# Mathematics for Fusion Power part 1

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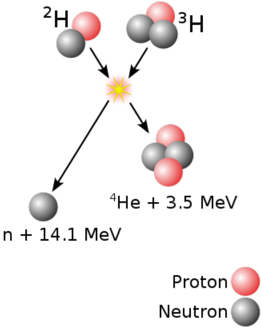
# Introduction: Preamble

- Ideal if you know exterior calculus and Hamiltonian dynamics
- But I'll summarise essentials
- Not necessary to know any physics

# DT fusion

- from wikipedia




2g $D$ + 3g $T$ per hour makes 470MW power.

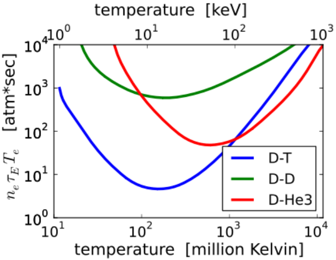
- $D$ from sea-water (1 in 5000 D:H). Breed $T$ from $Li$ blanket:

$$n + {}^6Li \rightarrow T + {}^4He + 4.8MeV, \quad n + {}^7Li + 2.5MeV \rightarrow T + {}^4He + n.$$

Get $Li$ from rocks or sea-water (95% $^7Li$).

# Sustained fusion

- Requires Lawson product $nT\tau_E \geq 3 \times 10^{21} keV\, s\, m^{-3}$ for $T \approx 14 keV (160 MK)$, where $n =$ electron number density, $T =$ ion temperature, $\tau_E =$ energy confinement time (for charged particles).



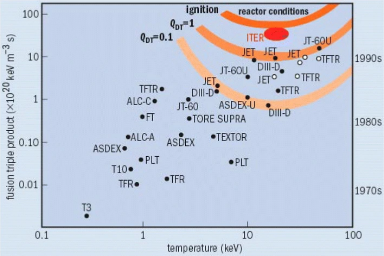

- Unavoidable energy loss by EM radiation, but slow if no high $Z$ impurities.
- Most important to confine the charged particles, and after that to reduce their transfer of kinetic energy.

# Confinement schemes

- Various ideas, e.g.

# Magnetic confinement

- At the envisaged conditions, gases are fully ionised: a plasma of ions (nuclei) & electrons.
- Aim to confine particles by magnetic field. Progress so far:




- $Q$ is ratio of power-out to power-in. Ignition is $Q = \infty$.
- This module will be about mathematics of confinement of charged particles by magnetic fields.
- Motion of charged particles creates magnetic fields, so in principle have to solve confinement self-consistently, but will largely restrict attention to given magnetic fields.

# Magnetic fields: Different faces

- Three ways to view a magnetic field:
  1. A volume-preserving 3D vector field $B$. Write $\Omega$ for volume-form, then $B$ volume-preserving is $L_B\Omega = 0$, i.e. $di_B\Omega = 0$. By Stokes' theorem, if $S$ is a closed surface bounding a volume then $\int_S i_B\Omega = 0$. Require this also for closed surfaces $S$ that do not bound a volume, e.g. boundaries of $T^2 \times I$.
  2. A closed 2-form $\beta = i_B\Omega$, which gives the magnetic flux $\int_S \beta$ through any surface $S$. Again, strengthen the definition from closed ($d\beta = 0$) to exact, i.e. $\beta = d\alpha$ for some 1-form $\alpha$. Given a Riemannian metric $g$, can define a vector field $A$ by $g(v, A) = i_v\alpha$ for all vectors $v$, called a vector potential for $B$. Can write $A = \alpha^\sharp$ or $\alpha = A^\flat$.
  3. A 1-form $B^\flat$, defined by $i_v B^\flat = g(v, B)$ for all vectors $v$. This view relates $B$ to electric current density $J = \mathrm{curl}\, B$ (in units with $\mu_0 = 1$): $i_J\Omega = dB^\flat$.
- In coordinates $x^i$, $B = \sum_i B^i \partial_{x_i}$, $\beta = \mathcal{J} \sum B^i dx^j \wedge dx^k$ over cyclic permutations of 123, where $\mathcal{J} = \Omega(\partial_{x^1}, \partial_{x^2}, \partial_{x^3})$, $B^\flat = \sum_i B_i dx^i$ and $B_i = g_{ij}B^j$ where $g_{ij} = g(\partial_{x^i}, \partial_{x^j})$. Components are called: $B^i$ *contravariant*, $B_i$ *covariant*, $B_{(i)} = B^i |\partial_{x^i}|$ *physical*.

