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(This paper is for the Special Issue edited by Prof. Gregoire Nicolis, Prof. Marko Robnik, Dr. Vassilis Rothos and Dr. Haris Skokos) MATHEMATICAL EXAMPLES OF SPACE-TIME PHASES

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The space-time phases of a complex dynamic system are the probability distributions for state as a function of space and time which arise by evolving initial probability distributions from the distant past. Variants of Toom's majority voter probabilistic cellular automaton are presented which are proved here to exhibit a variety of types of space-time phase. These examples are expected to serve as useful steps on the way to a general theory of space-time phases.

Keywords: space-time phase; emergence; complex system; probabilistic cellular automaton.

1. Introduction

The concept of "emergence", promulgated by J.S.Mill [Mill, 1843], has experienced a resurgence of interest under the banner of "complexity science". As proposed in [MacKay, 2005, 2008], a way to put "emergence" in complex dynamic systems onto a firm foundation is to use the notion of "space-time phase". For deterministic or stochastic dynamics of networks of units, which are considered to form "space", the *space-time phases* are the limit points[1] of probability distributions for state as a function of space and time which can occur by evolving suitable[2] initial probability distributions for state as a function of space from initial time going to minus infinity.

In its weakest sense, we say a space-time phase exhibits *emergence* if it is not a product of probability distributions for its units, i.e. the probability distributions for the histories of its units are not independent. The *amount of emergence* is the distance from the space-time phase to the set of product probability distributions over units, using a metric on spaces of multivariate probability distributions [MacKay, 2007, 2011], that is recalled in Section 4 and we henceforth call "Dobrushin metric".

[1]Let us say with respect to the topology of weak convergence with respect to continuous functions which are independent of the state outside a bounded subset of space-time.

[2]It is probably premature to make this more precise, because for example in the deterministic case one needs at least to require the initial probability distributions to have absolutely continuous marginals for finite subsets of space and some good behaviour of their densities; but in the case of probabilistic cellular automata, any Borel probability with respect to product topology suffices.

A system exhibits *strong emergence* if it has more than one space-time phase, yet is "indecomposable" in the sense of [Gielis & MacKay, 2000; MacKay, 2005], to be recalled in Section 2. The *amount of strong emergence* is the diameter of the set of space-time phases, using Dobrushin metric. The set of space-time phases is closed and convex, so we generally consider only the extremal phases: those which are not convex combinations of two others.

Here, a variety of examples of complex dynamic systems are constructed that we prove to have interesting space-time phases. They are probabilistic cellular automata (PCA), in particular, variants of Toom's majority voter PCA [Toom, 1980; Toom et al, 1990] and the proofs are simple corollaries of Toom's fundamental work.

The results are illustrated with snapshots from realisations of finite versions of the systems on a toroidal lattice. Although these finite versions have unique phase (being finite-state Markov processes with a single communicating component), the phases of the infinite system map into 'metastable" phases of very long lifetime for large finite versions. Also the illustrations are often for parameter values which are not rigorously known to be in the claimed regime, but chosen to make the phenomena more apparent.

2. Toom's majority voter probabilistic cellular automaton

Toom's majority voter PCA [Toom, 1980; Toom et al, 1990] is a discrete-time stochastic dynamic system consisting of a $\mathbb{Z}^2$ lattice of units each of which can be in one of two states, say $+$ and $-$ (or $+1$ and -1, or red and white). At each time step each site computes the majority state over its north and east neighbours and itself (called the "NEC neighbourhood" for north, east and centre), and with independent probability $1-\lambda$ it is updated to the majority state or with the remaining probability λ ("error rate") to the opposite state.

Although artificial, the advantage of this example is that a lot of mathematical results have been proved for it, in contrast to many other examples which numerically exhibit interesting space-time phases but whose rigorous status is unclear. To make progress, it is important to build on a firm foundation.

The main result is that for all λ near enough to $\frac{1}{2}$ ($\lambda \in (\frac{1}{2}, \frac{2}{3})$ suffices) there is a unique space-time phase, whereas for $\lambda > 0$ small enough ($\lambda < \frac{1}{2}48^{-8}$ suffices) there are at least two. Numerically there is a $\lambda_c \approx 0.09$ such that the above statements apply for all $\lambda \in [\lambda_c, 1-\lambda_c]$ and $\lambda \in (0, \lambda_c)$ respectively [Bennett & Grinstein, 1985], but the existence of such a λ_c is not proved.

