alexeyev, V.M.: Sur l'allure finale du mouvement dans le problème des trois corps. Actes du Congrès Int. Math. **2**, 893–907 (1970)

Arnold, V.I., Kozlov, V.V., Neishtadt, A.I.: Mathematical Aspects of Classical and Celestial Mechanics, Encyclopedia of Mathematical Sciences, vol. 3, Springer-Verlag, Berlin (1989)

Aubry, S., MacKay, R.S., Baesens, C.: Equivalence of uniform hyperbolicity for symplectic twist maps and phonon gap for Frenkel–Kontorova models. Physica D **56**, 123–134 (1992)

Bolotin, S.: Second species periodic orbits of the elliptic 3-body problem. Celest. Mech. Dyn. Astron. **93**, 345–373 (2005)

Bolotin, S.: Shadowing chains of collision orbits. Discr. Conts. Dyn. Syst. **14**, 235–260 (2006)

Bolotin, S.V., MacKay, R.S.: Periodic and chaotic trajectories of the second species for the n-centre problem. Celest. Mech. Dyn. Astron. **77**, 49–75 (2000)

Bruno, A.D.: On periodic flybys to the Moon. Celest. Mech. **24**, 255–268 (1981)

Font, J., Nunes, A., Simo, C.: Consecutive quasi-collisions in the planar circular RTBP. Nonlinearity **15**, 115–142 (2002)

Gomez, G., Olle, M.: Second species solutions in the circular and elliptic restricted three body problem, I and II. Celest. Mech. Dyn. Astron. **52**, 107–146, 147–166 (1991)

Hénon, M.R.: Generating Families in the Restricted Three Body Problem, Lect. Notes in Phys. Monographs, **52**, Springer, Berlin (1997)

Henrard, J.: On Poincaré second species solutions. Celest. Mech. **21**, 83–97 (1980)

Levy, P.: Editorial notes. In: Poincaré, H Oeuvres Tome VII, p. 629, Gauthiers-Villars, Paris (1952)

MacKay, R.S.: Chaos in three physical systems. In: Dumortier, F., Broer, H., Mawhin, J., Vanderbauwhede, A., Verduyn Lunel S. (eds.) Equadiff 2003, pp. 59–72, World Science Press, Singapore (2005)

Marco, J.-P., Niederman, L.: Sur la construction des solutions de seconde espèce dans le problème plan restreint des trois corps, Ann. Inst. H. Poincaré Phys. Théor. **62**, 211–249 (1995)

Perko, L.M.: Second species solutions with an $O(\mu^\nu)$, $0 < \nu < 1$ near-Moon passage. Celest. Mech. **24**, 155–171 (1981)

Poincaré, H.: Les Méthodes Nouvelles de la Mécanique Céleste, Tome III, Gauthiers-Villars, Paris (1899)

Xia, Z.: Arnold diffusion and oscillatory solutions in the planar three-body problem. J. Diff. Eq. **110**, 289–321 (1994)