# Crash course in exterior calculus

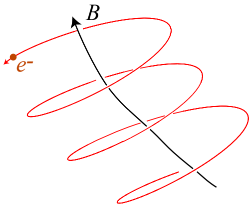
- Two views of a vector field on a smooth manifold:
  1. Field of tangent vectors $v$, representing velocities of parametrised curves. Induces a local flow $\phi$ by $\frac{d}{dt}\phi_t(x) = v(\phi_t(x))$.
  2. Linear operator $L_v$ on smooth functions $f$, satisfying Leibniz rule $L_v(fg) = (L_v f)g + fL_v g$. Relation $L_v f(x_0) = \frac{d}{dt} f(x(t))_{|t=0}$ for smooth curves $x$ with $x(0) = x_0$, $\dot{x}(0) = v(x_0)$.
- A differential $k$-form $\omega$ is an antisymmetric $k$-linear map from tangent space at each point to $\mathbb{R}$. It can be integrated over a smooth $k$-surface $S$ to give a scalar $\int_S \omega$.
- For vector field $X$ and $k$-form $\omega$, $i_X\omega$ is the $(k-1)$-form given by inserting $X$ as the first argument.
- For function $f$, derivative $Df$ is a 1-form $df$. Note $L_v f = i_v df$.
- For $k$-form $\omega$, $d\omega$ is $(k+1)$-form s.t. $\int_V d\omega = \int_{\partial V} \omega$ $\forall$ $(k+1)$-volumes $V$. As $\partial\partial V = \emptyset$, $d^2 = 0$.
- Pushforward $h_* u$ of a vector $u$ by a diffeo $h$ is the derivative of $h(x)$ as $x$ moves with velocity $u$, i.e. $h_* u = Dh\, u$. Pullback of a $k$-form $\omega$ is $h^*\omega(v_1, \dots v_k) = \omega(h_* v_1, \dots h_* v_k)$. Extend $L_v$ to $k$-forms $\omega$ by $\frac{d}{dt}\phi_t^*\omega_{|t=0}$. On forms, $L_v = i_v d + d i_v$.
- RS MacKay, Differential forms for plasma physics, J Plasma Phys 86 (2020) 925860101

# Some more

- A volume-form is a non-degenerate top-dimensional form $\Omega$. A form $\omega$ is *non-degenerate* if $i_v\omega = 0 \implies v = 0$.
- A Riemannian metric is a positive-definite symmetric covariant 2-tensor $g$. It induces a norm $|v| = \sqrt{g(v,v)}$ on vectors $v$, and on covectors $|\lambda| = \sqrt{g(\lambda^\sharp, \lambda^\sharp)}$. Also, for any function $f$, $g$ induces vector field $\nabla f = (df)^\sharp$.
- Say $\Omega$ is compatible with $g$ if $\Omega(v_1, .., v_n)^2 = \det[g(v_i, v_j)]$.
- For $k$-form $\alpha$ and $l$-form $\beta$, $\alpha \wedge \beta(v_1, \dots v_{k+l}) = \sum_{\pi \in Sh(k,l)} \varepsilon_\pi \alpha(v_{\pi(1)}..v_{\pi(k)})\beta(v_{\pi(k+1)}..v_{\pi(k+l)})$, where $Sh(k,l)$ (shuffles) is the set of permutations of $\{1, ..k+l\}$ such that $\pi(1) < .. < \pi(k)$ and $\pi(k+1) < .. < \pi(k+l)$.
- In 3D, cross-product $u \times v$ of vectors is defined by $(u \times v)^\flat = i_v i_u \Omega$. For compatible $\Omega$, $i_{u \times v}\Omega = u^\flat \wedge v^\flat$.
- Commutator $[u, v]$ of vector fields is defined by $L_{[u,v]} = L_u L_v - L_v L_u$. Equivalently, $[u, v] = L_u v = \frac{d}{dt}\phi_t^{u*} v_{|t=0}$.
- R.S.MacKay, Use of Stokes' theorem for plasma confinement, Phil Trans Roy Soc A 378 (2020) 20190519