First we elaborate on the regime of unique phase. For $\lambda = \frac{1}{2}$ the phase is just Bernoulli$(\frac{1}{2}, \frac{1}{2})$ for each site in space-time, because the majority state is irrelevant to the update rule, but for $\lambda \neq \frac{1}{2}$ it has non-trivial spatio-temporal correlations, because the update probability distribution depends on the state of the neighbourhood. See Figure 1. Numerically, this produces positive spatial correlations, though the best we have been able to prove so far is non-negative spatial correlations.

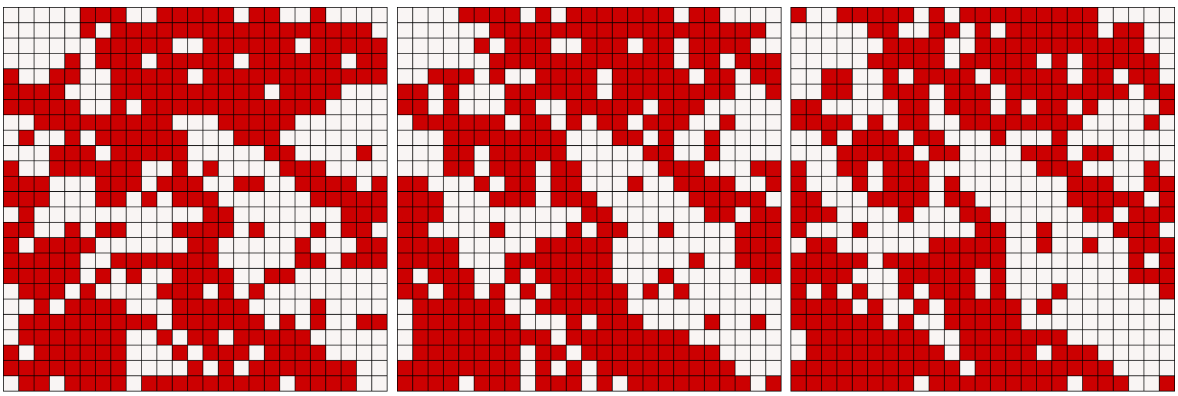

Fig. 1. Snapshots of a realisation of Toom's NEC voter PCA for $\lambda = 0.1$ at three successive times.

The uniqueness of space-time phase in this regime is a corollary of the uniqueness of stationary probability distribution and the fact that it attracts all initial probability distributions uniformly. So there is a unique space-time phase, given by time evolution from the unique stationary probability. Note that for $\lambda \in (\frac{1}{3}, \frac{2}{3})$, the future state of any unit is "weakly dependent" on the current state of the lattice, in the terminology of Dobrushin, so there is a unique stationary probability distribution and it attracts all initial probability distributions exponentially (see [MacKay, 2007, 2011] for a presentation). The exponential attraction is probably true for all λ in the interior of the set with unique stationary probability, but in any case to get from unique stationary probability to unique space-time phase requires only uniform convergence, which is assured by monotonicity of the PCA, a property to be recalled in Section 4.

Next we elaborate on the regime of non-unique phase. This is a consequence of Toom's result that for λ small enough there is a function $c(\lambda) < \frac{1}{2}$ and at least two stationary probability distributions, for one of which the probability of any given site being $+$ is $c(\lambda)$ and for the other it is $1 - c(\lambda)$. Each generates a space-time phase by time evolution. We call the phases "ferromagnetic" because of their resemblance to the phases of the 2D Ising model. See Figure 2.

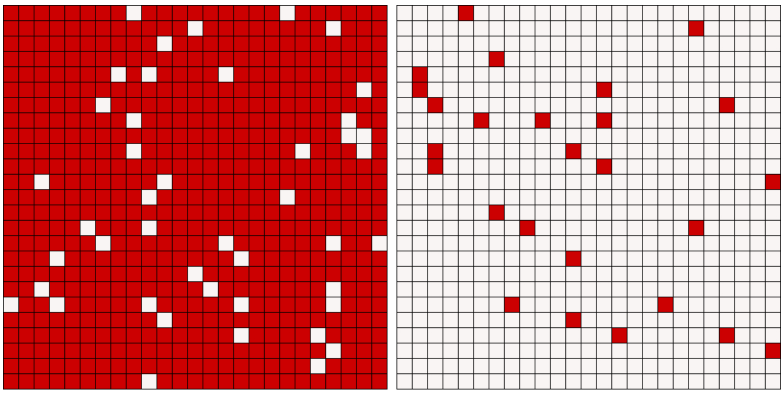

Fig. 2. Snapshots of the two ferromagnetic phases of Toom's PCA at $\lambda = 0.05$.

