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Topics in Partial Differential Equations

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Chapter 1

Homogenization Theory

In a very abstract way one might say that Analysis is theory of solving infinite-dimensional systems

$$\mathcal{F}(u) = 0,$$

where $\mathcal{F} : X \to Y$ is a possibly nonlinear operator and X, Y are Banach spaces. The most famous example is the Poisson equation where $F(u) = \Delta u + f$, $X = H_0^1(U)$ and $Y = H^{-1}(U)$. Homogenization theory provides a strategy of constructing effective equations in the sense that if $\mathcal{F}^\epsilon(u^\epsilon) = 0$, then there exists an effective operator $\mathcal{F}^{\mathrm{eff}}$ such that $\lim_{\epsilon\to 0} u^\epsilon = u^{\mathrm{eff}}$ and $\mathcal{F}^{\mathrm{eff}}(u^{\mathrm{eff}}) = 0$. The success of the homogenization theory can be traced back to the multiscale structure induced by the ϵ-dependency.

1.1 Functional analytic setting

Recall the weak form of elliptic partial differential equations. Let $d \in \{1, 2, \dots\}$, $U \subset \mathbb{R}^d$ open and bounded, and for a continuously differentiable function $u \in C^1(\bar{U})$ we define

$$\|u\|_{H^1(U)}^2 = \sum_{i=1}^d \int_U \left(\frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}x + \int_U u^2 \,\mathrm{d}x.$$

The Sobolev spaces $H^1(U)$ and $H_0^1(U)$ are defined as

$$H^1(U) = \left\{ u \in L^2(U) \;:\; u \text{ is weakly differentiable and } \max_{i=1\dots d} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2} < \infty \right\}.$$
$$H_0^1(U) = \overline{\{u \in C^1(U) \;:\; u|_{\partial U} = 0\}}^{H^1(U)}.$$

It is not hard to see that $H_0^1(U)$ is strictly smaller than $H^1(U)$ because the functions satisfy a boundary condition (prove it!). The space $H_0^1(U)$ is one of corner stone of the theory of elliptic partial differential equations.

The weak method works applied the Dirichlet problem works as follows. Let $b \in L^2(U)^d$, $c \in L^2(U)$ and $A \in L^\infty(\mathbb{R}^{d\times d})$ be a coefficient matrix such that there exists $c > 0$ with the property

$$\inf \left\{ \sum_{i,j=1}^d A_{i,j}(x)\xi_i\,\xi_j \;:\; \xi \in \mathbb{R}^d \text{ and } |\xi| = 1 \right\} \geq c. \tag{1.1}$$

A function $u \in H_0^1(U)$ is a weak solution of the boundary value problem

$$\begin{cases} -\mathrm{div}\,(A\nabla u) + b\cdot\nabla u + c\,u = f & x \in U, \\ u = 0 & x \in \partial U \end{cases}$$

if the following integral equation holds

$$B[u,\varphi] = \int_U f\,\varphi\,\mathrm{d}x \text{ for all } \varphi \in H_0^1(U). \tag{1.2}$$

with

$$B[u,\varphi] = \int_U (\nabla\varphi \cdot A\,\nabla u + \varphi\, b \cdot \nabla u + c\,\varphi\, u)\,\mathrm{d}x.$$

Existence and uniqueness can be established with the Lax-Milgram theorem.

Now we consider the inhogmogenous Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(A\nabla u) + b\cdot\nabla u + c\,u = f & x \in U, \\ u = g & x \in \partial U \end{cases}$$

where $g : \partial U \to \mathbb{R}$. If

i $A \in C^1(U)^{d\times d}$ and

ii There exists $v \in C^1(\bar{U})$ such that $v(x) = g(x)$ for all $x \in \partial U$,

then we can define $u = v + w$ and look for weak solutions $w \in H_0^1(U)$ of the homogeneous Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(A\nabla w) - (\mathrm{div}\,(A) + b)\cdot\nabla w + c\,w = -b\cdot\nabla v - cv + f & x \in U, \\ w = 0 & x \in \partial U \end{cases}$$

This approach is rather clumsy as neither assumption (i) nor assumption (ii) is particularly natural. It would be much better if we worked instead with the Sobolev space

$$H_g^1(U) = \left\{u \in H^1(U) \;:\; u|_{\partial U} = g\right\}.$$

The definition only makes sense if we can define boundary values of Sobolev functions. Clearly this does not work if we replace $H_0^1(U)$ with the space $L^2(U)$.

Definition 1.1.1. We say the boundary ∂U is Lipschitz if there exists $r > 0$ with the property that for each point $x \in \partial U$ there is Lipschitz function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that – upon relabeling and reorienting the coordinate axes if necessary – we have

$$U \cap B(x,r) = \{y \in B(x,r) \;:\; y_d > \gamma(y_1,\ldots,y_{d-1})\}.$$

Theorem 1.1.1 (Trace theorem)**.** *If ∂U is Lipschitz, then there exists bounded linear operator*

$$T : H^1(U) \to L^2(\partial U)$$

such that

$$Tu = u|_{\partial U}$$

if $u \in C^1(\bar{U})$.

Proof. We only establish the boundedness of T for the case where $U \subset \mathbb{R}^d$ is the half space: $U = \mathbb{R}^{d-1} \times [0,\infty)$. The full proof can be found in [1].

Let T^0 be the linear map defined by

$$T^0 : u \in C_0^1(\mathbb{R}^{d-1}\times[0,\infty)) \to u|_{\mathbb{R}^{d-1}},$$

with the convention that $C_0^1(U)$ denotes the set of those functions with the property that the support of the trivially extended function $\bar{u} : \mathbb{R}^d \to \mathbb{R}$ is compact. The trivial extension $\bar{u}$ assumes the value 0 outside U. Let us first show that

$$\|T^0(u)\|_{L^2(\mathbb{R}^{d-1})} \le \|u\|_{H^1(\mathbb{R}^{d-1}\times[0,\infty))}. \tag{1.3}$$

Since u has compact support, we have

$$|u(x',0)|^2 = -\int_0^\infty \frac{\partial}{\partial x_d}\left(u(x',x_d)^2\right)\,\mathrm{d}x_d = -\int_0^\infty 2\,u(x',x_d)\,\frac{\partial}{\partial x_d}u(x',x_d)\,\mathrm{d}x_d.$$

Therefore, by Young's inequality

$$|u(x',0)|^2 \le \int_0^\infty u(x',x_d)^2\,\mathrm{d}x + \int_0^\infty \left|\frac{\partial}{\partial x_d}u(x)\right|^2\,\mathrm{d}x_d.$$

Integrating over $\mathbb{R}^{d-1}$ in x' and using Fubini's theorem one obtains

$$\int_{\mathbb{R}^{d-1}} |u(x',0)|^2\,\mathrm{d}x \le \int_{\mathbb{R}^{d-1}\times[0,\infty)} u^2\,\mathrm{d}x + \int_{\mathbb{R}^{d-1}\times[0,\infty)} \left|\frac{\partial}{\partial x_d}u(x)\right|^2\,\mathrm{d}x,$$

which gives (1.3).

Suppose now that $u \in H^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)$. By standard density results there exists a sequence $u_n \in C^1(\mathbb{R}^{d-1}\times[0,\infty))$ converging to $u \in H^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)$. By (1.3) and the linearity of T^0, we have

$$\|T^0(u_n) - T^0(u_m)\|_{L^2(\mathbb{R}^{d-1})} \le \|u_n - u_m\|_{H^1(\mathbb{R}^{d-1}\times[0,\infty))} \quad \forall m,n\in\mathbb{N}.$$

Consequently, $T^0(u_n)$ is a Cauchy sequence in the complete space $L^2(\mathbb{R}^{d-1})$, and it has a limit $u_0 \in L^2(\mathbb{R}^{d-1})$. Define $T(u) = u_0$. By construction $T(u) = T^0(u)$ if $u \in C_0^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)$ so that T is a linear extension of T^0 to $H^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)$. By construction T is uniquely determined and linear and continuous from $H^1(\mathbb{R}^{d-1}\times[0,\infty))$ to $L^2(\mathbb{R}^{d-1})$. $\square$

One can prove that T is not onto $L^2(\partial U)$ (i.e. $L^2(\partial U)\setminus T(H^1(U)) \neq \emptyset$). This leads to the following definition.

Definition 1.1.2. Suppose that ∂U is Lipschitz. The space $H^{\frac{1}{2}}(\partial U)$ is the range of T, i.e. $H^{\frac{1}{2}}(\partial U) = T(H^1(U))$.

Proposition 1.1.3. *Suppose that ∂U is Lipschitz continuous. Then $H^{\frac{1}{2}}(\partial U)$ is a Banach space for the norm defined by*

$$\|u\|^2_{H^{\frac{1}{2}}(\partial U)} = \int_{\partial U} |u(x)|^2\,\mathrm{d}\mathcal{H}^{d-1}(x) + \int_{\partial} \mathrm{d}\mathcal{H}^{d-1}(x) \int_{\partial U} \mathrm{d}\mathcal{H}^{d-1}(y)\,\frac{|u(x)-u(y)|^2}{|x-y|^d}.$$

The proof can be found in [1].

Corollary 1.1.4. *If ∂U is Lipschitz and $u \in C_{\mathrm{Lip}}(\partial U)$, the $u \in H^{\frac{1}{2}}(\partial U)$*

Proof. Since u is Lipschitz there exists $C > 0$ such that

$$|u(x) - u(y)| \le C\,|x-y| \text{ for all } x,y\in\partial U. \tag{1.4}$$

Thus,

$$\frac{|u(x)-u(y)|^2}{|x-y|^d} \le C^2\,|x-y|^{2-d}.$$

Fix now $x \in \partial U$, we can assume without loss of generality that $x = 0$. Then (1.4) implies that

$$\begin{aligned}
&\int_{\partial U} \mathrm{d}\mathcal{H}^{d-1}(y)\, \frac{|u(y)-u(0)|^2}{|y|^d} \\
=&\int_{\partial U\cap B(0,r)} \mathrm{d}\mathcal{H}^{d-1}(y)\, \frac{|u(y)-u(0)|^2}{|y|^d} + \int_{\partial U\setminus B(0,r)} \mathrm{d}\mathcal{H}^{d-1}(y)\, \frac{|u(y)-u(0)|^2}{|y|^d} \\
\leq& C\left(\int_{\partial U} \mathrm{d}\mathcal{H}^{d-1}(y)\, |y|^{2-d} + \mathcal{H}^{d-1}(\partial U)\, r^d\right) \\
\leq& C\left(1 + \int_{B^{d-1}(0,r)} \underbrace{|\eta^2+\gamma(\eta)^2|^{\frac{2-d}{2}}}_{\leq \eta^2} \sqrt{1+|\nabla\gamma(\eta)|^2}\, \mathrm{d}\eta\right) \\
\leq& C\left(1 + \int_{B^{d-1}(0,r)} |\eta|^{2-d}\, \mathrm{d}\eta\right) = C\left(1+\int_0^r \mathrm{d}s\, s^{d-2}\, s^{2-d}\right) = C\,(1+r).
\end{aligned}$$

□

The key properties of $H^{\frac{1}{2}}(\partial U)$ are given by the following result

Proposition 1.1.5. *Suppose that ∂U is Lipschitz. Then $H^{\frac{1}{2}}(\partial U)$ has the properties*

1. *The imbedding $H^{\frac{1}{2}}(\partial U) \subset L^2(\partial U)$ is compact.*
2. $H^1_0(U) = \{u \in H^1(U) \;:\; T(u) \equiv 0\}$,
3. *There exists a linear continous map*
$$g \in H^{\frac{1}{2}}(\partial U) \to u_g \in H^1(U),$$
with $T\, u_g = g$.

It is well known, that if ∂U is Lipschitz continuous, then the unit outward normal vector to U is well defined almost everywhere [18].

Proposition 1.1.6 (Green formula)**.** *Suppose that ∂U is Lipschitz and $u, v \in H^1(U)$. Then*
$$\int_U u \frac{\partial v}{\partial x_i}\, \mathrm{d}x = -\int_U v\, \frac{\partial u}{\partial x_i}\, \mathrm{d}x + \int_{\partial U} T(u)(s)\, T(v)(s)\, \nu_i(s)\, \mathrm{d}\mathcal{H}^{d-1}(s)$$
for $i \in \{1, \dots, d\}$ and where $\nu = (\nu_1, \dots, \nu_d)$ denotes the outward normal vector to U.

We finish the discussion of boundary values by recalling an important result due to Lions and Magenes [15]. They observe that although a function $v \in L^2(U)$ does not have a trace on the boundary it is possible to give a sense to $v \cdot n$ if div $v \in L^2(U)$ as well.

Definition 1.1.7. Suppose that ∂U is Lipschitz. $H^{-\frac{1}{2}}(\partial U)$ is the dual space of $H^{\frac{1}{2}}(\partial U)$ equipped with the norm
$$\|F\|_{H^{-\frac{1}{2}}(\partial U)} = \sup_{u \in H^{\frac{1}{2}}(\partial U)\setminus\{0\}} \frac{{}_{H^{-\frac{1}{2}}}\langle F, u\rangle_{H^{\frac{1}{2}}}}{\|u\|_{H^{\frac{1}{2}}(\partial U)}}.$$

Proposition 1.1.8. *Suppose the ∂U is Lipschitz. The space $H^{-\frac{1}{2}}(\partial U)$ has the following properties:*

1. $L^2(\partial U) \subset H^{-\frac{1}{2}}(\partial U)$.
2. *Define the space*
$$H_{\mathrm{div}}\,(U) = \{v \;:\; v \in L^2(U, \mathbb{R}^d) \textit{ and } \mathrm{div}\; v \in L^2(U)\}.$$
Then, $v \cdot \nu \in H^{-\frac{1}{2}}(\partial U)$ and the map
$$v \in H(U, \mathrm{div}\,) \to v \cdot \nu \in H^{-\frac{1}{2}}(\partial U)$$
is linear and continuous. Moreover, if $v \in H_{\mathrm{div}}\,(U)$ and $w \in H^1(U)$, then
$$-\int_U (\mathrm{div}\;\, v)\, w\, \mathrm{d}x = \int_U v \cdot \nabla w\, \mathrm{d}x + {}_{H^{-\frac{1}{2}}}\langle v \cdot \nu, w\rangle_{H^{\frac{1}{2}}}.$$

Periodic setting

In this section we introduce a notion of periodicity for function in the Sobolev space H^1. Let

$$U_{\text{per}} = \mathbb{R}^d / Y\mathbb{Z}^d = Y\,[0,1]^d$$

be the periodic cell and define

$$C^\infty_{\text{per}}(\mathbb{R}^d) = \left\{ u \in C^\infty(\mathbb{R}^d) \;:\; u(x) = u(y) \text{ if } Y^{-1}(x-y) \in \mathbb{Z}^d \right\}.$$

The space $H^1_{\text{per}}(U_{\text{per}})$ is the closure of $C^\infty_{\text{per}}(\mathbb{R}^d)|_{U_{\text{per}}}$ with respect to the H^1-norm. From this definition and the proof of Theorem 1.1.1 it is obvious that $H^1_{\text{per}}(U_{\text{per}})$ has the following property.