# Charged particle motion

- Treat classically: Lorentz force $F = ev \times B$ on charge $e$ with velocity $v$. Momentum $p = mv^\flat$, Newton's law $\frac{dp}{dt} = F^\flat$.
- In constant field $B = |B|\hat{z}$:
  1. $v^z = \text{cst}$, $q^z(t) = q^z(0) + v^z t$.
  2. $m\dot{v}^x = ev^y|B|$, $m\dot{v}^y = -ev^x|B|$, so horizontal velocity rotates, $v(t) = R_{\Omega t}v(0)$, with "gyrofrequency" $\Omega = -e|B|/m$.
  3. Then position $q(t) = Q(t) + \rho(t)$, with $Q(t) = Q(0) + v^z\hat{z}t$ ("guiding centre"), $\rho(t) = R_{\Omega t}\rho(0)$, $\rho = \frac{v \times b}{\Omega}$ ("gyroradius vector"), where $b = \frac{B}{|B|}$.
- In general field, define $\rho = \frac{v \times b}{\Omega}$, $Q = q - \rho$, $v_\parallel = v \cdot b$ and seek evolution of $Q, \rho, v_\parallel$.



# Hamiltonian formulation

- In canonical coordinates $(q^i, p_i)$, $\dot{q}^i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q^i}$.
- For one particle in magnetic field, choose a vector potential $A$, and let $p = m\dot{q}^\flat + eA^\flat(q)$, $H(q,p) = \frac{1}{2m}|p - eA^\flat(q)|^2$.
- Better to use symplectic formulation. A *Hamiltonian system* is a vector field $X$ on a manifold $M$ such that $i_X\omega = dH$ for some function $H : M \to \mathbb{R}$ and symplectic form $\omega$ on $M$
- A *symplectic form* is a non-degenerate closed 2-form; implies $\dim M = 2n$ even, $n$ is called *number of degrees of freedom (DoF)*.

# Example

- $M = T^*Q$, the cotangent bundle of a manifold $Q$. Can write a cotangent as $(q, p)$ where $q \in Q$ and $p$ is a covector at $q$, i.e. a linear map $T_qQ \to \mathbb{R}$.
- Let $\pi : T^*Q \to Q$ be the natural map.
- $T^*Q$ has a natural 1-form $\alpha$ defined by $\alpha(v) = p(\pi_* v)$. So it has a natural symplectic form $\omega = -d\alpha$.
- In any local coordinate system $q^i$ for $Q$, can choose associated coordinates for $p$ so that $p(\dot{q}) = \sum_i p_i \dot{q}^i$. Then $\alpha = \sum_i p_i dq^i$ and $\omega = \sum_i dq^i \wedge dp_i$.
- A simple mechanical system on $T^*Q$ is defined by this $\omega$ and $H(q, p) = \frac{1}{2}|p|^2 + V(q)$ with respect to some Riemannian metric on $Q$ (which incorporates masses and moments of inertia). Produces $\nabla_{\dot{q}}\dot{q} = -\nabla V(q)$.