An open question is whether Toom's NEC PCA possesses any more space-time phases in the regime of non-uniqueness. It is an open question even whether it has more stationary probability distributions. One possibility for other space-time phases might be "interface phases" (either horizontal or vertical or diagonally from NW to SE and drifting with some speed, but these might require one more dimension), e.g. Figure 3.

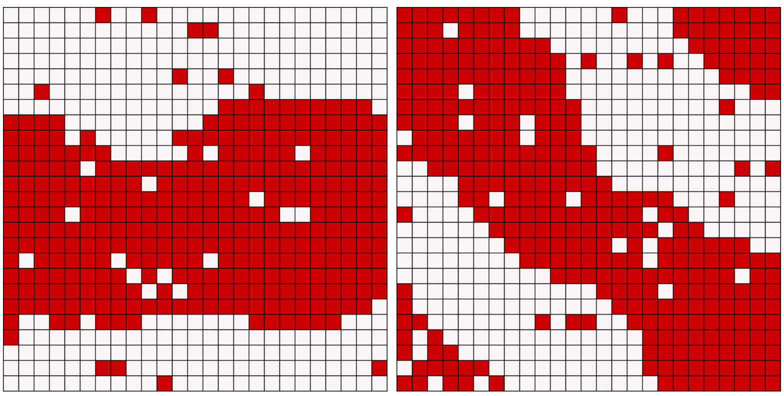

Fig. 3. Possible evidence for interface phases at $\lambda = 0.05$.

Toom's PCA is indecomposable in the sense of [Gielis & MacKay, 2000; MacKay, 2005], and thus provides a key example of strong emergence. A PCA is said to be *indecomposable* if there is a $D \in \mathbb{R}_+$ such that for any two subsets A, B of space-time separated by a distance of at least D and any two realisations x, y there is a realisation z agreeing with x on A and with y on B. The point of the definition is to exclude trivial systems with more than one phase, when the dynamics has more than one communicating component.

Glauber dynamics (or discrete-time analogues) of the 2D Ising model provides another example of strong emergence, but Toom's has the advantage of robustness. For example the symmetry of Toom's PCA between + and − can be broken yet the non-uniqueness of phase persists (this was proved by Toom and illustrated in [Bennett & Grinstein, 1985]), whereas breaking this symmetry in the Ising model by adding a magnetic field yields a unique phase.

The aim of this paper is to construct variants of Toom's PCA with various interesting space-time phases.

3. Previously proposed variants

First are recalled some variants that were already proposed in [Gielis & MacKay, 2000].

3.1. *Period-two phases*

By taking λ near 1, two space-time phases are obtained which are given simply by the ferromagnetic phases for $1-\lambda$ and interchanging + and − at odd times. Figure 4 illustrates this. It is the first proved example of which we are aware of the "nontrivial collective behaviour" of [Crutchfield & Kaneko, 1988], though perhaps [Grinstein et al, 1993] and some references cited there came close (and compare "asymptotic periodicity", e.g. [Losson & Mackey, 1996]).

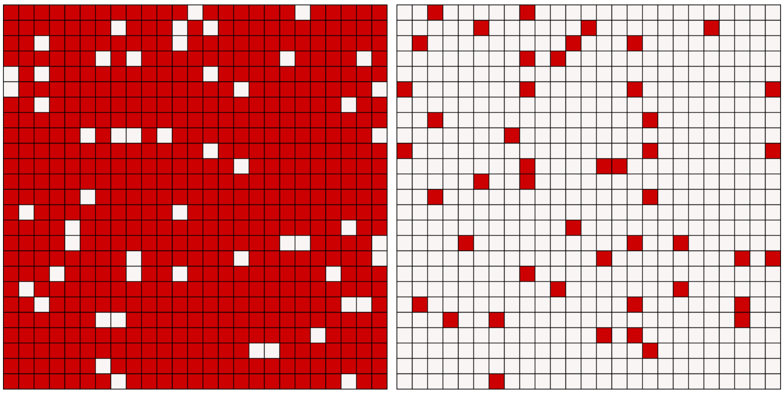

Fig. 4. Two successive snapshots from a period-2 phase of Toom's PCA, at $\lambda = 0.95$.