Proposition 1.1.9. *Let $u \in H^1_{\text{per}}(U_{\text{per}})$. Then u has the same trace on the opposing faces of U_{per}.*

Let g be a function defined on U_{per} and denote by $g^\#$ its periodic extension to the whole of $\mathbb{R}^d$, defined by

$$g(x + Y\,k) = g(x) \text{ for all } x \in U_{\text{per}}, k \in \mathbb{Z}^d.$$

Proposition 1.1.10. *Let $u \in H^1_{\text{per}}(U_{\text{per}})$ and $u^\#$ be its Y-periodic extension. Then $u^\#|_\omega \in H^1(\omega)$ for each open bounded subset $\omega \subset \mathbb{R}^d$.*

Proof. See [5], pp. 57–59. □

Definition 1.1.11. The quotient space $\dot{H}^1_{\text{per}}(U_{\text{per}})$ is defined as the space of equivalence classes with respect to the relation

$$u \sim v \Leftrightarrow u - v \text{ is a constant.}$$

We denote by $\dot{u}$ the equivalence class represented by u.

Proposition 1.1.12. *The quantity*

$$\|\dot{u}\|_{\dot{H}^1_{\text{per}}} = \|\nabla u\|_{L^2(U_{\text{per}})}$$

defines a norm on $\dot{H}^1_{\text{per}}$.

Boundary value problems

The main tool is the Lax-Milgram theorem.

Theorem 1.1.2. *Let X be a Hilbert space Let $B : X \times X \to \mathbb{R}$ be a bilinear form for which there exist constants $\alpha, \beta > 0$ such that*

$$B[u,v] \le \alpha \|u\| \|v\| \textit{ for all } u, v \in X,$$

and

$$B[u,u] \ge \beta \|u\|^2 \textit{ for all } u \in H.$$

Finally, let $f \in X^*$ be a bounded linear functional on X. Then, there exists a unique element $u \in X$ such that

$$B[u,v] = \langle f, v\rangle \textit{ for all } v \in X. \tag{1.5}$$

If $A \in L^\infty(U, \mathbb{R}^{d\times d})$ such that there exists $c > 0$ with the property

$$\xi \cdot A(x)\xi \ge c\,|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d \text{ and a.e. } x \in U,$$

then the Lax-Migram theorem provides existence and uniqueness of solutions u of elliptic problems

$$-\text{div}\,(A\nabla u) = f \tag{1.6}$$

with the following standard boundary conditions

Dirichlet condition $u = g$ on ∂U,

Neumann condition $\frac{\partial u}{\partial \nu_A} = g$ on ∂U, where $\frac{\partial}{\partial \nu_A} = \nu \cdot A\nabla$

Robin condition $\frac{\partial u}{\partial \nu_A} + \lambda u = g$ on ∂U for some $\lambda > 0$,

Periodic condition $u(x) = u(y)$ if $x - y \in Y\mathbb{Z}^d$.

Proposition 1.1.13 (Dirichlet problems). *Suppose that ∂U is Lipschitz and that $A \in L^\infty(U, \mathbb{R}^{d\times d})$ satisfies (1.1). Define*

$$B[u,v] = \int_U \nabla v \cdot A\nabla u \, \mathrm{d}x.$$

There exists $C(U,A) > 0$ such that for every $f \in H^{-1}(U)$ and $g \in H^{\frac{1}{2}}(\partial U)$ there exists a unique function $u \in H^1(U)$ with the property $u|_{\partial U} = g$,

$$B[u,v] = {}_{H^{-1}(U)} \langle f, v\rangle_{H^1(U)} \text{ for all } v \in H^1_0(U)$$

and

$$\|u\|_{H^1(U)} \leq C \left(\|f\|_{H^{-1}(U)} + \|g\|_{H^{\frac{1}{2}}(\partial U)} \right). \tag{1.7}$$

Proof. If $g = 0$, then the result follows from Prop. 1.1.5.2 and Theorem 6.2.3 in [11].

Since $g \in H^{\frac{1}{2}}(\partial U)$, Prop. 1.1.5.3 implies that there exists $G \in H^1(U)$ such that $T\,G = g$ and

$$\|G\|_{H^1(U)} \leq C \|g\|_{H^{\frac{1}{2}}(\partial U)}. \tag{1.8}$$

Observe that $f + \mathrm{div}\,(A\nabla G) \in H^{-1}$. Hence, the case $g = 0$ implies that the homogeneous Dirichlet problem with $B[z,v] = \int_U \nabla v \cdot A\nabla u \, \mathrm{d}x$

$$\begin{cases} \text{Find } z \in H^1_0(U) \text{ such that} \\ B[z,v] = {}_{H^{-1}(U)} \langle f + \mathrm{div}\,(A\nabla G), v\rangle_{H^1_0(U)} \text{ for all } v \in H^1_0(U) \end{cases}$$

admits a unique solution $z \in H^1_0(U)$. Moreover,

$$\|z\|_{H^1(U)} \leq \frac{1}{\beta} \|f + \mathrm{div}\,(A\nabla G)\|_{H^{-1}(U)}.$$

Set $u = z + G$. From Prop. 1.1.5.2 and the linearity of T one has $T\,u = g \in H^{\frac{1}{2}}(\partial U)$. Further, choosing $v \in H^1_0(U)$ as a testfunction, one obtains

$$\begin{aligned} \langle -\mathrm{div}\,(A\nabla u), v\rangle &= \int_U \nabla v \cdot A\nabla u \, \mathrm{d}x = B[u,v] = B[z,v] + B[G,v] \\ &= \langle f + \mathrm{div}\,(A\nabla G), v\rangle + \int_U \nabla v \cdot A\nabla G \, \mathrm{d}x = \langle f, v\rangle \end{aligned}$$

which means that $-\mathrm{div}\,(A\nabla u) = f$ in the weak sense, and hence u satisfies (1.6).

We now make use of estimate (1.8) to derive (1.7). Prop 1.1.5.3 and the Poincaré inequality imply that

$$\begin{aligned} &\|u\|_{H^1(U)} \leq \|u - G\|_{H^1(U)} + \|G\|_{H^1(U)} \\ \leq &\|z\|_{L^2(U)} + \|\nabla z\|_{L^2(U)} + \|G\|_{H^1(U)} \\ \leq &|z\|_{H^1_0(U)} + C \|g\|_{H^{\frac{1}{2}}(\partial U)} \\ \leq &\frac{1}{\beta} \left(\|f\|_{H^{-1}(U)} + \|\mathrm{div}\,(A\nabla G)\|_{H^{-1}(U)} \right) + C \|g\|_{H^{\frac{1}{2}}(\partial U)}. \end{aligned}$$

On the other hand, by Cauchy-Schwarz and again Prop 1.1.5.3

$$\langle \mathrm{div}\,(A\nabla G), v\rangle = \int_U \nabla v \cdot A\nabla G \, \mathrm{d}x \leq C \|g\|_{H^{\frac{1}{2}}(\partial U)} \|\nabla v\|_{L^2(U)}.$$

This, together with the above estimate implies that

$$\|u\|_{H^1(U)} \leq \frac{1}{\beta}\|f\|_{H^{-1}(U)} + C\,\|g\|_{H^{\frac{1}{2}}(\partial U)}.$$

The proof of (1.7) is finished. □

Now we consider Neumann boundary conditions. In the case of Dirichlet boundary conditions we constructed search spaces which automatically satisfy boundary conditions. This trick does not work in the case of Neumann problems because the boundary conditions are too singular. The solution is to construct a suitably adapted variational formulation.

If we multiply (1.6) with a function $v \in H^1(U)$ and formally integrate by parts we obtain that variational formulation

$$\begin{cases} \text{Find } u \in H^1(U) \text{ such that} \\ B[u,v] = \langle f, v\rangle + \langle g, v\rangle \text{ for all } v \in H^1(U). \end{cases} \tag{1.9}$$

Since only the derivatives of u are relevant it is clear that we generate new solutions by adding a constant. The implies that uniqueness can only be expected in

$$\dot{H}^1(U) = H^1(U)/\sim$$

where $u \sim v$ if $u - v = \text{const}$.

Lemma 1.1.14. *Let $U \subset \mathbb{R}^d$ be open, connected and ∂U is Lipschitz. Then $\dot{H}^1(U)$ is a Hilbert space with the inner product*

$$(u,v)_{\dot{H}^1} = \int_U \nabla u \cdot \nabla v \,\mathrm{d}x.$$

Proof. We have to show that $(u,u)_{\dot{H}^1} = 0$ implies that $u \equiv \text{const}$. This follows immediately from the Poincaré-Wirtinger inequality [11, Theorem 5.8.1.1]: There exists $C(U)$ such that

$$\left\| u - \frac{1}{|U|}\int_U u \,\mathrm{d}x \right\| \leq C(U)\,\|\nabla u\|_{L^2(U)}.$$

□

Proposition 1.1.15 (Neumann problem)**.** *Suppose that U is open, connected and ∂U is Lipschitz. Then there exists $C(U,A)$ with the property that for every $f \in L^2(U)$ and for every $g \in H^{-\frac{1}{2}}(\partial U)$ such that*

$$\int_U f \,\mathrm{d}x + {}_{H^{-\frac{1}{2}}(\partial U)}\langle g, 1\rangle_{H^{\frac{1}{2}}(\partial U)} = 0 \tag{1.10}$$

there exists a unique $u \in \dot{H}^1(U)$ satisfying (1.9) such that

$$\|u\|_{H^1(U)} \leq C\left(\|f\|_{L^2(U)} + \|g\|_{H^{-\frac{1}{2}}(\partial U)}\right). \tag{1.11}$$

Proof. We define the functional $F \in H^1(U)^*$

$$F(v) = \int_U f\,v \,\mathrm{d}x + {}_{H^{-\frac{1}{2}}}\langle g, v\rangle_{H^{\frac{1}{2}}}.$$

The key observation is that thanks to (1.10) we have $F(v+c) = F(v)$ if c is constant. This implies that $F \in \dot{H}^1(U)^*$. Similarly, the bilinear form

$$B[u,v] = \int_U \nabla v \cdot A\nabla u \,\mathrm{d}x$$

is continuous on $\dot{H}^1(U) \times \dot{H}^1(U)$. The ellipticity estimate (1.1) implies that B is coercive and therefore the Lax-Milgram theorem provides existence and uniqueness of functions $u \in H^1(U)$ with the property $B[u,v] = F(v)$ for all $v \in H^1(U)$. □

To construct solutions for Robin problems we have define a suitable adapted bilinear form. After multiplication with $v \in H^1(U)$ and partial integration eqn. (1.6) reads

$$\int_U \nabla v \cdot A \nabla u \,\mathrm{d}x - \int_{\partial U} v \,\frac{\partial u}{\partial \nu_A}\,\mathrm{d}\mathcal{H}^{d-1} = \langle f, v\rangle. \tag{1.12}$$

The boundary condition $\frac{\partial u}{\partial \nu_A} = g - \lambda u$ allows us to rewrite (1.12) so that we obtain

$$B[u,v] = \langle f, v\rangle + {}_{H^{-\frac{1}{2}}}\langle g, v\rangle_{H^{\frac{1}{2}}} \text{ for all } v \in H^1(U), \tag{1.13}$$

with

$$B[u,v] = \int_U \nabla v \cdot A \nabla u + \lambda \int_{\partial U} u\, v \,\mathrm{d}\mathcal{H}^{d-1}.$$

Proposition 1.1.16 (Robin problems)**.** *Suppose that U is connected and ∂U is Lipschitz. Then, there exists $C > 0$ such that for every $f \in L^2(U)$ and $g \in H^{-\frac{1}{2}}(\partial U)$ there exists a unique function $u \in H^1(U)$ such that (1.13) and*

$$\|u\|_{H^1(U)} \leq C \left(\|f\|_{L^2(U)} + \|g\|_{H^{-\frac{1}{2}}} \right)$$

holds

Proof. As in the proof of Prop. 1.1.15, let $F \in H^1(U)^*$ be defined by

$$F(v) = \int_U f\, v \,\mathrm{d}x + {}_{H^{-\frac{1}{2}}}\langle g, v\rangle_{H^{\frac{1}{2}}}.$$

We will again apply the Lax-Milgram theorem with $X = H^1(U)$. As a consequence of the trace-theorem (Thm. 1.1.1) the bilinear form B is continuous. We are done once the coercivity of B is established. It suffices to show that there exists a constant $C > 0$ such that

$$\|u\|_{L^2(U)} \leq C \, \left(\|\nabla u\|_{L^2(U)} + \|u\|_{\partial U} \right). \tag{1.14}$$

The existence of C is established via a standard contradiction argument. Assume that there exists a sequence $u_n \in L^2(U)$ such that $\|u_n\|_{L^2(U)} = 1$ and

$$\lim_{n\to\infty} \left(\|\nabla u_n\|_{L^2(U)} + \|u_n\|_{\partial U} \right) = 0.$$

Then $\limsup_{n\to\infty} \|u_n\|_{H^1(U)} < \infty$. By Banach-Alaoglu there exists a subsequence (not relabelled) and $v \in H^1(U)$ such that $\|\nabla v\|_{L^2(U)} = 0$ and thus v is constant. Moreover, Rellich's theorem and Prop. 1.1.5.1 imply that u_n and $u_n|_{\partial U}$ converge strongly in $L^2(U)$ and $L^2(\partial U)$, resp. This implies that $\|v\|_{L^2(U)} = 1$ and thus $v \equiv |U|^{-\frac{1}{2}}$. We have obtained a contradiction to $\|v\|_{L^2(\partial U)} = 0$. □

Finally we mention that the same idea also establishes the existence and uniqueness of solutions in the case of periodic boundary conditions.