# Charged particle Hamiltonian

- For one particle of charge $e$, mass $m$, in $(Q^3, g)$ with magnetic flux form $\beta$, take $\omega = -d\alpha - e\pi^*\beta$ on $T^*Q$ and $H = \frac{1}{2m}|p|^2$.
- Equations of motion are given by solving $\omega((\dot{q}, \dot{p}), (\xi_q, \xi_p)) = \frac{1}{m} i_{p^\sharp}\xi_p$ for all $\xi \in T(T^*Q)$.
- In Euclidean case and Cartesian coordinates, this gives $\dot{q}^i = \frac{p_i}{m}$, so $p = m\dot{q}^\flat$, and $-\dot{p}\xi_q - e\beta(\dot{q}, \xi_q) = 0$ for all $\xi_q$, so using $\beta(\dot{q}, \xi_q) = \Omega(B, \dot{q}, \xi_q)$, we get $\dot{p} = e(\dot{q} \times B)^\flat$.

# Advantages of symplectic formulation

- $H$ is conserved along $X$: $i_X dH = i_X i_X \omega = 0$ by antisymmetry
- $\omega$ is conserved along $X$: $L_X \omega = i_X d\omega + d i_X \omega = 0 + d^2 H = 0$; hence Poincaré invariant $\int_D \omega$ is conserved for any disk $D$ flowing with $X$, which has many consequences (see later).
- and ...

# Continuous symmetry leads to conservation & reduction

- Say vector field $u$ on $M$ is a *continuous symmetry* of Hamiltonian system $(H, \omega)$ if $L_u H = 0, L_u \omega = 0$.
- **Theorem** [Noether]: If Hamiltonian system $(H, \omega)$ has a continuous symmetry $u$ then it conserves a local function $K$.
- **Proof**: $d\omega = 0$ so $di_u\omega = 0$, so $u$ is locally Hamiltonian, i.e. $i_u\omega = dK$ for some local function $K$ (Poincaré lemma). $i_X dK = i_X i_u \omega = -i_u dH = 0$, so $K$ is conserved by $X$. □
- Often $K$ is global, e.g. if $H_1(M)$ is spanned by closed trajectories $\gamma$ of the set of vector fields of the form $au + bX$ for functions $a, b$ (or by asymptotic cycles), because $\int_\gamma i_u\omega = \int i_u\omega(au + bX)\, dt = \int \omega(u, au) + \omega(u, bX)\, dt = \int 0 - b\, dH(u)\, dt = 0.$ So can reduce to level sets $K^{-1}(k)$.
- Also, $i_u dK = 0$, and $[u, X] = 0$ because $\omega$ non-degenerate and $i_{[X,u]}\omega = i_X L_u \omega - L_u i_X \omega = 0 - L_u dH = dL_u H = 0.$ So if orbit-space of flow $\phi^u$ on $K^{-1}(k)$ is a manifold then can quotient by $\phi^u$ to reduce the dynamics on $K^{-1}(k)$ by one more dimension.
- The resulting vector field is Hamiltonian with respect to the reductions of $\omega$ and $H$.