3.2. *Antiferromagnetic phases*

We make an antiferromagnetic variant of Toom's PCA by taking the majority of one's own vote and the negatives of those of the north and east neighbours with probability $1-\lambda$, λ small. Then one obtains antiferromagnetic phases from Toom's ferromagnetic ones, by interchanging + and − at all odd sites in space (meaning those for which the sum of the north and east components is odd). See Figure 5.

3.3. *Period-two antiferromagnetic phases*

Taking λ near 1 in the antiferromagnetic model produces phases that can be obtained from the original ferromagnetic phases by interchanging + and − at all odd sites in space-time. See Figure 6.

4. Variants with more than two phases

Next we show how to make variants with arbitrarily many phases, in particular at least 2^n for any positive integer n, following Theorem 3 of [Toom, 1980].

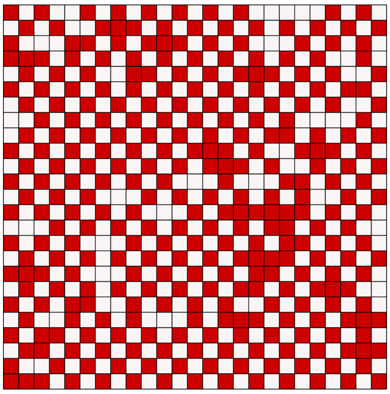

Fig. 5. A snapshot of an antiferromagnetic phase, with $\lambda = 0.05$.

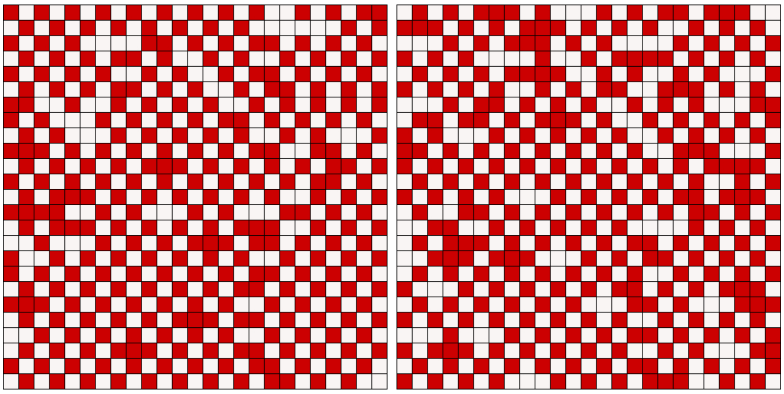

Fig. 6. Successive snapshots from a period-2 antiferromagnetic phase at $\lambda = 0.95$.

Replace the NEC neighbourhood by the vertices of any triangle in $\mathbb{Z}^2$. Then the triangle has area $\frac{n}{2}$ for some positive integer n. Space-time $\mathbb{Z}^3$ decomposes into n independent sublattices under the resulting PCA, on each of which the dynamics is equivalent to the NEC case. Thus for $\lambda < \lambda_c$ each sublattice has at least two phases, and so the whole lattice has at least 2^n phases. They look like moving wallpaper patterns, with errors. See Figure 7.

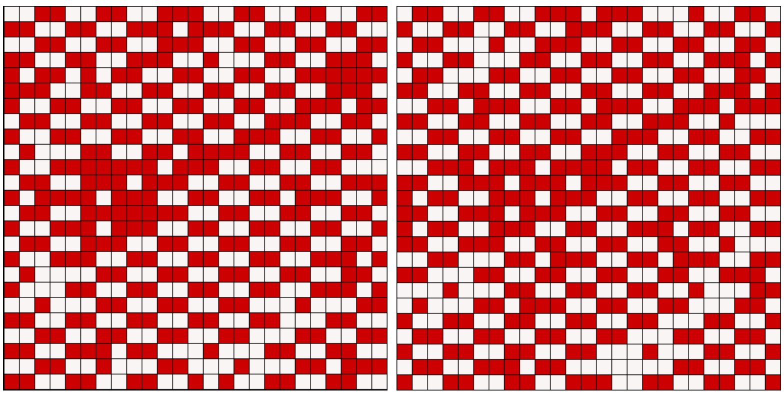

Fig. 7. Two successive snapshots of a moving wallpaper phase, using neighbourhood $\{E, NW, SW\}$ and $\lambda = 0.03$.