Proposition 1.1.17 (Periodic problems)**.** *Let $f \in H^1_{\mathrm{per}}(U)^*$ such that $\langle f, 1\rangle = 0$. Then there exists a unique function $u \in \dot{H}^1_{\mathrm{per}}(U_{\mathrm{per}})$ such that*

$$B[u,v] = \langle f, v\rangle \textit{ for all } v \in \dot{H}^1_{\mathrm{per}},$$

where

$$B[u,v] = \int_{U_{\mathrm{per}}} \nabla v \cdot A \nabla u \,\mathrm{d}x.$$

Energy minimization

Let us consider first the case of Dirichlet boundary conditions. A very important principle which paves the way for the analysis of non-linear problems is the fact that eqn. (1.6) is a necessary condition for u to minimize the functional

$$I[u] = \frac{1}{2}\int_U \nabla u \cdot A\nabla u \,\mathrm{d}x - \int_U f\, u \,\mathrm{d}x$$

if A is symmetric. It is not hard to see that (1.6) is also sufficient if suitable boundary conditions are applied.

Proposition 1.1.18 (Dirichlet boundary conditions)**.** *Assume that U is open such that ∂U is Lipschitz, A is symmetric, $u \in H^1(U)$ and $g = u|_{\partial U} \in H^{\frac{1}{2}}(\partial U)$. If*

$$B[u,v] = \langle f, v\rangle \text{ for all } v \in H_0^1(U) \tag{1.15}$$

with

$$B[u,v] = \int_U \nabla v \cdot A\nabla u \,\mathrm{d}x,$$

then u is the unique minimizer of I subject to the constraint $u|_{\partial U} = g$.

Proof. Let $v \in H^1(U)$ such that $v|_{\partial U} = u|_{\partial U}$ and define $h = v - u \in H_0^1(U)$. Then

$$\begin{aligned}
I[v] =& I[u+h] = \frac{1}{2}\int_U \nabla(u+h)\cdot A\nabla(u+h)\,\mathrm{d}x - \langle f, u+h\rangle \\
=& \frac{1}{2}\int_U \nabla u \cdot A\nabla u - \langle f, u\rangle + \underbrace{B[u,h] - \langle f, h\rangle}_{=0 \text{ by (1.15) since } h\in H_0^1(U)} + \frac{1}{2}\int_U \nabla h \cdot A\nabla h\,\mathrm{d}x \\
\geq& I[u] + \frac{1}{C}\|\nabla h\|_{L^2(U)} \geq I[u] + \frac{1}{C}\|h\|^2_{L^2(U)}
\end{aligned}$$

by Poincaré. This shows that $I[v]$ is minimal if and only if $h \equiv 0$, i.e. $v = u$. □

In the case of Neumann-boundary conditions we obtain a similar result.

Proposition 1.1.19 (Neumann boundary conditions)**.** *Assume that the conditions of Proposition 1.1.15 are satisfied. Then u is a minimizer of*

$$I[u] = \frac{1}{2}\int_U \nabla u \cdot A\nabla u\,\mathrm{d}x - \int_U f\,u\,\mathrm{d}x - \int_{\partial U} g\,u\,\mathrm{d}\mathcal{H}^{d-1}.$$

Proof. Exercise. □

1.2 Homogenization Theory

The aim of homogenization theory is to establish the macroscopic behaviour of a system which is 'microscopically' heterogeneous. This means that the heterogeneous material is replace by a fictitious, homogeneous medium. In the context of elliptic pdes we assume that the coefficients $A \in L^\infty\left(\mathbb{R}^d, \mathbb{R}^{d\times d}\right)$ oscillate very rapidly. This may serve as a model for problems where the material properties have a complicated microstructure. Examples are composite materials (eg. plywood), or distributed inclusions. A key aspect of homogenization theory lends itself to obvious generalizations which cover situations where the kinematics of microscopic scale and the macroscopic scale are quite different. Examples of such systems are atomistic models that are approximated by continuum models.

We will assume that A is periodic and account for the presence of two scales by setting

$$A^\epsilon(x) = A(x/\epsilon).$$

The objective is to study the asymptotic behavior of solutions u^ϵ of the elliptic boundary value problem

$$\begin{cases} -\text{div } (A^\epsilon \nabla u^\epsilon) = f & \text{in } U, \\ u^\epsilon = 0 & \text{on } \partial U \end{cases} \tag{1.16}$$

as $\epsilon \to 0$. We will always assume that $f \in H^{-1}(U)$ is given, and the Y-periodic matrix $A \in L^\infty\left(\mathbb{R}^d, \mathbb{R}^{d\times d}\right)$ satisfies the bound (1.1). A function a is Y-periodic if $x - y \in Y\,\mathbb{Z}^d$ implies that $a(x) = a(y)$.

Proposition 1.1.13 implies that there exists a constant $C > 0$ with the property that (1.16) admits a unique solution $u^\epsilon \in H^1_0(U)$ such that

$$\|u_\epsilon\|_{H^1_0(U)} \leq C\,\|f\|_{H^{-1}(U)}. \tag{1.17}$$

In particular, the right hand-side does not depend on ϵ. Banach-Alaoglu implies that there exists $u^{\text{hom}} \in H^1_0(U)$ such that

$$u^\epsilon \rightharpoonup u^{\text{hom}} \text{ as } \epsilon \to 0.$$

Observe that *a priori* the limit u^0 depends on the subsequence.

At this point the central question is:

- Does u^0 satisfy some boundary value problem in U?
- Is u^0 unique?

In order to investigate this question, let us introduce the (stress) vector

$$\xi^\epsilon = A^\epsilon \nabla u^\epsilon,$$

which satisfies

$$\int_U \xi^\epsilon \nabla v \,\mathrm{d}x =_{H^{-1}(U)} \langle f, v\rangle_{H^1_0(U)} \text{ for all } v \in H^1_0(U). \tag{1.18}$$

It follows from (1.17) that there exists ξ^0 and a subsequence (not relabeled) such that

$$\xi^\epsilon \rightharpoonup \xi^0 \text{ in } L^2(U) \text{ as } \epsilon \to 0.$$

We can pass to the limit in (1.18) and obtain that

$$\int_U \xi^0 \nabla v \,\mathrm{d}x =_{H^{-1}(U)} \langle f, v\rangle_{H^1_0(U)} \text{ for all } v \in H^1_0(U),$$

i.e.

$$-\text{div}\,(\xi^0) = f \text{ in } U. \tag{1.19}$$

If A^ϵ converged strongly to $\hat{A}$ in $L^\infty(U, \mathbb{R}^{d\times d})$ (which it does not), then one could easily characterize the relation between u^0 and ξ^0. Indeed,

$$\lim_{\epsilon\to 0}\int_U \varphi\cdot A^\epsilon \nabla u^\epsilon \,\mathrm{d}x = \lim_{\epsilon\to 0}\int_U \varphi\cdot (A^\epsilon - A^0)\nabla u^\epsilon \,\mathrm{d}x + \lim_{\epsilon\to 0}\int_U \varphi\cdot A^0 \nabla u^\epsilon \,\mathrm{d}x = \int_U \varphi \cdot A^0 \nabla u^0 \,\mathrm{d}x.$$

The last equation holds because the last term converges thanks to the weak convergence of u^ϵ and

$$\begin{aligned}&\lim_{\epsilon\to 0}\left|\int_U \varphi\cdot (A^\epsilon - A^0)\nabla u^\epsilon \,\mathrm{d}x\right| \\ \leq&\|\varphi\|_{L^2(U)} \underbrace{\lim_{\epsilon\to 0}\|A^\epsilon - A^0\|_{L^\infty(U)}}_{=0} \underbrace{\limsup_{\epsilon\to 0}\|\nabla u^\epsilon\|_{L^2(U)}}_{<\infty} = 0.\end{aligned}$$

We will show that there exists a unique, x-independent matrix $A^{\text{hom}} \in \mathbb{R}^{d\times d}$ such that ∇u^ϵ converges weakly in $L^2(U)$ to a gradient ∇u, where u is a solution of

$$\begin{cases} -\text{div }\left(A^{\text{hom}} \nabla u\right) = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases} \tag{1.20}$$

It is remarkable that limiting problem (1.20) is local.

1.2.1 A one dimensional example

How can we find the effective matrix A? Let us consider a simple, 1-dimensional example which was studied first by Spagnolo in 1967. The result will provide a justification of (1.20) and at the same time demonstrate that the identification of A^{hom} is not a trivial task.

Let $U = (\alpha, \beta)$ be an interval in $\mathbb{R}$ and consider the ordinary differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(A(x/\epsilon)\,\frac{\mathrm{d}}{\mathrm{d}x}u^\epsilon(x)\right) = f \text{ in } (\alpha,\beta), \tag{1.21}$$

$$u^\epsilon(\alpha) = u^\epsilon(\beta) = 0. \tag{1.22}$$

We assume that A is periodic, with period-length Y. Equation (1.21) can be integrated twice, and we find that

$$u^\epsilon(x) = \int_\alpha^x \frac{1}{A(s/\epsilon)}\,(F(s) + c^\epsilon)\ \mathrm{d}s,$$

where $F(x) = \int_\alpha^x f(s)\,\mathrm{d}s$, and c^ϵ is chosen such that $u^\epsilon(b) = 0$, i.e.

$$c^\epsilon = -\frac{\int_\alpha^\beta \frac{1}{A(x/\epsilon)}\,F(x)\,\mathrm{d}x}{\int_\alpha^\beta \frac{1}{A(x/\epsilon)}\,\mathrm{d}x}.$$

It is not hard to construct a closed expression for the limit $u(x) = \lim_{\epsilon\to 0} u^\epsilon(x)$. First, we show that c^ϵ converges as $\epsilon \to 0$. This is a consequence of the fact that

$$\frac{1}{A(\cdot/\epsilon)} \rightharpoonup \frac{1}{A^{\mathrm{hom}}} \text{ in } L^2(U), \tag{1.23}$$

with

$$A^{\mathrm{hom}} = \frac{Y}{\int_0^Y \frac{1}{A(x)}\,\mathrm{d}x}. \tag{1.24}$$

Expression (1.24) is also called the harmonic average of A. The proof of (1.23) is an exercise.

The weak convergence (1.24) implies that

$$\lim_{\epsilon\to 0} c^\epsilon = c^0 = -\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathrm{d}x\,F(x) = -\int_\alpha^\beta \mathrm{d}x \int_\alpha^x \mathrm{d}s\,f(s),$$

and

$$u(x) = \frac{1}{A^{\mathrm{hom}}}\int_\alpha^x \mathrm{d}s\ \left(F(s) + c^0\right) = \frac{1}{A^{\mathrm{hom}}}\int_\alpha^x \mathrm{d}s\ \left(F(s) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathrm{d}t\,F(t)\right).$$

It is an easy exercise to check that that u indeed satisfies (1.20). A two-dimensional example by Murat and Tartar [17] shows that for $d > 1$ the homogenized matrix A^{hom} is in general different from inverse of the average of A^{-1}.

Auxiliary periodic problems

Proposition 1.1.17 implies that for each $\lambda \in \mathbb{R}^d$ there exists solutions $\chi_\lambda \in H^1_{\mathrm{per}}(U_{\mathrm{per}})$ and $w_\lambda \in H^1(U_{\mathrm{per}})$ (w_λ is *not* periodic) which solve the cell problems

$$-\mathrm{div}\ (A^*\,\nabla\chi_\lambda) = -\mathrm{div}\ (A^*\,\lambda) \ \text{ in } H^1_{\mathrm{per}}(U_{\mathrm{per}}), \tag{1.25}$$

and

$$\begin{cases} \mathrm{div}\,(A^*\nabla w_\lambda) = 0 & \text{in } U_{\mathrm{per}}, \\ w_\lambda - \lambda\cdot y \in H^1_{\mathrm{per}}(U_{\mathrm{per}}). \end{cases} \tag{1.26}$$

It is easy to see that $w_\lambda(y) = \lambda \cdot y - \chi_\lambda(y)$.

Note that two solutions χ_λ^1 and χ_λ^2 of (1.25) only differ by a constant. It will turn out that this constant is irrelevant for our purposes as we are only interested in $\nabla\chi_\lambda$.

By linearity it suffices to solve (1.25) for $\lambda = e_i$ and define

$$\chi_\lambda = \sum_{i=1}^d \lambda_i \, \chi_{e_i}.$$

The main convergence results

Theorem 1.2.1. *Let $f \in H^{-1}(U)$ and u^ϵ, u be the solutions of (1.16) and (1.20). Then, as $\epsilon \to 0$,*

$$\begin{cases} u^\epsilon \rightharpoonup u & \text{weakly in } H_0^1(U), \\ A^\epsilon \, \nabla u^\epsilon \rightharpoonup A^{\mathrm{hom}} \nabla u & \text{weakly in } L^2(U), \end{cases}$$

where A^{hom} is constant, elliptic and given by

$$(A^{\mathrm{hom}})^* \, \lambda = \frac{1}{|Y|} \int_{\mathbb{R}^d/Y} \mathrm{d}y \, A^* \, \nabla w_\lambda,$$

and w_λ solves (1.26).

The above theorem is very precise regarding the spatial oscillation of u_ϵ, but does not offer any information of the convergence rate. The next theorem provides an expansion of u_ϵ in powers of ϵ, but the error bounds are far from being optimal. The origin of the suboptimality is fact that u_ϵ satisfies boundary conditions and it is not easy to construct good approximations which also satisfy boundary conditions. On the other hand, the theorem provides a very good interior approximation.

Theorem 1.2.2. *Let $f \in H^{-1}(U)$ and u^ϵ be the solution of* (1.16) *with $A^\epsilon(x) = A(x/\epsilon)$ and A is periodic. Then u^ϵ admits the following asymptotic expansion*

$$u^\epsilon = u_0 - \epsilon \sum_{i=1}^d \hat{\chi}_i(x/\epsilon) \, \frac{\partial u_0}{\partial x_i} + \epsilon^2 \sum_{k,l=1}^d \hat{\theta}_{kl}(x/\epsilon) \, \frac{\partial^2 u_0}{\partial x_k \partial x_l} + \ldots,$$

where $\hat{\chi}_i$ and $\hat{\theta}_{kl}$ are Y-periodic solutions of the cell problems

$$\begin{aligned} -\mathrm{div}\,(A\nabla\hat{\chi}_i) &= -\sum_{j=1}^d \frac{\partial A_{ij}}{\partial y_j}, \\ -\mathrm{div}\,(A\nabla\hat{\theta}_{ij}) &= -A_{ij} - \frac{\partial}{\partial y_i}\left(\hat{\chi}_j \sum_{l=1}^d A_{il}\right) - \sum_{k=1}^d a_{i,k} \frac{\partial(\hat{\chi}_j - y_l)}{\partial y_k}. \end{aligned}$$

Moreover, if $f \in C^\infty(\bar{U})$, ∂U is of class C^∞ and, furthermore

$$\hat{\chi}_i, \hat{\theta}_{kl} \in W^{1,\infty}(U_{\mathrm{per}}), i, k, l = 1 \ldots d,$$

then there exists $C > 0$ such that

$$\left\| u^\epsilon - \left(u_0 - \epsilon \sum_{i=1}^d \hat{\chi}_i(x/\epsilon) \, \frac{\partial u_0}{\partial x_i} + \epsilon^2 \sum_{k,l=1}^d \hat{\theta}_{kl}(x/\epsilon) \, \frac{\partial^2 u_0}{\partial x_k \partial x_l} \right) \right\| \leq C \sqrt{\epsilon}.$$

Proof of Theorem 1.2.1. The proof, which goes back to Tartar [20] relies on the construction of oscillating test functions obtained by periodizing the solutions of the problem (1.25). As we will see in the proof, the fact that (1.25) contains the adjoint operator $-\mathrm{div}\,(A^*\nabla)$ is the key point in this method. Indeed, when trying to identify the limit ξ^0 in (1.19), this essential fact allows us to eliminate all the terms containing a product of two weakly convergent sequences. By this method we naturally obtain the homogenized matrix $A^{\hom}$ and the cell problems (1.25).