# Poincaré lemma

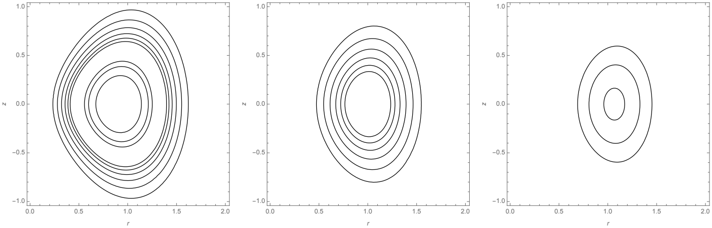
- **Theorem**: If a $k$-form ($k \geq 1$) $\beta$ is closed on a contractible open subset $U$ of a manifold then $\beta = d\alpha$ for some $(k-1)$-form $\alpha$ on $U$.
- **Proof**: $U$ contractible implies there is a vector field $X$ on $U$ with forward flow $\phi$ that maps $U$ into itself and $\phi_t U$ contracts to a point as $t \to \infty$. Define $\alpha = -\int_0^\infty i_X \phi_t^* \beta \, dt$. Then $d\alpha = -\int_0^\infty d i_X \phi_t^* \beta \, dt = -\int_0^\infty L_X \phi_t^* \beta - i_X d \phi_t^* \beta \, dt = -\int_0^\infty \partial_t \phi_t^* \beta \, dt + \int_0^\infty i_X \phi_t^* d\beta \, dt$. The second integral is 0 because $d\beta = 0$. The first is the change $\phi_0^* \beta - \phi_\infty^* \beta = \beta$. □
- Converse of Noether theorem: if $X_H$ conserves a function $K$ (or $i_{X_H} \alpha = 0$ for a closed 1-form $\alpha$), then $X_K$ is a cts symmetry of $(H, \omega)$.
- **Proof**: $X_K$ is defined by $i_{X_K} \omega = dK$ (or $\alpha$). So $L_{X_K} H = i_{X_K} dH = i_{X_K} i_{X_H} \omega = -i_{X_H} i_{X_K} \omega = -i_{X_H} dK = 0$. And $L_{X_K} \omega = d i_{X_K} \omega = ddK = 0$. □

# Example: Charged particle in axisymmetric field

- Let $\omega = -d\alpha - e\beta$, $H = \frac{1}{2m}|p|^2$, $D = \mathbb{R}^3 \setminus \{r = 0\}$ in cylindrical coordinates $(r, \phi, z)$, $u$ lift to $T^*D$ of $\partial_\phi = r\hat{\phi}$.
- Choose coordinates $(p_r, p_\phi, p_z)$ so that $\alpha = \sum_i p_i dq^i$. Then $L_u\alpha = 0$, and $H = \frac{1}{2m}(p_r^2 + r^{-2}p_\phi^2 + p_z^2)$ so $L_u H = 0$.
- Say $B$ is *axisymmetric* if $L_u\beta = 0$. Then $L_u\omega = 0$ so Noether gives a conserved quantity.
- First, $L_u\beta = 0 \implies di_u\beta = 0$, so $i_u\beta = d\psi$ for some local function $\psi$ on $D$. $D$ contains a closed orbit of $u$ so $\psi$ is global. $i_u d\psi = 0$ so $\psi$ independent of $\phi$. $\Omega(\partial_r, \partial_\phi, \partial_z) = r$ and $i_u i_B \Omega = d\psi$ imply $B^r = \frac{1}{r}\partial_z\psi, B^z = -\frac{1}{r}\partial_r\psi$. Magnetic flux $\int_S i_B\Omega$ through any annulus $S$ spanning two $u$-circles is $2\pi[\psi]$, so $\psi$ is called poloidal flux function.
- Note $\beta = r(B^\phi dz \wedge dr + B^r d\phi \wedge dz + B^z dr \wedge d\phi)$, so $d\beta = 0$ implies $B^\phi$ independent of $\phi$.
- Then $i_u\omega = dp_\phi - e\,d\psi = dL$ with $L = p_\phi - e\psi$. So $L$ is conserved. Reduced system: $H = \frac{1}{2m}(p_r^2 + p_z^2 + \frac{(L+e\psi)^2}{r^2})$, $\omega = dr \wedge dp_r + dz \wedge dp_z + erB^\phi dr \wedge dz$.
- Note $p_\phi = rp_{(\phi)}$, $rB^\phi = B_{(\phi)}$.

# continued

- If $\psi/r$ grows with $(r - r_0, z)$ then get confinement by $|L + e\psi| \leq \sqrt{2mH}\, r$. Starting principle of "tokamak".
- e.g. $\psi = (r^2 - 1)^2 + \frac{1}{2}(3 + r^2)z^2$; contours of $\frac{(L+e\psi)^2}{e^2 r^2} = 0.05, 0.1, 0.2, 0.4, 0.8, 1.6$ for $\frac{L}{e} = -0.6, 0, +0.6$:




- But requires current density $J_{(\phi)} = rJ^\phi = \frac{1}{r}\partial_z^2\psi + \partial_r(\frac{1}{r}\partial_r\psi) = \frac{3}{r} + 9r$ in the plasma, which needs driving and promotes instabilities.