One might object that since the sublattices are independent, this does not qualify as strong emergence, though in fact the system is still indecomposable in the sense of [Gielis & MacKay, 2000; MacKay, 2005], because the notion refers to the state space, not to space-time. Nevertheless, it is a good idea to modify the example to make the sublattices dependent and so remove any possible objection. One can add any small dependence between the sublattices and still deduce there are at least 2^n phases. The key to this is a comparison argument using a partial order, which we now explain.

A *partial order* on a set X is a relation $\preceq$ such that for all $x, y, z \in X$ then (i) $x \preceq x$, (ii) $x \preceq y$ & $y \preceq x \implies x = y$, and (iii) $x \preceq y$ & $y \preceq z \implies x \preceq z$. A partial order on states $x = (x_s)_{s\in S}$ of a network S is often defined by choosing a partial order on the states of single units, say that generated by $- \prec +$ for a 2-state unit, and saying $x \preceq y$ if $x_s \preceq y_s$ for all $s \in S$. A real-valued function f of state is called *non-decreasing* if $x \preceq y$ implies $f(x) \le f(y)$. The *transition operator* P on real-valued functions of state is defined by $(Pf)(x) = \int f(x')p(dx'|x)$, where $p(.|x)$ is the transition probability distribution for the state x' at the next time given current state x. A PCA is *monotone* (some authors say "attractive") with respect to a partial order $\preceq$ on the states of the system if for all non-decreasing real-valued functions f of state, Pf is non-decreasing. For a probability distribution ρ and a real-valued function f, $\rho(f)$ denotes the mean of f with respect to ρ. The transition operator P induces an operator on probability distributions ρ, for which we use the same name P but acting on the right, by $(\rho P)(f) = \rho(Pf)$ for all functions f. The partial order on states induces a partial order on probability distributions by $\rho \preceq \sigma$ if $\rho(f) \le \sigma(f)$ for all non-decreasing functions f. So the PCA is monotone if and only if $\rho \preceq \sigma \implies \rho P \preceq \sigma P$.

Toom's proof works by showing that if one starts from δ_-, the probability distribution supported on all $-$, then the probability distribution $\delta_- P^t$ can not evolve very far for positive t and small λ: it is bounded above in the partial order by some probability distribution μ^- with

$$\mu^-(x_s = +) = c(\lambda) < \frac{1}{2} \text{ for all } s \in S.$$

Suppose the update rule is modified to make the error rate λ depend on the state of a larger neighbourhood, but bounded by some λ_1 to which Toom's proof applies. Then the modified rule is a perturbation of the Toom rule with parameter λ_1 for which if one starts from probability distribution δ_- then $\delta_- P^t$ remains bounded above by $\mu^-(\lambda_1)$ in the partial order. This is because the deviation from the rule with λ_1 is always towards more minuses (this style of proof can be formalised by the concept of "joining" of two stochastic processes, called "coupling" by many authors). The same argument applies to starting in the probability distribution δ_+ with all $+$. Hence, there are at least two stationary probability distributions.

Applied to the model with the generalised NEC neighbourhood above and any dependence of error rate on other sites bounded by λ_1, we obtain the 2^n phases claimed.

Incidentally, the statement that μ^- is not very far from δ_- can be quantified in Dobrushin metric:

$$D(\mu^-, \delta_-) = 2c(\lambda).$$

This is a useful example of explicit computation of the Dobrushin distance between two probability distributions, so we spell it out. First we recall the definition of the *Dobrushin metric* given in [MacKay, 2007, 2011]:

$$D(\rho, \rho') = \sup_{f \in F \setminus C} \frac{\rho(f) - \rho'(f)}{\|f\|},$$