The characterization of the key steps is really Luc Tartar's contribution. He always thought that Applied Mathematics can significantly benefit from the identification of underlying structures. This is opposed to the view that Applied Mathematics is just a collection of methods.

Luc Tartar's PhD supervisor was Jacques-Louis Lions who died in 2001. It can be argued that the mathematical field 'Applied Analysis' largely owes its existence to J.L. Lions. He had immense vision and wrote more than 400 scientific publications including many books.

Let us briefly recall the weak convergence framework. There exists a subsequence (not relabelled) such that as $\epsilon \to 0$ the following convergences hold:

$$\begin{cases} u^\epsilon \rightharpoonup u^0 \text{ in } H_0^1(U), \\ \nabla u^\epsilon \rightharpoonup \nabla u^0 \text{ in } L^2(U), \\ \xi^\epsilon = A^\epsilon \nabla u^\epsilon \rightharpoonup \xi^0 \in L^2(U, \mathbb{R}^d). \end{cases} \tag{1.27}$$

Recall that ξ^ϵ satisfies

$$\int_U \xi^\epsilon \cdot \nabla v \,\mathrm{d}x = \langle f, v \rangle \text{ for all } v \in H_0^1(U). \tag{1.28}$$

Theorem 1.2.1 is proven if we show that

$$\xi^0 = A^{\hom} \nabla u^0 \tag{1.29}$$

since

$$\int_U \nabla v \cdot \xi^0 \,\mathrm{d}x = \lim_{\epsilon \to 0} \int_U \nabla v \cdot \xi^\epsilon \,\mathrm{d}x = \langle f, v \rangle.$$

As the solutions χ and u of (1.25) and (1.20) are unique (χ is unique up to an irrelevant constant) the convergences in (1.27) do not depend on the choice of the subsequence.

Since $\chi_\lambda \in L^2(U_{\text{per}})$ we find that

$$w_\lambda^\epsilon(x) = \lambda \cdot x - \epsilon \chi_\lambda(x/\epsilon)$$

satisfies the following convergences

$$\begin{cases} w_\lambda^\epsilon \rightharpoonup \lambda \cdot x & \text{weakly in } H_0^1(U), \\ w_\lambda^\epsilon \to \lambda \cdot x & \text{strongly in } L^2(U). \end{cases} \tag{1.30}$$

Next, we introduce the vector function

$$\eta_\lambda^\epsilon(x) = (A^\epsilon)^* \nabla w_\lambda^\epsilon = (A^* \nabla_y w_\lambda)(x/\epsilon).$$

A standard result for periodic functions states that if $f : \mathbb{R}^d \to \mathbb{R}$ is Y-periodic such that $f \in L^p(U_{\mathrm{per}})$ ($p \in [1, \infty]$) then for every open and bounded $\omega \subset \mathbb{R}^d$ the sequence $f^\epsilon \in L^p(\omega)$ which is defined by $f^\epsilon(x) = f(x/\epsilon)$ converges weakly (weak-* if $p = \infty$) to the constant function

$$m = \frac{1}{U_{\mathrm{per}}} \int_U f \, \mathrm{d}x.$$

The periodicity of η_λ^ϵ together with the the definition of A^{hom} implies that

$$\eta_\lambda^\epsilon \rightharpoonup (A^{\mathrm{hom}})^* \lambda \text{ weakly in } L^2(U, \mathbb{R}^d). \tag{1.31}$$

Next we observe that (1.26) implies that $\mathrm{div}\,(\eta_\lambda^\epsilon) = 0$ holds in the weak sense, i.e.

$$\int_U \eta_\lambda^\epsilon \cdot \nabla v \, \mathrm{d}x = 0 \text{ for all } v \in H_0^1(U). \tag{1.32}$$

Let now $\varphi \in C_c^\infty(U)$ and choose $\varphi\, w_\lambda^\epsilon \in H_0^1(U)$ as a testfunction in (1.28) and $\varphi\, u^\epsilon \in H_0^1(U)$ as a testfunction in (1.32). Then we obtain the following identities:

$$\int_U \xi^\epsilon \cdot \nabla w_\lambda^\epsilon \, \varphi \, \mathrm{d}x + \int_U \xi^\epsilon \cdot \nabla \varphi \, w_\lambda^\epsilon \, \mathrm{d}x = \langle f, \varphi\, w_\lambda^\epsilon \rangle, \tag{1.33}$$

$$\int_U \eta_\lambda^\epsilon \cdot \nabla u^\epsilon \, \varphi \, \mathrm{d}x + \int_U \eta_\lambda^\epsilon \cdot \nabla \varphi \, u^\epsilon \, \mathrm{d}x = 0. \tag{1.34}$$

The definitions of ξ^ϵ and η_λ^ϵ implies that

$$\xi^\epsilon \cdot \nabla\, w_\lambda^\epsilon = A^\epsilon \, \nabla u^\epsilon \cdot \nabla w_\lambda^\epsilon = (A^\epsilon)^* \nabla w_\lambda^\epsilon \cdot \nabla u^\epsilon = \eta_\lambda^\epsilon \cdot \nabla u^\epsilon.$$

Subtraction of (1.34) from (1.33) yields

$$\int_U \xi^\epsilon \cdot \nabla \varphi \, w_\lambda^\epsilon \, \mathrm{d}x - \int_U \eta_\lambda^\epsilon \cdot \nabla \varphi \, u^\epsilon \, \mathrm{d}x = \langle f, \varphi\, w_\lambda^\epsilon \rangle. \tag{1.35}$$

We will show next that due to the strong L^2-convergence of w_λ^ϵ and u^ϵ we can pass to the limit in this identity.

Let us point out here the main idea of Tartar's method, namely the use of the adjoint problem in the definition. It is precisely this fact which allows one to cancel the two terms where one cannot identify the limit of the product of two only weakly converging sequences. We will show now that we can pass to the limit for the other terms and the limit expression easily delivers the claimed identity (1.29).

Take $\epsilon \to 0$ in (1.35). Since ξ^ϵ converges weakly in $L^2(U)$ to ξ^0 (eq. (1.27)) and w_λ^ϵ converges strongly to $\lambda \cdot x$ in $L^2(U)$ (eq. (1.30)), one obtains that

$$\lim_{\epsilon \to 0} \int_U w^\epsilon \, \xi^\epsilon \cdot \nabla \varphi \, \mathrm{d}x = \int_U (\lambda \cdot x)\, \xi^0 \cdot \nabla \varphi \, \mathrm{d}x.$$

Furthermore, since η_λ^ϵ converges weakly in L^2 to $(A^{\mathrm{hom}})^* \lambda$ (eq. (1.31)) and u^ϵ converges strongly in L^2 to u^0 (eq. 1.27) one finds that

$$\lim_{\epsilon \to 0} \int_U \eta_\lambda^\epsilon \cdot \nabla \varphi \, u^\epsilon \, \mathrm{d}x = \int_U (A^{\mathrm{hom}})^* \lambda \cdot \nabla \varphi \, u^0.$$

Finally, (1.35) and the weak convergence in $H^1(U)$ of w_λ^ϵ to $\lambda \cdot x$ (eq. (1.30)) implies that

$$\int_U (\lambda \cdot x)\, \xi^0 \cdot \nabla \varphi \, \mathrm{d}x - \int_U (A^{\mathrm{hom}})^* \lambda \cdot \nabla \varphi \, u^0 \, \mathrm{d}x = \langle f, (\lambda \cdot x)\, \varphi \rangle \text{ for all } \varphi \in C_0^\infty(U),$$

which can be rewritten in the form

$$\int_U \xi^0 \cdot \nabla[(\lambda \cdot x)\,\varphi]\,\mathrm{d}x - \int_U \xi^0 \cdot \lambda\,\varphi\,\mathrm{d}x - (A^{\mathrm{hom}})^*\lambda \cdot \nabla\varphi\,u^0\,\mathrm{d}x = \langle f, (\lambda \cdot x)\,\varphi\rangle \text{ for all } \varphi \in C_0^\infty(U).$$

This gives, by using the testfunction $v(x) = (\lambda \cdot x)\,\varphi$ in the equation $\int_U \xi^0 \cdot \nabla v\,\mathrm{d}x = \langle f, v\rangle$, the identity

$$\int_U \xi^0 \cdot \lambda\,\varphi\,\mathrm{d}x = -\int_U (A^{\mathrm{hom}})^*\lambda \cdot \nabla\varphi\,u^0\,\mathrm{d}x.$$

Integration by part yields

$$\int_U \xi^0 \cdot \lambda\,\varphi\,\mathrm{d}x = \int_U (A^{\mathrm{hom}})^*\lambda \cdot \nabla u^0\,\varphi\,\mathrm{d}x.$$

As φ is arbitrary this implies that the integrands coincide, i.e.

$$\xi^0 \cdot \lambda = (A^{\mathrm{hom}})^*\lambda \cdot \nabla u_0 = A^{\mathrm{hom}}\nabla u^0 \cdot \lambda,$$

which gives (1.29) since $\lambda \in \mathbb{R}^d$ is arbitrary. The proof of Theorem 1.2.1 is finished. □

Homogenization theory is mostly concerned with the study of the properties of the homogenized coefficients A^{hom}. Here we establish only the most basic facts.

Proposition 1.2.1. *1. The components of A^{hom} admit the representation formula*

$$A_{ij}^{\mathrm{hom}} = \frac{1}{|Y|}\int_{U_{\mathrm{per}}} \nabla w_{e_j} \cdot A\nabla w_{e_i}\,\mathrm{d}x. \tag{1.36}$$

2. The homogenized matrix A^{hom} is elliptic, i.e. there exists a constant $\beta^{\mathrm{hom}} > 0$ such that

$$\xi \cdot A^{\mathrm{hom}}\xi \geq \beta^{\mathrm{hom}}\,|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d. \tag{1.37}$$

Proof. 1. Define $w_j = w_{e_j}$ and recall that $w_j - y_j \in H^1_{\mathrm{per}}(U_{\mathrm{per}})$. Test the equation $\operatorname{div}(A^*\nabla w_\lambda) = 0$ with $w_j - y_j$:

$$\begin{aligned}
0 =& -\frac{1}{|U_{\mathrm{per}}|}\int_{U_{\mathrm{per}}} (w_j - y_j)\operatorname{div}(A^*\nabla w_\lambda)\,\mathrm{d}y \\
=&\frac{1}{|U_{\mathrm{per}}|}\int_{U_{\mathrm{per}}} (\nabla w_j - e_j) \cdot A^*\nabla w_\lambda\,\mathrm{d}y - \underbrace{\int_{\partial U} (w_j - y_j)\nu \cdot A^*\nabla w_\lambda\,\mathrm{d}\mathcal{H}^{d-1}}_{=0 \text{ since } w_i - y_i \text{ is periodic}} \\
=&\frac{1}{|U_{\mathrm{per}}|}\int_U \nabla w_j \cdot A^*\nabla w_\lambda\,\mathrm{d}y - \lambda \cdot A^{\mathrm{hom}}e_j.
\end{aligned}$$

This implies that

$$\int_U \nabla w_j \cdot A^*\nabla w_\lambda\,\mathrm{d}y = \lambda \cdot A^{\mathrm{hom}}e_j$$

and thus formula (1.36) if $\xi = e_i$.

2. The first statement and the fact that $\chi_\lambda = \lambda \cdot y - w_\lambda \in H^1_{\mathrm{per}}(U_{\mathrm{per}})$ delivers the formula

$$\begin{aligned}
\xi \cdot A^{\mathrm{hom}}\xi =& \frac{1}{|Y|}\int_{U_{\mathrm{per}}} \nabla w_\xi \cdot A\nabla w_\xi\,\mathrm{d}y \geq \frac{\beta}{|Y|}\int_{U_{\mathrm{per}}} |\nabla w_\xi|^2\,\mathrm{d}y \\
=&\frac{\beta}{|Y|}\int_{U_{\mathrm{per}}} |\xi|^2\mathrm{d}y - 2\frac{\beta}{|Y|}\underbrace{\int_{U_{\mathrm{per}}} \nabla\chi_\xi \cdot \xi\,\mathrm{d}y}_{=0} + \frac{\beta}{|Y|}\int_{U_{\mathrm{per}}} \underbrace{|\nabla\chi_\xi|^2}_{\geq 0}\,\mathrm{d}y \geq \beta\,|\xi|^2.
\end{aligned}$$

□

An interesting consequence of Theorem 1.2.1 is the convergence of the energy associated to problem (1.16), namely the quantity

$$E^\epsilon[u^\epsilon] = \int_U \nabla u^\epsilon \cdot A \nabla u^\epsilon.$$

Proposition 1.2.2. *Let u^ϵ be the solution of (1.16). Then*

$$\lim_{\epsilon\to 0} E^\epsilon[u^\epsilon] = E^0[u^0] = \int_U \nabla u^0 \cdot A^0 \nabla u^0 \, \mathrm{d}x,$$

where A^{hom} and u^0 are given by Theorem 1.2.1.