# Guiding-centre motion

- Approximate symmetry of a Hamiltonian system leads to an "adiabatic invariant" and approximate reduced system.
- If $B(x(t))$ seen by the particle changes by a factor at most $\varepsilon$ small during one gyroperiod $T = \frac{2\pi}{\Omega}$ ($\Omega = -\frac{e|B|}{m}$) then have approximate symmetry by rotation of the gyroradius vector about guiding centre.
- Verification: (i) Define $Q, \rho, p_{||}$ from $q, p$ by $q = Q + \rho$, $\rho \cdot b = 0$, $p^\sharp = eB(Q) \times \rho + p_{||} b(Q)$. This can be solved for $Q, \rho, p_{||}$ if $B$ changes slowly in distance $|p||B|/e$. Choose slowly varying frame for $\rho, b$. Let $u = (0, 0, 0, -\rho_2, \rho_1, 0)$. Then $H = \frac{1}{2m}(p_{||}^2 + e^2|B(Q)|^2|\rho|^2)$ is (exactly) $u$-invariant.

# (ii) $L_u\omega$

- For $\omega = -d\alpha - e\beta$: write $B$ for $|B(Q)|$.
  $\alpha = \sum_i p_i dq^i = eB(\rho_2(dQ_1 + d\rho_1) - \rho_1(dQ_2 + d\rho_2)) + p_{\|} dQ_3$.
- So $d\alpha = eB(d\rho_2 \wedge dQ_1 - d\rho_1 \wedge dQ_2 - 2d\rho_1 \wedge d\rho_2) + dp_{\|} \wedge dQ_3 + edB \wedge (\rho_2(dQ_1 + d\rho_1) - \rho_1(dQ_2 + d\rho_2))$. Then
  $i_u d\alpha = eB(\rho_1 dQ_1 + \rho_2 dQ_2 + d|\rho|^2) + e|\rho|^2 dB$. So
  $L_u d\alpha = eB(d\rho_1 \wedge dQ_1 + d\rho_2 \wedge dQ_2) + edB \wedge (\rho_1 dQ_1 + \rho_2 dQ_2)$.
  Second term is $O(\varepsilon)$.
- Or compute $L_u\alpha = eB(\rho_1 dQ_1 + \rho_2 dQ_2)$ and take $dL_u\alpha$.
- $\beta = |B(Q+\rho)| d(Q_1 + \rho_1) \wedge d(Q_2 + \rho_2)$. So
  $i_u\beta = -|B(Q+\rho)|(\rho_1 dQ_1 + \rho_2 dQ_2 + \rho_1 d\rho_1 + \rho_2 d\rho_2)$. Then
  $L_u\beta = -|B(Q)|(d\rho_1 \wedge dQ_1 + d\rho_2 \wedge dQ_2) + O(\varepsilon)$.
- So $L_u\omega = O(\varepsilon)$.

# Adiabatic invariant

- Treating $u$ as an approximate symmetry, get approximate conserved quantity $K$ from $i_u\omega \approx dK$. From above, $i_u\omega \approx -e|B|(\rho_1 d\rho_1 + \rho_2 d\rho_2) = -\frac{e}{2}|B|d|\rho|^2$, so take $K = -\frac{e}{2}|B||\rho|^2$.
- Conventional to write $K = -\frac{m}{e}\mu$ with "magnetic moment" $\mu = \frac{e^2}{2m}|B||\rho|^2 = \frac{m|v_\perp|^2}{2|B|}$.
- This makes $\mu$ an *adiabatic invariant*: $\forall k > 0\ \exists \varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ the change in $\mu$ during any time-interval of length $\leq T/\varepsilon$ is at most $k$.
- Theory of adiabatic invariants shows that for $C^r$ system, there is an asymptotic series for a circle action $u(\varepsilon)$ (gyro-rotation being the first term) and associated $\mu(\varepsilon)$ with first term $\mu$, such that truncating at the $r^{th}$ term, the errors are $O(\varepsilon^r)$.