where F denotes the set of (continuous in product topology) real-valued functions f of state with finite value of $\|f\| = \sum_{s\in S} \Delta_s(f)$, where $\Delta_s(f)$ is the Lipschitz constant of f with respect to changes in x_s, and C denotes the constant functions. First, take $f(x) = x_{00}$, the value of the state at site $(0,0)$, considered as ± 1. Then $\mu^-(f) - \delta_-(f) = 2c(\lambda)$ (the factor of 2 is because $+1 - (-1) = 2$), and the Dobrushin semi-norm $\|f\| = \sum_{s\in S} \Delta_s(f) = 1$ (taking the convention $d(+,-) = 2$), so $D(\mu^-, \delta_-) \ge 2c(\lambda)$. Next, to upper bound the Dobrushin distance it suffices to take all functions f which are independent of the state outside a finite subset of space and which are 0 on the all minus state. Denote the subset of dependency for f by I. Then $f(x)$ can be written as the sequential sum of changes in the value of f starting from the all minus state, on changing from $-$ to $+$ at sites where $x_s = +$, working in some enumeration of the sites of I. This is bounded above by $\sum_{s:x_s=+} 2\Delta_s(f)$. So

$$\mu^-(f) - \delta_-(f) = \sum_{x \text{ on } I} \mu^-(x) f(x) \le \sum_{x \text{ on } I} \mu^-(x) \sum_{s:x_s=+} 2\Delta_s(f)$$

$$= \sum_{s \in I} 2\Delta_s(f) \sum_{x \text{ on } I : x_s = +} \mu^-(x) = \sum_s 2\Delta_s(f) c(\lambda) \leq 2c(\lambda)\|f\|.$$

For comparison, the distance between δ_- and δ_+ is 2, thus for $c(\lambda)$ small, μ^- is much closer to δ_- than to δ_+.

5. Variants with more than two states per site

One can allow $n > 2$ states per site but to apply the Toom majority rule one needs to decide what to do if there is not a majority state in the NEC neighbourhood and which state to choose when voting against the majority. Choices that work are to give the centre site the deciding vote about what constitutes the majority and to divide the error probability equally between the non-majority states (but there is a lot of freedom for changes in the latter rule). Then one can prove existence of at least n phases by Toom's method as follows. Consider starting from all in one state, call it red. Then the probability of any site at any future time being not red is bounded by the same expression as for Toom's PCA, so we deduce that red dominates for all time. Figure 8 gives an illustration.

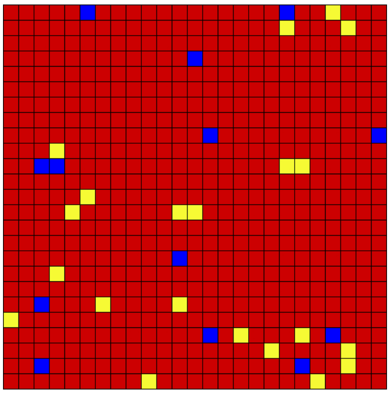

Fig. 8. Snapshot of a phase from a 3-state NEC voter PCA with $\lambda = 0.05$

[Menezes & Toom, 2006] studied variants of the majority voter PCA with more than 2 states per site, but their main point was to make an example with the eroder property and yet unique stationary probability for all positive error rate.

6. Variants with dependence on other neighbours

Toom's proof works with any neighbourhood for which the deterministic limit has the "eroder" property of [Toom, 1980; Toom et al, 1990]. It is proved that the eroder property fails for symmetric neighbourhoods [Pippenger, 1994], so it might seem that this restriction on neighbourhood is severe. Nevertheless, one can allow some dependence on arbitrary other neighbours, for example in the error rate and still keep non-unique phase. As in Section 4 any weak dependence on other neighbours works, simply by comparing with the Toom PCA at a different parameter value.

Finally, numerically one observes non-unique phase even for symmetric neighbourhoods and it would be nice to be able to prove this. One potential strategy is via the zero-temperature dynamics studied by [Spirin, Krapivsky & Redner, 2002].

7. Coupled map lattices

Using the trick of [Gielis & MacKay, 2000], all the above examples of PCA can be turned into examples of coupled map lattices exhibiting equivalent space-time phases. Thus all the phenomena occur also in deterministic systems.

8. Conclusion

Examples with a variety of types of space-time phase and multiplicities have been constructed.

Note that all the examples can be made inhomogeneous, e.g. the error rate λ can depend somewhat on site in space-time. The same phenomenology is obtained. What is not so clear is how much the $\mathbb{Z}^2$ lattice can be changed. The theory of unique phase for weakly dependent PCA applies to arbitrary networks, but Toom's results on non-unique phase require a certain amount of structure which might not be easy to generalise (though he has formulations in continuous space [Toom, 2002]).

It would be good to study the effects of control on space-time phases, and to make examples for other types of system, e.g. continuous-time and mobile units.

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