Proof. The weak formulation of (1.16) is

$$B^\epsilon[u^\epsilon, v] = \int f \, v \, \mathrm{d}x \text{ for all } v \in H_0^1(U),$$

with $B^\epsilon[u,v] = \int_U \nabla u^\epsilon \cdot A^\epsilon \nabla u^\epsilon \, \mathrm{d}x$. Since $u^\epsilon \in H_0^1(U)$ we can choose $v = u^\epsilon$ and find that

$$\lim_{\epsilon\to 0} B[u^\epsilon, u^\epsilon] = \lim_{\epsilon\to 0} \int f \, u^\epsilon \, \mathrm{d}x = \int_U f \, u^0 \, \mathrm{d}x. \tag{1.38}$$

On the other hand we can test (1.20) with $u^0 \in H_0^1(U)$ and find that

$$B^0[u^0, u^0] = \int_U f \, u^0 \, \mathrm{d}x,$$

with $B^0[u,v] = \int_U \nabla u \cdot A^{\mathrm{hom}} \nabla v$. Together with (1.38) this implies the claim

$$\lim_{\epsilon\to 0} B^\epsilon[u^\epsilon, u^\epsilon] = B^0[u^0, u^0].$$

□

Actually it is not hard to see that the energy density converges, not just the energy.

Proposition 1.2.3. *Let u^ϵ be the solution of (1.16) and $\varphi \in C^1(\bar{U})$ such that $\varphi = 0$ on ∂U. Then*

$$\lim_{\epsilon\to 0} \int_U \varphi \nabla u^\epsilon \cdot A^\epsilon \nabla u^\epsilon \, \mathrm{d}x = \int_U \varphi \nabla u^0 \cdot A^0 \nabla u^0 \, \mathrm{d}x, \tag{1.39}$$

where A^{hom} and u^0 are given by Theorem 1.2.1.

Proof. Using $\varphi \, u^\epsilon$ as a testfunction in the variational formuation (1.28) yields

$$\begin{aligned}
\int_U \varphi \nabla u^\epsilon \cdot A^\epsilon \nabla u^\epsilon \, \mathrm{d}x &= \int_U \nabla(\varphi \, u^\epsilon) \cdot A^\epsilon \nabla u^\epsilon \, \mathrm{d}x - \int_U u^\epsilon \, \nabla\varphi \cdot A^\epsilon \, \nabla u^\epsilon \, \mathrm{d}x \\
&= {}_{H^{-1}}\langle f, \varphi \, u^\epsilon \rangle_{H^1} - \int_U u^\epsilon \, \nabla\varphi \cdot A^\epsilon \nabla u^\epsilon \, \mathrm{d}x \\
&= {}_{H^{-1}}\langle f, \varphi \, u^\epsilon \rangle_{H^1} - \int_U u^\epsilon \, \nabla\varphi \cdot \xi^\epsilon \, \mathrm{d}x
\end{aligned}$$

Recall that $\varphi \, u^\epsilon \rightharpoonup \varphi \, u^0$ weakly in $H^1(U)$ as $\epsilon \to 0$. Thanks to Rellich's theorem this implies that $\varphi \, u^\epsilon \to \varphi \, u^0$ strongly in L^2. Thus we can pass to the limit in the above equations and find that

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_U \varphi \nabla u^\epsilon \cdot A^\epsilon \nabla u^\epsilon \, \mathrm{d}x &= {}_{H^{-1}}\langle f, \varphi \, u^0 \rangle_{H^1} - \int_U u^0 \, \nabla\varphi \cdot \xi^0 \, \mathrm{d}x \\
&= {}_{H^{-1}}\langle f, \varphi \, u^0 \rangle_{H^1} - \int_U \nabla(\varphi \, u^0) \cdot \xi^0 \, \mathrm{d}x + \int_U \varphi \nabla u^0 \cdot \xi^0 \, \mathrm{d}x
\end{aligned} \tag{1.40}$$

Next, we test eqn. (1.20) with φu^0 and obtain

$$\int_U \nabla(\varphi u^0) \cdot \xi^0 \, \mathrm{d}x = \langle f, \varphi u^0 \rangle,$$

together with (1.40) this implies that

$$\lim_{\epsilon \to 0} \int_U \nabla u^\epsilon \cdot A^\epsilon \nabla u^\epsilon \, \mathrm{d}x = \int_U \varphi \nabla u \cdot \xi^0 \, \mathrm{d}x.$$

Eqn. (1.39) has been established since $\xi^0 = A^{\mathrm{hom}} \nabla u^0$ (see (1.29). □

An very interesting generalization of Theorem 1.2.2 is given by the famous Div–Curl Lemma by Murat and Tartar.

Notation: If $w \in L^2(U, \mathbb{R}^d)$ we define curl $w \in H^{-1}(U, \mathbb{R}^{d\times d})$ by

$$(\mathrm{curl}\ w)_{i,j} = \frac{\partial}{\partial x_j} w_i - \frac{\partial}{\partial x_i} w_j, \quad i, j = 1, \ldots, d.$$

Theorem 1.2.4 (Div-Curl Lemma). *Assume that ∂U is C^2 and $v^n \rightharpoonup v$ and $w^n \rightharpoonup w$ weakly in $L^2(U, \mathbb{R}^d)$ and satisfy*

$$\{\mathrm{div}\ v^n \ : \ n = 1 \ldots \infty\} \text{ is precompact in } H^{-1}(U), \tag{1.41}$$

$$\{\mathrm{curl}\ w^n \ : \ n = 1 \ldots \infty\} \text{ is precompact in } H^{-1}(U, \mathbb{R}^{d\times d}). \tag{1.42}$$

Then

$$\lim_{n\to\infty} \int_U (v^n \cdot w^n - v \cdot w)\, \varphi \, \mathrm{d}x = 0$$

for all $\varphi \in C^1(\bar{U})$ such that $\varphi = 0$ on ∂U.

Proof. 1. Define the sequence $u^n \in H^2(U, \mathbb{R}^n)$ as the unique solution of the Poisson equation

$$\begin{cases} -\Delta u^n = w^n & \text{in } U, \\ u^n = 0 & \text{on } \partial U. \end{cases} \tag{1.43}$$

This requires an application of regularity theory which states that under rather mild assumptions weak solutions of (1.43) are in fact in $H^2(U)$ (i.e. strong solutions), see e.g. [11, Theorem 6.3.4]. The regularity result also provides the existence of $C > 0$ such that $\|u^n\|_{H^2(U)} \le C \|w^n\|_{L^2(U)}$.

2. Now set $z^n = -\mathrm{div}\ u^n$ and $y^n = w^n - \nabla z^n$. Then $\limsup_{n\to\infty} \|z^n\|_{H^1(U)} < \infty$. Additionally

$$y_i^n = w_i^n - \frac{\partial z^n}{\partial x_i} = -\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} u_i^n + \sum_{j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} u_j^n = \sum_{j=1}^d \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_i} u_j^n - \frac{\partial}{\partial x_j} u_i^n \right). \tag{1.44}$$

In view of assumption (1.42) we infer from (1.43) that curl u^n lies in a compact subset of $H^1_{\mathrm{loc}}(U, \mathbb{R}^{d\times d})$. Thus from (1.44) it follows that y^n is contained in a compact subset of $L^2_{\mathrm{loc}}(U, \mathbb{R}^d)$.

3. We may suppose, upon passing to subsequences as necessary, that

$$z^n \rightharpoonup z \text{ weakly in } H^1(U) \text{ and } y^n \to y \text{ strongly to } L^2_{\mathrm{loc}}(U, \mathbb{R}^d), \tag{1.45}$$

where $z = -\mathrm{div}\, u$, $y = w - \nabla z$ for $u \in H^2(U, \mathbb{R}^d)$ solving

$$\begin{cases} -\Delta u = w & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

4. Now observe

$$\int_U v^n \cdot w^n \,\varphi \,\mathrm{d}x = \int_U v^n \cdot (y^n + \nabla z^n)\,\varphi \,\mathrm{d}x.$$

According to (1.45)

$$\lim_{n\to\infty} \int_U v^n \cdot y^n \,\varphi \,\mathrm{d}x = \int_U v \cdot y \,\varphi \,\mathrm{d}x.$$

In addition, assumption (1.41) and (1.45) allow us to compute

$$\lim_{n\to\infty} \int_U v^n \cdot \nabla z^n \,\varphi \,\mathrm{d}x = - \lim_{n\to\infty} \left(\int_U v^n \cdot \nabla\varphi \, z^n \,\mathrm{d}x + \langle \mathrm{div}\ v^n, z^n \,\varphi \rangle \right) \tag{1.46}$$

$$= - \int_U v \cdot \nabla\varphi \, z \,\mathrm{d}x - \langle \mathrm{div}\ v, z \,\varphi \rangle = \int_U v \cdot \nabla z \,\varphi \,\mathrm{d}x. \tag{1.47}$$

Thus

$$\lim_{n\to\infty} \int_U v^n \cdot w^n \,\varphi \,\mathrm{d}x = \int_U v \cdot (y + \nabla z)\,\varphi \,\mathrm{d}x = \int_U v \cdot w \,\varphi \,\mathrm{d}x$$

and the proof is finished.

□

Chapter 2

Calculus of Variations

An important tool for the study of PDE are scalar quantities which depend on the solutions. Examples are Liapunov functions, or energies whose minimizers satisfy PDE. Calculus of Varations provides the framework where the link between many PDE and scalar quantities can be studied in detail.

2.1 Convex Analysis

Definition 2.1.1.

1. A set $\Omega \subset \mathbb{R}^d$ is convex if for every $x, y \in \Omega$ and every $\lambda \in [0,1]$ we have $\lambda\, x + (1-\lambda)\, y \in \Omega$

2. Let $\Omega \subset \mathbb{R}^d$ be convex. The function $f : \Omega \to \mathbb{R}$ is convex if

$$f(\lambda\, x + (1-\lambda)\, y) \leq \lambda\, f(x) + (1-\lambda)\, f(y)$$

for all $x, y \in \Omega$, $\lambda \in [0,1]$.

Theorem 2.1.2. *Let $\Omega \subset \mathbb{R}^d$ be open, convex and $f : \Omega \to \mathbb{R}$.*

1. *f is convex if and only if for every open and bounded set $U \subset \Omega$ and every $u \in L^1(U)$* **Jensen's inequality**

$$f\left(\frac{1}{|U|}\int_U u(x)\,\mathrm{d}x\right) \leq \frac{1}{|U|}\int_U f(u(x))\,\mathrm{d}x$$

holds.

2. *If $f \in C^1(\Omega)$ then f is convex if and only if*

$$f(x) \geq f(y) + \nabla f(y) \cdot (x-y)$$

for all $x, y \in \Omega$.

3. *If $f \in C^2(\Omega)$, then f is convex if and only if the Hessian $\nabla^2 f(x) \in \mathbb{R}^{d\times d}$ is positive definite.*

Proof. Exercise. □

We study now functionals of the form

$$I[u] = \int_U W(x, u(x), \nabla u(x))\,\mathrm{d}x,$$

where $W : U \times \mathbb{R} \times \mathbb{R}^d$ is an energy density.

The central problem is motivated by the definition

$$m = \inf\{I[u] \ : \ u \in W_0^{1,p}(U)\},$$

where $W_0^{1,p}(U) = \overline{C_0^1(\bar{U})}_{\|\cdot\|_{W^{1,p}(U)}}$ and

$$\|u\|^p_{W^{1,p}(U)} = \int_U |u|^p \, \mathrm{d}x + \sum_{i=1}^d \int_U \left|\frac{\partial u}{\partial x_i}\right|^p \, \mathrm{d}x.$$

We will address the following questions:

1. Existence and uniqueness: Is m a minimum? If yes, is the minimum unique?
2. Properties of minimizers: If $I[u] = m$, what are the properties of u? Does u satisfy necessary conditions?

2.2 Euler-Lagrange equations

First we demonstrate the minimizers of I satisfy the Euler-Lagrange equations.

Theorem 2.2.1. *Let $U \subset \mathbb{R}^d$ be open and bounded such that ∂U is Lipschitz $p > 1$ and $W \in C^1(\bar{U} \times \mathbb{R} \times \mathbb{R}^d)$ such that*

$$|W_u(x,u,\xi)|\,, |W_\xi(x,u,\xi)| \le C\left(1 + |u|^{p-1} + |\xi|^{p-1}\right) \tag{H3}$$

for each $(x,u,\xi) \in \bar{U} \times \mathbb{R} \times \mathbb{R}^d$, where $W_u = \frac{\partial W}{\partial u}$ and $W_\xi = (\frac{\partial f}{\partial \xi_i})_{i=1\ldots d}$.

1. *If $u \in W_0^{1,p}(U)$ is a minimizer of I (i.e. $m = I[u]$), then u satisfies the weak Euler-Lagrange equations*

$$\int_U \left(\varphi \, W_u(x,u,\nabla u) + \nabla\varphi \cdot W_\xi\right) \mathrm{d}x = 0 \text{ for all } \varphi \in W_0^{1,p}(U). \tag{2.1}$$

2. *if $W \in C^2(\bar{U} \times \mathbb{R} \times \mathbb{R}^d)$ and $u \in C^2(\bar{U})$, then u satisfies the Euler-Lagrange equation*

$$-\mathrm{div}\left(W_\xi(x,u,\nabla u)\right) + W_u(x,u,\nabla u) = 0 \text{ in } U. \tag{2.2}$$

Conversely, if $(u,\xi) \to W(x,u,\xi)$ is convex for every $x \in U$ and if u solves either (2.1) or (2.2), then u is a minimizer of I.

Remark 2.2.2.
1. Assumption (H3) implies that $\varphi \, W_u$ and $\nabla\varphi \cdot W_\xi$ are both in $L^1(U)$. Without (H3) statement (2.1) does not make sense.
2. (2.2) implies (2.1). The converse holds if u is sufficiently regular.