# Approximate reduced system

- $H(Q, p_{\|}) = \frac{1}{2m} p_{\|}^2 + \mu |B(Q)|$, and $\omega = -d(p_{\|} b^\flat) - e\beta = b^\flat \wedge dp_{\|} - p_{\|} db^\flat - e i_B \Omega$. Let $c = \mathrm{curl}\, b$, so $i_c \Omega = db^\flat$, then $\omega = b^\flat \wedge dp_{\|} - e i_{\tilde{B}} \Omega$ with $\tilde{B} = B + \frac{p_{\|}}{e} c$.
- Equations of motion: $i_{(\dot{Q}, \dot{p}_{\|})} \omega = dH$ says $i_{(\dot{Q}, \dot{p}_{\|})} (b^\flat \wedge dp_{\|} - e i_{\tilde{B}} \Omega) = \frac{p_{\|}}{m} dp_{\|} + \mu d|B|$. Apply to $(0, \delta p_{\|})$:

$$\dot{Q}_{\|} = p_{\|}/m. \tag{1}$$

Apply to $(\xi, 0)$: $e(\tilde{B} \times \dot{Q}) \cdot \xi = \xi \cdot (\mu \nabla |B| + \dot{p}_{\|} b)$, so

$$e\tilde{B} \times \dot{Q} = \mu \nabla |B| + \dot{p}_{\|} b. \tag{2}$$

The case $\xi = \tilde{B}$ gives (avoiding $\tilde{B} \cdot b = 0$)

$$\dot{p}_{\|} = -\mu \frac{\tilde{B} \cdot \nabla |B|}{\tilde{B} \cdot b}. \tag{3}$$

Lastly, take $b \times (2)$ and use (1):

$$\dot{Q} = \frac{1}{\tilde{B} \cdot b} \left( \frac{\mu}{e} b \times \nabla |B| + \frac{p_{\|}}{m} \tilde{B} \right). \tag{4}$$

- (3,4): the (first-order) guiding-centre equations in Hamiltonian form.

# Things to note

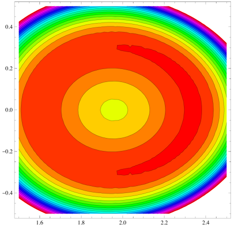
- If $|B|$ has a well with minimax $B_w$ then it confines particles with $H \leq \mu B_w$; but does not help for small $\mu$.
- $\mu$ is (up to a scaling) the Poincaré invariant of the disk spanned by a gyro-orbit: $\int_D \omega = -e|B|\,\pi|\rho|^2$.
- The parallel motion sees a force roughly $-\mu b \cdot \nabla|B|$.
- There are small perpendicular drifts roughly $\frac{\mu}{e|B|} b \times \nabla_\perp |B|$ and $p_\parallel^2 b \times \frac{\kappa}{me}$, where $\kappa = \nabla_b b$ is the curvature vector of the fieldline (from $c$ in $\tilde{B}$ and $c_\perp = b \times \kappa$).
- Higher-order approximate symmetries produce higher-order GC Hamiltonian systems.