Proof. Assumption (H3) and the observation that

$$W(x,u,\xi) = W(x,0,0) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} W(x,t\,u,t\,\xi)\,\mathrm{d}t$$

implies that there exists $C > 0$ such that

$$|W(x,u,\xi)| \le C\,\left(1 + |u|^p + |\xi|^p\right) \text{ for all } (x,u,\xi) \in U \times \mathbb{R} \times \mathbb{R}^d.$$

In particular

$$|I[u]| < \infty \text{ for all } u \in W^{1,p}(U).$$

Next we prove that

$$\lim_{h\to 0} \frac{1}{h} \left(I[u + h\,\varphi] - I[u] \right) = \int_U \left(\varphi\, W_u(x, u, \nabla u) + \nabla\varphi \cdot W_\xi(x, u, \nabla u) \right) \mathrm{d}x. \tag{2.3}$$

First, define

$$g(x, h) = \frac{1}{h} \left(W(x, u(x) + h\,\varphi(x), \nabla u(x) + h\,\nabla\varphi(x)) - W(x, u(x), \nabla u(x)) \right).$$

Clearly

$$\lim_{h\to 0} g(x, h) = g(x, 0) = \varphi(x)\, W_u(x, u, \nabla u) + \nabla\varphi(x) \cdot W_\xi(x, u, \nabla u).$$

Since $W \in C^1$ we have for almost every $x \in U$ that $h \mapsto g(x, h)$ is C^1 and therefore there exists $\theta(x) \in [-|h|, |h|]$ such that

$$g(x, h) = \frac{\partial}{\partial h} \eta(x, \theta(x)),$$

with $\eta(x, h) = W(x, u(x) + h\,\varphi(x), \nabla u(x) + h\,\nabla\varphi(x))$. Moreover, another application of (H3) shows that there exists $C > 0$ such that

$$|g(x, h)| \leq \left| \frac{\partial g}{\partial h}(x, \theta) \right| = C\,(1 + |u|^p + |\varphi|^p + |\nabla u|^p + |\nabla\varphi|^p) = G(x) \text{ for all } x \in U.$$

Note that $G \in L^1(U)$ because $u, \varphi \in W^{1,p}(U)$. Summarizing the results we have that

$$\begin{aligned} g(x, h) &\in L^1(U), \\ |g(x, h)| &\leq G(x) \text{ with } G \in L^1(U), \\ \lim_{h\to 0} g(x, h) &= g(x, 0) \text{ a.e. in } U. \end{aligned}$$

Lebesgue's dominated convergence theorem implies that (2.3) holds.

Now we are in a position to derive (2.1) and (2.2). The minimality of u implies that

$$\frac{1}{h} \left(I[u + h\,\varphi] - I[u] \right) \geq 0 \text{ for each } h \in \mathbb{R}, \varphi \in W_0^{1,p}(U).$$

Taking the limit $h \to 0$ and using (2.3) delivers (2.1).

To get (2.2) it remains to integrate by parts and to find

$$\int_U \left[W_u(x, u, \nabla u) - \operatorname{div} W_\xi(x, u, \nabla u) \right] \varphi\, \mathrm{d}x \text{ for all } \varphi \in W_0^{1,p}(U).$$

The fundamental lemma of the Calculus of Variations implies that (2.2) holds.

Now we prove the converse. Assume that (2.1) holds. From the convexity of W we deduce that

$$W(x, v, \nabla v) \geq W(x, u, \nabla u) + W_u(x, u, \nabla u) \cdot (v - u) + W_\xi(x, u, \nabla u) \cdot (\nabla v - \nabla u).$$

Integrating, using (2.1) and the fact that $u - \bar{u} \in W_0^{1,p}(U)$ we get immediately that $I[v] \geq I[u]$. □

Examples:

1. If $A \in L^\infty(U, \mathbb{R}^{d\times d})$ is symmetric a.e. $x \in U$, $f \in L^2(U)$ and $W(x, u, \xi) = \frac{1}{2}\xi \cdot A(x)\xi - f\,u$, then the associated Euler-Lagrange equation reads

$$-\mathrm{div}\,(A\nabla u) = f.$$

Clearly I has no minimizers if A is not positive semi-definite. If A is the identity matrix, then the identifcation of minimizers of I with solutions of the Poisson equation is called Dirichlet's principle.

2. If $W(x,u,\xi) = \frac{1}{2}|\xi|^2 - G(u)$, $G \in C^1(\mathbb{R})$, then the Euler-Lagrange equation reads
$$-\Delta u = g(u),$$
where $g = G'$.

3. If $W(x,u,\xi) = \frac{1}{p}|\xi|^p - f\,u$, then the Euler-Lagrange equation reads
$$-\Delta_p(u) = f,$$
where $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian.

4. If $W(x,u,\xi) = \sqrt{1+|\xi|^2}$, then
$$I[u] = \int_U \sqrt{1+|\nabla u|^2}\,\mathrm{d}x$$
is the $d-1$ dimensional area of the graph of u. The associated Euler-Lagrange equation
$$\operatorname{div}\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\nabla u\right) = 0$$
is called Minimal surface equation.

2.3 The direct method

We address now the question in which cases I attains the minimum.

Theorem 2.3.1. *Let $U \subset \mathbb{R}^d$ be open and bounded such that ∂U is Lipschitz and assume that $W \in C(\bar{U}\times\mathbb{R}\times\mathbb{R}^d)$ has the following properties*

$$\xi \to W(x,u,\xi) \text{ is convex for every } (x,u) \in U\times\mathbb{R}, \tag{H1}$$
$$\exists\, p > q \geq 1, \alpha_1 > 0, \alpha_2, \alpha_3 \in \mathbb{R} \text{ such that } W(x,u,\xi) \geq \alpha_1|\xi|^p + \alpha_2|u|^q + \alpha_3 \text{ for all } x,u,\xi. \tag{H2}$$

If $m = \inf_{u\in W_0^{1,p}(U)} I[u] < \infty$, then there exists a minimizer $u \in W_0^{1,p}(U)$ such that $m = I[u]$.

If $(u,\xi) \to W(x,u,\xi)$ is strictly convex, i.e.
$$\lambda\, f(x,u,\xi) + (1-\lambda)\, f(x,v,\eta) > f(x, \lambda u + (1-\lambda)\, v, \lambda\,\xi + (1-\lambda)\,\eta)$$
for every $\lambda \in (0,1)$, $x \in U$, then the minimizer is unique.

Remark 2.3.2.
1. The assumptions of the theorem are nearly optimal in the sense that weakening any of them leads to a counterexample to the existence of minima (see below). The only exception is the continuity of f with respect to x.
2. Uniqueness holds if $(u,\xi) \to W(x,u,\xi)$ is strictly convex.
3. The theorem also holds in the vectorial case where $u \in W^{1,p}(U,\mathbb{R}^m)$. However, if $d, m > 1$, then assumption (H1) is far from optimal.
4. The case $p = 1$ is not covered by the theorem because the Sobolev space $W^{1,1}(U)$ is not reflexive.

Examples

1. If $p > 1$, $f \in L^{p'}(U)$ and $W(x,u,\xi) = \frac{1}{p}|\xi|^p - f(x)\,u$, then the assumptions of Theorem are satisfied. If $p = 2$ this proves existence of weak solutions of the Poisson equation
$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

2. The Minimal surface integrand $W(\xi) = \sqrt{1+|\xi|^2}$ satisfies (H1) but not (H2) as $p = 1$ is not permitted. This observation means that existence of solutions of the minimal surface equation requires more sophisticated methods. In 1936 Jesse Douglas won the very first Fields medal for achieving this result.

3. We demonstrate that without convexity in ξ it cannot be expected that I admits minimizers. The standard example which illustrates the problem is due to Bolza. Let $W(x,u,\xi) = (1-|\xi|^2)^2 + u^2$.. It is not hard to see that for
$$\inf_{u \in W_0^{1,p}(U)} I[u] = 0,$$
irrespective of the value of d and the domain U. To keep the presentation simple we choose $d = 1$ and $U = (0,1)$. Define
$$u_n(x) = \begin{cases} x - \frac{k}{n} & \text{if } x \in \left[\frac{2k}{2n}, \frac{2k+1}{2n}\right], \\ \frac{k+1}{n} - x & \text{if } x \in \left[\frac{2k+1}{2n}, \frac{2k+2}{2n}\right). \end{cases}$$
Clearly $u_n'(x) \in \{\pm 1\}$ for a.e. $x \in U$ and $\|u_n\|_{L^\infty(U)} = \frac{1}{2n}$. This implies that $I[u_n] \leq \frac{1}{4\,n^2}$ and thus $\lim_{n\to 0} I[u_n] = 0$. Since $I[u] \geq 0$ for every $u \in W^{1,4}(U)$ this implies that $\inf I[u] = 0$.

   On the other hand, if $I[u] = 0$ then $\|u\|^2_{L^2(U)} = 0$, this implies that $u \equiv 0$. But since $I[0] = 1$, this is a contradiction.

Proof. We will prove the theorem only after making some simplifying assumptions. The full proof can be found in [6].

We assume that $W \in C^1(U \times \mathbb{R} \times \mathbb{R}^n)$ and

$$(u,\xi) \to W(x,u,\xi) \text{ is convex for each } x \in U, \tag{H1+}$$
$$\exists p > 1, \alpha_1 > 0, \alpha_3 \in \mathbb{R} \text{ s.t. } W(x,u,\xi) \geq \alpha_1 |\xi|^p + \alpha_3 \text{ for all } x,u,\xi \tag{H2+}$$
$$\exists \beta > 0 \text{ s.t. } |W_u(x,u,\xi)| + |W_\xi(x,u,\xi)| \leq \beta \left(1 + |u|^{p-1} + |\xi|^{p-1}\right) \text{ for all } x,u,\xi. \tag{H3+}$$

First we establish precomactness of minimizing sequences. Let $u_n \in W_0^{1,p}(U)$ be a minimizing sequence, i.e.
$$\lim_{n\to\infty} I[u_n] = m.$$
Assumption (H2+) implies that
$$m + 1 \geq I[u_n] \geq \alpha_1 \|\nabla u_n\|^p_{L^p} - \alpha_3 |U|$$
and we conclude that
$$\|u_n\|_{L^p(U)} \leq C \text{ for all } n.$$
Poincaré's inequality implies that
$$\|u_n\|_{W_0^{1,p}(U)} \leq C \text{ for all } n.$$
Banach-Alaoglu's theorem implies that there exists a subsequence (not relabeled) and $u \in W_0^{1,p}(U)$ such that
$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(U).$$
Next, we show that I is weakly lower semincontiuous, in the sense that
$$I[u] \leq \liminf_{n\to\infty} I[u_n] \tag{2.4}$$
for *any* sequence u_n that converges to u (not just minimizing sequences).

To see this we observe that thanks to assumption $W \in C^1$ and Theorem 2.1.2.2 we have the inequality
$$I[u_n] \geq \int_U W(x,u(x),\nabla u(x))\,\mathrm{d}x + \int_U W_u(x,u(x),\nabla u(x))\,(u_n - u)\,\mathrm{d}x + \int_U W_\xi(x,u(x),\nabla u(x)) \cdot (\nabla u_n - \nabla u)\,\mathrm{d}x$$
$$= I[u] + J_n^0 + J_n^1.$$

If we manage to show that

$$\lim_{n\to\infty} J_n^0 = \lim_{n\to\infty} J_n^1 = 0, \tag{2.5}$$

then

$$m = \lim_{n\to\infty} I[u_n] \le I[u].$$

Together with the assumption that u_n is a minimzing sequence the shows that u is a minimizer of I.

In order to see that (2.5) holds we first have to check that J_1 and J_2 are well defined, this follows if $W_u(x, u, \nabla u), W_\xi(x, u, \nabla u) \in L^{p'}(U)$. To see this we observe that

$$\begin{aligned} \int_U |W_u(x, u(x), \nabla u(x))|^{p'} \,\mathrm{d}x &\le \beta^{p'} \int_U \left(1 + |u|^{p-1} + |\nabla u|^{p-1}\right)^{\frac{p}{p-1}} \,\mathrm{d}x \\ &\le C \,\left(1 + \|u\|_{W^{1,p}}^p\right). \end{aligned}$$

For $\int_U |W_\xi(x, u(x), \nabla u(x))|^{p'}$ one obtains a similar estimate.

Now we observe that $u_n \rightharpoonup u$ weakly in $W^{1,p}(U)$ implies that $\nabla u_n \rightharpoonup u$ weakly in $L^p(U)$ and $u_n \rightharpoonup u$ weakly in $L^p(U)$, thus $\lim\limits_{n\to\infty} J_0^0 = \lim\limits_{n\to\infty} J_n^1 = 0$. This shows that (2.4) indeed holds.

Finally we establish uniqueness. Assume that $u, v \in W_0^{1,p}(U)$ are both minimizers such that $u \neq v$ and $I[u] = I[v]$. To construct a contradiction define $w = \frac{1}{2}(u + v)$ and

$$V \subset U = \{x \,:\, u(x) = v(x)\}^c.$$

Clearly $|V| > 0$ since $u \neq v$. Furthermore, define

$$V_\epsilon = \{x \in V \,:\, \frac{1}{2} W(x, u(x), \nabla u(x)) + \frac{1}{2} W(x, u(x), \nabla u(x)) - W(x, w(x), \nabla w(x))\}.$$

Then

$$I[w] \le I[u] - \epsilon\,|V_\epsilon|. \tag{2.6}$$

The strict convexity of W implies that $\lim_{\epsilon\to 0} |V_\epsilon| = |V| > 0$, and therefore $|V_{\epsilon_0}| > 0$ for some $\epsilon_0 > 0$. Hence (2.6) implies that $I[w] < I[u]$ which contradicts the assumption the $I[u]$ is a minimizer. $\square$

2.4 Γ-convergence

Like in chapter 1 we study sequences of minimization problems which are give by functional I_ϵ. Recall the strategy we used to study homogenization problems. Find A^{hom} such that $u^\epsilon \in H_0^1(U)$ and

$$-\mathrm{div}\, A^\epsilon \nabla u^\epsilon = f \text{ in the weak sense}$$

implies that $u^\epsilon \rightharpoonup u^0$ weakly in $H_0^1(U)$ as $\epsilon \to 0$ and

$$-\mathrm{div}\, A^{\mathrm{hom}} \nabla u^0 = f \text{ in the weak sense.}$$

In words, we study the asymptotic behavior of solutions. The disadvantage of this approach is that requires the existence of solutions. This can be cumbersome and it is desirable to construct definitions of limiting problems with does not involve solutions of the intermediate problems.

We define asymptotic minimization problems in a way which only involves approximate solutions (minimizers).

Definition 2.4.1. Let (X, d) be a metric space and $I_n : X \to \mathbb{R} \cup \infty$. We say that $I : X \to \mathbb{R} \cup \{+\infty\}$ is the Γ-limit of I_n if for every $x \in X$

(i) For every sequence x_n converging to x

$$I[x] \leq \liminf_{n\to\infty} I_n[x_n] \text{ (lim inf inequality).}$$

(ii) There exists a recovery sequence $y_n \in X$ such that

$$I[y] \geq \limsup_{n\to\infty} I_n[y_n].$$

The usefulness of Definition 2.4.1 is a consequence of the following key properties:

1. Γ-limits are unique.
2. Minimizers converge to minimizers.
3. Γ-limits exist under very weak assumptions.

Theorem 2.4.2. *Assume that (X, d) is a metric space and $I_n : X \to \mathbb{R} \cup \{\infty\}$ a sequence.*

1. *If I is the Γ-limit of I $I_n[x_n] = \inf_{x\in X}[x]$ and $\lim_{n\to\infty} x_n = y$, then$I[y] = \inf_{x\in X}[x]$.*
2. *If $I, J : X \to \mathbb{R} \cup \{\infty\}$ are both Γ-limits of I_n. Then $I = J$.*
3. *If I is the Γ-limit of I_n, then I is lower semi-continuous.*
4. *If (X, d) is separable, then there exists a subsequence (not relabelled) and $I : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ such that I is the Γ-limit of I_n.*

Proof. □

It is important to realize that Γ-limits are in general smaller than pointwise limits. We illustrate this by a simple example. Let $X = \mathbb{R}$ and $I_n(x) = x^2 + \cos(n\,x)$ and $I(x) = x^2 - 1$.

Indeed, since $I_n \geq I$ it suffices to construct recovery sequences. But this is trivial since for every $x \in \mathbb{R}$ there exists $k_n \in \mathbb{Z}$ such that $\lim_{n\to\infty} \frac{\pi + 2\pi\, k_n}{n} = x$. Then $I_n(x_n) = I(x_n)$.

2.4.1 Periodic homogenization

We consider Y-periodic integrands $W(x, \xi)$, i.e. $W(x, \xi) = W(y, \xi)$ if $x - x \in Y\,\mathbb{Z}^d$. This setting covers the quadratic case $W(x, \xi) = \frac{1}{2}\xi \cdot A^\epsilon(x)\xi$, with $A^\epsilon \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d\times}$ symmetric and periodic.

Theorem 2.4.3. *Assume that $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ has the properties*

$$\alpha\,|\xi|^p \leq W(x, \xi) \leq \beta\,(1 + |\xi|^p) \text{ for all } x \in \mathbb{R}^d, \xi \in \mathbb{R}^d \tag{2.7}$$

$$W(x, \xi) = W(y, \xi) \text{ for all } \xi \in \mathbb{R}^d\; x, y \in \mathbb{R}^d \text{ s.t. } y - x \in Y\,\mathbb{R}^d, \tag{2.8}$$

for some $0 < \alpha \leq \beta$. If U is open and bounded, $X = W_0^{1,p}(U)$ and

$$I_\epsilon[u] = \int_U W(x/\epsilon, \nabla u)\,\mathrm{d}x,$$

then the Γ-limit of I_ϵ as $\epsilon \to 0$ is given by

$$I[u] = \int_U W^{\mathrm{hom}}(\nabla u)\,\mathrm{d}x,$$

where

$$W^{\mathrm{hom}}(\xi) = \inf\left\{\frac{1}{|U_{\mathrm{per}}|}\int_{U_{\mathrm{per}}} W(x, \xi + \nabla u)\,\mathrm{d}x \;:\; u \in W^{1,p}_{\mathrm{per}}(U_{\mathrm{per}})\right\}. \tag{2.9}$$

Proof. Our strategy is as follows: We will establish the following, much more general and less explicit formula

$$W^{\mathrm{hom}}(\xi) = \liminf_{n\to\infty} \left\{ |n\,U_{\mathrm{per}}|^{-1} \int_{n\,U_{\mathrm{per}}} W(x, \xi + \nabla v)\,\mathrm{d}x \;:\; v \in W^{1,p}_{\mathrm{per}}(n\,U_{\mathrm{per}}) \right\}. \tag{2.10}$$

It is not hard to see that the cell problem (2.9) is a consequence of (2.10) if $\xi \mapsto W(x,\xi)$ is convex for every x. Indeed, by definition

$$\inf \left\{ |n\,U_{\mathrm{per}}|^{-1} \int_{n\,U_{\mathrm{per}}} W(x, \xi + \nabla v)\,\mathrm{d}x \;:\; v \in W^{1,p}_{\mathrm{per}}(j\,U_{\mathrm{per}}) \right\} \le W^{\mathrm{hom}}(\xi). \tag{2.11}$$

On the other hand, if $v \in W^{1,p}(n\,U_{\mathrm{per}})$, then we can define the convex combination

$$u = |I|^{-1} \sum_{i\in I} v(\cdot - Y\,i) \in W^{1,p}(U_{\mathrm{per}}),$$

with $I = \{0, 1, \dots, n-1\}^d$.

The function u belongs to $W^{1,p}_{\mathrm{per}}(U_{\mathrm{per}})$ (Exercise).

Moreover, by the convexity and the periodicity of W,

$$\begin{aligned}
|U_{\mathrm{per}}|^{-1} \int_{U_{\mathrm{per}}} W(x, \xi + \nabla u)\,\mathrm{d}x &= |n\,U_{\mathrm{per}}|^{-1} \int_{n\,U_{\mathrm{per}}} W(x, \xi + \nabla u)\,\mathrm{d}x \\
=&|n\,U_{\mathrm{per}}|^{-1} \int_{n\,U_{\mathrm{per}}} W\left(x, |I|^{-1} \sum_{i\in I} (\xi + \nabla v(x - Y\,i))\right) \mathrm{d}x \\
\le&|n\,U_{\mathrm{per}}|^{-1}\,|I|^{-1} \sum_{i\in I} \int_{n\,U_{\mathrm{per}}} W\left(x, \xi + \nabla v(x - Y\,i)\right)\mathrm{d}x \\
=&|n\,U_{\mathrm{per}}|^{-1}\,|I|^{-1} \sum_{i\in I} \int_{n\,U_{\mathrm{per}}} W\left(x, \xi + \nabla v(x)\right)\mathrm{d}x = |n\,U_{\mathrm{per}}|^{-1} \int_{n\,U_{\mathrm{per}}} W\left(x, \xi + \nabla v(x)\right)\mathrm{d}x.
\end{aligned}$$

Thus

$$\inf \left\{ |n\,U_{\mathrm{per}}|^{-1} \int_{n\,U_{\mathrm{per}}} W(x, \xi + \nabla v)\,\mathrm{d}x \;:\; v \in W^{1,p}_{\mathrm{per}}(j\,U_{\mathrm{per}}) \right\}$$

and together with (2.11) we find that

$$W^{\mathrm{hom}}(\xi) = |U_{\mathrm{per}}|^{-1} \min \left\{ \int_{U_{\mathrm{per}}} W(x, \xi + \nabla u)\,\mathrm{d}x \;:\; u \in W^{1,p}_{\mathrm{per}}(U_{\mathrm{per}}) \right\}.$$

Now we address the question why W^{hom} actually characterizes the Γ-limit. The point is that the Γ-limit might not be a local functional at all. We have to establish an integral representation result and define $\mathcal{A}(U)$ to denote the family of open subsets of U.

Theorem 2.4.4. *Let $U \subset \mathbb{R}^d$ be open and bounded, and let $1 \le p < \infty$. Assume that the functional $F : W^{1,p}(U) \times \mathcal{A}(U) \to [0,\infty)$ satisfies*

1. *F is local, i.e. $F(u,V) = F(v,V)$ if $u = v$ a.e. on $V \in \mathcal{A}(U)$.*
2. *For all $u \in W^{1,p}(U)$ the set function $F(u,\cdot)$ is the restriction of a Borel measure to $\mathcal{A}(U)$.*
3. *There exists $c > 0$ and $a \in L^1(U)$ such that*

$$F(u,U) \le c \int_U \left(a(x) + |\nabla u|^p\right) \mathrm{d}x$$

for all $u \in W^{1,p}(U)$ and $V \in \mathcal{A}(U)$,

4. *$F(u+z,U) = F(u,U)$ for all $z \in \mathbb{R}$, $u \in W^{1,p}(U)$ and $U \in \mathcal{A}(U)$,*
5. *$F(\,\cdot\,,U)$ is sequentially lower semi-continuous with respect to the weak convergence in $W^{1,p}(U)$ for each $U \in \mathcal{A}(U)$.*

Then there exists a (Caratheodory) function $\varphi : U \times \mathbb{R}^d \to [0,\infty)$ satisfying the growth condition

$$0 \le f(x,\xi) \le C\ (a(x) + |\xi|^p)$$

such that

$$F(u,U) = \int_U \varphi(x, \nabla u(x))\,\mathrm{d}x.$$

The function φ does not depend on x if F is translation invariant in the sense that

$$F(u, B(x,\rho))) = F(u, B(y,\rho))$$

for all $x, y \in U$, $\rho < 0$ s.t. $B(x,\rho) \cup B(y,\rho) \subset U$ and all affine functions u ($u(x) = F\,x$ for some $F \in \mathbb{R}^d$).

Proof. See Theorem 9.1 in [3]. □

To apply this theorem we have to argue that

1. The Γ-limit of I_ϵ exists and satisfies the assumptions of the theorem.
2. The integrand φ is convex.

Once we have completed those steps we can establish formula (2.10) as follows. First we note that $\int_{U_{\mathrm{per}}} \nabla u\,\mathrm{d}x = 0$ for each $u \in W_0^{1,p}(U_{\mathrm{per}})$ (Exercise). The convexity of φ together with Jensen's inequality implies that

$$\varphi(\xi) = \varphi\left(\frac{1}{|U_{\mathrm{per}}|}\int_{U_{\mathrm{per}}} (\xi + \nabla u)\,\mathrm{d}x\right) \le \frac{1}{|U_{\mathrm{per}}|}\int_{U_{\mathrm{per}}} \varphi(\xi + \nabla u)\,\mathrm{d}x \tag{2.12}$$

Since $u \equiv 0 \in W_0^{1,0}(U_{\mathrm{per}})$ this implies that

$$\begin{aligned}\varphi(\xi) &= \min\left\{\frac{1}{|U_{\mathrm{per}}|}\int_{U_{\mathrm{per}}} \varphi(\xi + \nabla u)\,\mathrm{d}x \ :\ u \in W_0^{1,p}(U)\right\}\\ &= \liminf_{n\to\infty} \min\left\{\frac{1}{|n\,U_{\mathrm{per}}|}\int_{n\,U_{\mathrm{per}}} \varphi(\xi + \nabla u)\,\mathrm{d}x \ :\ u \in W_0^{1,p}(n\,U_{\mathrm{per}})\right\}.\end{aligned}$$

The second equality holds because Γ-convergence implies that minima converge to minima.

Step 1 is almost straightforward. We consider sequences $F_n : L^p(U) \times \mathcal{A}(U) \to [0,\infty]$, $n = 1, 2, \ldots$ Since $L^p(U)$ is separable an easy compactness argument implies that there exists a Γ-limit $F : L^p(U) \times \mathcal{A}(U) \to [0,\infty]$. It is easy to check that F satisfies assumptions 1,3,4,5 of Theorem 2.4.4.

We have to verify that F satisfies assumption 2 of Theorem 2.4.4 and use notions from measure theory.

Definition 2.4.5. A set function $\mathcal{A}(\Omega) \to [0,\infty]$ is called

Increasing If $\alpha(\emptyset) = 0$ and $\alpha(V) \le \alpha(U)$ if $V \subset U$.

Supadditive If $\alpha(U \cup V) \le \alpha(U) + \alpha(V)$ for all $U, V \in \mathcal{A}(\Omega)$.

Superadditive If $\alpha(U \cup V) \ge \alpha(U) + \alpha(V)$ for all $U, V \in \mathcal{A}(\Omega)$, $U \cap V = \emptyset$.

Inner regular If

$$\alpha(U) = \sup\{\alpha(V)\ :\ V \in \mathcal{A}(\Omega), V \subset\subset U\}$$

for all $U \in \mathcal{A}(\Omega)$.

Proposition 2.4.6. *Let $\alpha : \mathcal{A}(\Omega) \to [0, \infty]$ be an increasing set function. Then the following statements are equivalent:*

1. *α is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω.*
2. *α is subadditive, superadditive and inner regular.*

We say that the functional $F : L^p(U) \times \mathcal{A}(U) \to [0, \infty]$ satisfies the fundamental L^p estimate if for every $\sigma > 0$, $U, U', V \in \mathcal{A}(\Omega)$ with $U' \subset\subset U$ ($\overline{U'}$ is compact and $\overline{U'} \subset U$) there exists a cutoff function $\varphi \in C^\infty(\Omega)$ such that

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in U', \\ 0 & \text{if } x \in \Omega \setminus U, \end{cases}$$

and

$$F(\varphi\, u + (1 - \varphi)\, v, U' \cup V) \leq (1 + \sigma)(F(u, U) + F(v, V)) + M_\sigma \int_{(U \cap V) \setminus U'} |u - p|^p \, \mathrm{d}x + \sigma.$$

Exercise: Show that $F(u, U) = \int_U W(x, u, \nabla u) \, \mathrm{d}x$ satisfies the fundamental estimate if there exists $\alpha, \beta > 0$ such that

$$\alpha\, |\xi|^p \leq W(x, u, \xi) \leq \beta\, (1 + |\xi|^p) \text{ for all } x, u, \xi. \tag{2.13}$$

Proposition 2.4.7. *Let $F_n : L^p(U) \times \mathcal{A}(U) \to [0, \infty]$ be a sequence of functionals such that*

$$F(u, U) = \int_U W(x, u(x), \nabla u(x)) \, \mathrm{d}x$$

and (2.13) holds for some $\alpha, \beta > 0$.

If F is the Γ-limit of F_n, then $F(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}(U)$.

□

Chapter 3

An introduction to nonlinear elliptic equations

Ω is a bounded domain in $\mathbb{R}^n$.

Theorem 3.0.8 (*Dominated convergence theorem*)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. Let g_j be a sequence of functions in $L^p(\Omega)$ with*

$$||g_j||_{L^p(\Omega)} \leq C \quad \forall \quad j.$$

If $g \in L^p(\Omega)$ and $g_j \to g \quad a.e.$ then

$$g_j \to g \quad in \quad L^p(\Omega).$$

3.1 Elementary functions on function spaces

Since we are interested in nonlinear partial differential equations it is necessary to introduce $f(u)$ for $u \in L^p(\Omega)$. Given $f \in C(\mathbb{R})$ we set $f(u)$ to be $f(u)(x) = f(u(x))$ $x \in \Omega$.

Lemma 3.1.1. *Let $\theta : \mathbb{R} \to \mathbb{R}$ be C^1 and $\theta' \in L^\infty(\mathbb{R})$. Then $\Theta : H^1(\Omega) \to H^1(\Omega)$ is continuous and $\Theta : H_0^1(\Omega) \to H_0^1(\Omega)$ is continuous in the case $\theta(0) = 0$ where $\Theta(u)(x) := \theta(u(x))$ It holds that*

$$\frac{\partial}{\partial x_i}\theta(u) = \theta'(u)\frac{\partial u}{\partial x_i} \tag{3.1}$$

where we use the notation $\theta(u)$ for convenience.

Proof. See [8, 4]. We have $|\theta'(\cdot)| \leq M$.