# GC motion in axisymmetric field

- Recall FGCM $H = \frac{1}{2m} p_{\|}^2 + \mu |B(Q)|$, $\omega = -d(p_{\|} b^\flat) - e\beta$.
- Recall $B$ axisymmetric means $L_u \beta = 0$ for $u = \partial_\phi$ in cylindrical coordinates. Note this implies $L_u |B| = 0$ and $L_u b^\flat = 0$ too, because $u$ is an isometry ($L_u g = 0$).
- Proof: First, $L_u \Omega = 0$ so $i_{[B,u]} \Omega = i_B L_u \Omega - L_u i_B \Omega = 0$, so $[u, B] = 0$. Second, $L_u g = 0$ and $B^\flat = i_B g$ imply $L_u B^\flat = L_u i_B g = i_B L_u g - i_{[B,u]} g = 0$ (identity is true even though $g$ is not antisymmetric). Then $2|B| L_u |B| = L_u i_B B^\flat = i_B L_u B^\flat - i_{[B,u]} B^\flat = 0$, so $L_u |B| = 0$. Last, $L_u B^\flat = L_u (|B| b^\flat) = (L_u |B|) b^\flat + |B| L_u b^\flat$, so $L_u b^\flat = 0$.
- Lift $u$ to $U = (\partial_\phi, 0)$ on $(Q, p_{\|})$. Thus $L_U H = 0$ and $L_U \omega = 0$. So $U$ is a continuous symmetry, $i_U \omega = dL$ for some local function $L$, and $L$ is conserved. Compute $i_U d(p_{\|} b^\flat) = L_U (p_{\|} b^\flat) - d(p_{\|} i_u b^\flat)$. Use $i_u b^\flat = b_\phi = r b_{(\phi)}$. Recall $i_u \beta = d\psi$. So $L = r b_{(\phi)} p_{\|} - e\psi$ (not same as before).
- A Hamiltonian system reducible to 1DoF is called *integrable*.

# Reduced system

- Quotient by $U$ to reduce to 1DoF in $(r, z)$:
  $H = \frac{1}{2m}\left(\frac{L+e\psi}{rb_{(\phi)}}\right)^2 + \mu|B|(r,z)$ and $\omega = eB_{(\phi)}dr \wedge dz$.
- Can write $|B| = \sqrt{r^{-2}|\nabla\psi|^2 + B_{(\phi)}^2}$ and $b_{(\phi)} = B_{(\phi)}/|B|$.
- If choose $\psi$ and $B_{(\phi)}$ to make $H$ have a local minimum for each $L$ and $\mu$ then the corresponding particles are confined. Full principle of tokamak. Confines more particles.
- Example: Solov'ev equilibrium $B_{(\phi)} = I(\psi)/r$, $I^2 = I_0^2 - 2E\psi$, $\psi = (Dr^2 - C)^2 + \frac{1}{2}(E + (F - 8D^2)r^2)z^2$ in $0 \leq \psi \leq p_0/F$, with $E, F, C, D, p_0 > 0$, $2Ep_0 \leq I_0^2 F$ $(p = p_0 - F\psi)$.
- Contours of $H$ for given $L, \mu$; note the banana orbit.



# Currents

- Note $rJ^z = \partial_r(rB_{(\phi)})$, $rJ^r = -\partial_z(rB_{(\phi)})$, so can achieve a strong $B_{(\phi)} = \frac{I_x}{2\pi r}$ from external poloidal current $I_x$, making small gyroradius for desired energies.
- But bounded contours of $H$ require a local max or min for $\psi$, and so current $J_{(\phi)} = rJ^\phi = \frac{1}{r}\partial_z^2\psi + \partial_r(\frac{1}{r}\partial_r\psi)$ in the plasma.
- There are some natural currents in a plasma, in particular "diamagnetic" current $J_\perp = B \times \nabla p/|B|^2$ to make MHS $J \times B = \nabla p$. It contributes little to $J_{(\phi)}$, but in general is not divergence-free: $\mathrm{div} J_\perp = -|B|^{-2} J_\perp \cdot \nabla |B|^2$.
- So it is accompanied by a parallel current $J_\parallel b$ s.t. $B \cdot \nabla \frac{J_\parallel}{|B|} = -\mathrm{div} J_\perp$ (a magnetic differential equation that restricts $B$). Contributions include a "bootstrap" current, driven by friction between circulating electrons and those on bananas, which could provide much of the required $J_{(\phi)}$.
- But still need some current drive, and toroidal current promotes dangerous instabilities.
- So can one do better than a tokamak?