Since $u \in H^1(\Omega)$ there exists a sequence u_m of $C^1(\Omega)$ functions such that u_m converges to u in $H^1(\Omega)$ and also u_m converges to u $a.e.$ in Ω. Obviously $\theta(u_m) \in C^1(\bar{\Omega})$ and since

$$|\theta(u_m) - \theta(u)| \leq M|u_n - u|$$

$\theta(u_m)$ converges to $\theta(u)$ in $L^2(\Omega)$.

On the other hand

$$\theta'(u_m)\frac{\partial u_m}{\partial x_i} = \frac{\partial}{\partial x_i}\theta(u_m) \to \theta'(u)\frac{\partial u}{\partial x_i} \quad \text{in} \quad L^2(\Omega).$$

This follows by:- (i)

$$\theta'(u_m)\frac{\partial u_m}{\partial x_i} - \theta'(u)\frac{\partial u}{\partial x_i} = \theta'(u_m)[\frac{\partial u_m}{\partial x_i} - \frac{\partial u}{\partial x_i}] + [\theta'(u_m) - \theta'(u)]\frac{\partial u}{\partial x_i} = A_m + B_m$$

and

(ii)

observing that A_m converges to zero in $L^2(\Omega)$ and B_m converges to zero a.e. in Ω and

$$|B_m|^2 \leq (2M)^2|]\frac{\partial u}{\partial x_i}|^2$$

so by the dominated convergence theorem, B_m converges to zero in $L^2(\Omega)$.

Since the derivatives in the sense of distributions of $\theta(u)$ are the limit in $L^2(\Omega)$ of $\frac{\partial}{\partial x_i}\theta(u_m)$ we have proved (3.1). □

Definition 3.1.2. Max, Min, and Mod in $H^1(\Omega)$ Set

$$u^+ := max(u,0) = \begin{cases} u(x) \text{ if } u(x) \geq 0 \\ 0 \text{ if } u(x) \leq 0 \end{cases} \tag{3.2}$$

$$u^- := max(-u,0) = -min(u,0) = \begin{cases} -u(x) \text{ if } -u(x) \geq 0 \\ 0 \text{ if } u(x) \geq 0 \end{cases} \tag{3.3}$$

Then

$$u = u^+ - u^-, \qquad |u| = u^+ + u^-.$$

$$\chi_{\{u>0\}}(x) \begin{cases} = 1 \text{ if } u(x) > 0 \\ = 0 \text{ if } u(x) \leq 0. \end{cases}$$

$$sign(u)(x) \begin{cases} = 1 \text{ if } u(x) > 0 \\ = -1 \text{ if } u(x) < 0 \\ = 0 \text{ if } u(x) = 0. \end{cases}$$

Theorem 3.1.3. *If Ω is a bounded domain and $u \in H^1(\Omega)$ then u^-, u^- and $|u| \in H^1(\Omega)$.*

Proof. Consider the global Lipschitz C^1 functions $\theta_\epsilon(\cdot) : \mathbb{R} \to \mathbb{R}$

$$\theta_\epsilon(\cdot) = \begin{cases} |\cdot|_\epsilon \\ (\cdot)^+_\epsilon \end{cases}$$

defined by

$$|r|_\epsilon := (r^2+\epsilon^2)^{\frac{1}{2}}, \quad (r)^+_\epsilon = \begin{cases} (r^2+\epsilon^2)^{\frac{1}{2}} - \epsilon \text{ if } r > 0 \\ 0 \text{ if } r \leq 0 \end{cases}$$

which have Lipschitz constant 1. Then

$$\frac{\partial}{\partial x_i}\theta_\epsilon(u) = \theta'_\epsilon(u)\frac{\partial u}{\partial x_i}$$

and

$$\theta_\epsilon(u) \in H^1(\Omega)$$

$$\theta_\epsilon(u) \to \theta(u) \ a.e. \text{ in } \Omega, \quad |\theta_\epsilon(u)| \leq |u|$$

so

$$\theta_\epsilon(u) \to \theta(u) \text{ in } L^2(\Omega).$$

Here $\theta(r) = |r|$ or $\theta(r) = (r)^+$.

Also we have

$$\frac{\partial}{\partial x_i}\theta_\epsilon(u) = \theta'_\epsilon(u)\frac{\partial u}{\partial x_i}$$

and

$$\theta'_\epsilon(u) \to \begin{cases} \chi_{\{u>0\}} \\ sign(u) \end{cases} \quad a.e. \text{ in } \Omega$$

so we deduce that

$$\frac{\partial}{\partial x_i}\theta_\epsilon(u) = \theta'_\epsilon(u)\frac{\partial u}{\partial x_i} \to \begin{cases} \chi_{\{u>0\}}\frac{\partial u}{\partial x_i} \\ sign(u)\frac{\partial u}{\partial x_i} \end{cases} \quad \text{in } L^2(\Omega).$$

□

Lemma 3.1.4. *Let $u \in L^1_{loc}(\Omega)$, $\frac{\partial u}{\partial x_i} \in L^1_{loc}(\Omega)$. Then u^-, u^- and $|u| \in L^1_{loc}(\Omega)$ and*

$$\frac{\partial u^+}{\partial x_i} = \chi_{\{u>0\}}\frac{\partial u}{\partial x_i}$$

where $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\}$.

Let c be a constant then for $u \in H^1(\Omega)$

$$\nabla u = 0 \quad a.e. \quad in \quad \{u(x) = c\}.$$

Definition 3.1.5. Let $v \in H^1(\Omega)$. Then we say that

$$v \leq 0 \quad \text{on} \quad \partial\Omega$$

if and only if $v^+ \in H^1_0(\Omega)$.

3.2 Weak maximum/comparison principle

Let $\boldsymbol{A}$ be an $n \times n$ matrix with coefficients

$$a_{ij} \in L^\infty(\Omega)$$

and for all $\boldsymbol{\xi} \in \mathbb{R}^n$

$$\sum_{i,j=1}^n \xi_i a_{ij}(x)\xi_j \geq a_0|\boldsymbol{\xi}|^2 \quad a.e. \text{ in } \Omega.$$

Set

$$a(u.v) := \int_\Omega \boldsymbol{A}\nabla u \cdot \nabla v dx.$$

Theorem 3.2.1. *Let $f_1, f_2 \in L^2(\Omega)$ and $\phi_1, \phi_2 \in H^1(\Omega)$. Suppoose*

$$f_1 \leq f_2 \ a.e. \quad in \quad \Omega, \quad \phi_1 \leq \phi_2 \quad on \quad \partial\Omega.$$

Then the unique solutions of the boundary value problem

$$u_i = \phi_i \quad on \quad \partial\Omega, \quad -div(\boldsymbol{A}\nabla u_i) = f_i \quad in \quad \Omega$$

written in variational form as

$$u_i - \phi_i \in H^1(\Omega): \ a(u_i, v) = (f, v) \quad \forall v \in H^1(\Omega)$$

satisfy

$$u_1 \leq u_2 \quad a.e. \quad in \quad \Omega.$$

Proof. By subtraction

$$a(u_1 - u_2, v) = (f_1 - f_2, v) \quad \forall v \in H^1(\Omega)$$

and since $(u_1 - u_2)^+ \in H^1_0(\Omega)$ it follows that

$$a(u_1 - u_2, (u_1 - u_2)^+) = (f_1 - f_2, (u_1 - u_2)^+) \leq 0.$$

The result follows by noting that

$$\int_\Omega \boldsymbol{A}\nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2)^+ dx = \int_\Omega \boldsymbol{A}\nabla(u_1 - u_2)^+ \cdot \nabla(u_1 - u_2)^+ dx.$$

□

3.3 A compactness and finite dimensional approximation method

Definition 3.3.1. Caretheodory function The function $a : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is said to be a *Caretheodory function* provided it satisfies

1. for $a.e.\ x \in \Omega, \quad u \to a(x,u)$ is continuous from $\mathbb{R}$ into $\mathbb{R}$
2. $\forall u \in \mathbb{R}, \quad x \to a(x,u)$ is measurable

Let $a : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be a *Caretheodory function* and satisfy

$$0 \leq A_0 \leq a(x,u) \leq A_M \quad a.e.\ x \in \Omega \quad \forall u \in \mathbb{R} \tag{3.4}$$

for positive constants A_0 and A_M.This is a nonlinear system of equations for the coefficients α_j

Consider the boundary value problem

$$\begin{cases} -\nabla(a(x,u)\nabla u) = f \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.5}$$

where $f \in L^2(\Omega)$ is given.

Theorem 3.3.2. *There exists a weak solution $u \in H_0^1(\Omega)$ of (3.5).*

Proof. We seek a solution in $V := H_0^1(\Omega)$ which, recalling Poincare's inequality,

$$||v||_{L^2(\Omega)} \leq C_P ||\nabla v||_{L^2(\Omega)} \quad \forall v \in V$$

we endow with the norm $||v||_V := ||\nabla v||_{L^2(\Omega)}$.

We use a *Galerkin method*, a fixed point theorem in finite dimensions and compactness. Let V_m be a finite dimensional subspace of $V := H_0^1(\Omega)$ with the approximation property that

$$\forall v \ \in V \ \exists \ v_m \in V_m \text{ such that } v_m \to v \ \text{ in } \ V.$$

V_m could be a finite element space or be spanned by eigenfunctions of a linear elliptic operator. Set

$$a(w : u, v) := \int_\Omega a(x,w)\nabla u \cdot \nabla v dx, \ u, v \ \in V$$

where w is given in V.

Variational problem

($\mathcal{P}$) Find $u \in V$ such that

$$a(u; u, v) = (f, v) \ \ \forall \ \ v \in V. \tag{3.6}$$

Finite dimensional approximation

($\mathcal{P}_\mathbf{m}$) Find $u_m \in V_m$ such that

$$a(u_m; u_m, v_m) = (f, v_m) \ \ \forall \ \ v_m \in V_m. \tag{3.7}$$

Fixed point problem in finite dimensions

Given $w_m = \sum_{j=1}^m \beta_j \phi_j^m \in V_m$ where the ϕ_j^m are the basis functions of V_m, set $U_m = \sum_{j=1}^m \alpha_j \phi_j^m \in V_m$ to be the unique solution of

$$U_m \in V_m : \quad a(w_m; U_m, v_m) = (f, v_m) \forall v_m \in V_m. \tag{3.8}$$

That U_m exists and is unique follows by the Lax-Milgram theorem and the following standard estimate holds

$$||\nabla U_m||_{L^2(\Omega)} \leq C_P \frac{||f||_{L^2(\Omega)}}{A_0} := c^*. \tag{3.9}$$

Thus we have constructed a map $G_m : V_m \to V_m$ by $G_m(w_m) := U_m$. and if we can show that G_m has a *fixed point* i.e.

$$G_m(u_m^*) = u_m^*$$

then we have constructed a solution of $\mathcal{P}_m$. In order to do this we will apply the following Brouwer fixed-point theorem. Of course it would be nice to use the contraction mapping theorem whenever it is applicable. However in general in this setting the mapping is not a contraction.

Theorem 3.3.3. Brouwer fixed-point theorem

Let $K \subset \mathbb{R}^n$ be a compact convex set and $F : K \to K$ be a continuous mapping. Then there exists a fixed point $u \in K$ of F i.e. $u = F(u)$.

Proof. A version is proved in [12].

□

To use this we formulate the fixed point theorem in terms of the coefficients α_j and β_j. Note that the discrete problem is

$$\sum_{j=1}^{m} \alpha_j \int_\Omega a(x, \sum_{j=1}^{m} \beta_j \phi_j^m) \nabla \phi_j^m \nabla \phi_i^m dx = (f, \phi_i^m) \;\; \forall i = 1, 2...m. \tag{3.10}$$

We may write this as

$$\mathcal{S}(\boldsymbol{\beta}) := \boldsymbol{\alpha}$$

and noting the a priori estimate (3.9) we define K_m and $\hat{K}_m$ to be the closed convex sets of H_0^1 and $\mathbb{R}^m$ by

$$K_m := \{v_m \in V_m \mid v_m := \sum_{j=1}^{m} \gamma_j \phi_j^m \text{ satisfies } ||\nabla v_m||_{L^2(\Omega)} \leq c^*\}$$

$$\hat{K}_m := \{\boldsymbol{\gamma} \mid v_m := \sum_{j=1}^{m} \gamma_j \phi_j^m \text{ satisfies } ||\gamma||_m := ||\nabla v_m||_{L^2(\Omega)} \leq c^*\}$$

and note that $\mathcal{S} : \hat{K}_m \to \hat{K}_m$.

We wish to apply the Brouwer fixed point theorem to obtain a fixed point of $\mathcal{S}$ and hence obtain a u_m solving $\mathcal{P}_\mathbf{m}$.

(1) It is straightforward to see that $\hat{K}_m$ is convex and compact. (Show that $|| \cdot ||_m$ is a norm on $\mathbb{R}^m$.)

(2) We now show that $\mathcal{S}(\cdot)$ is continuous.

Let $\boldsymbol{\beta}^n$ be a sequence converging to $\boldsymbol{\beta}$ then we may define $\boldsymbol{\alpha}^n$ given by

$$\sum_{l=1}^{m} \alpha_l^n \int_\Omega a(x, \sum_{j=1}^{m} \beta_j^n \phi_j^m) \nabla \phi_l^m \nabla \phi_i^m dx = (f, \phi_i^m) \;\; \forall i = 1, 2...m. \tag{3.11}$$

Continuity of $\mathcal{S}(\cdot)$ follows from showing that

$$\mathcal{S}(\boldsymbol{\beta}^n) = \boldsymbol{\alpha}^n \to \boldsymbol{\alpha}$$

where $S(\beta) = \alpha$.

Since $U^n = \sum_{j=1}^m \alpha_j^n \phi_j^m \in K_m$ and $\hat{K}_m$ is compact, there is a subsequence $\boldsymbol{\alpha}^{n_k}$ which converges to an $\boldsymbol{\alpha}^*$ and $U^{n_k} \to U^* = \sum \alpha_l^* \phi_l^m$ in V_m. Also since for $a.e.\ x \in \Omega$ $a(x,r)$ is continuous in r we have that

$$\int_\Omega a(x, \sum_{j=1}^m \beta_j^{n_k} \phi_j^m) \sum_l \alpha_l^{n_k} \nabla \phi_l^m \nabla \phi_i^m \to \int_\Omega a(x, \sum_{j=1}^m \beta_j \phi_j^m) \sum_l \alpha_l^* \nabla \phi_l^m \nabla \phi_i^m.$$

Thus U^* satisfies

$$U^* \in V_m : \quad a(w_m; U^*, v_m) = (f, v_m) \forall v_m \in V_m$$

and by the uniqueness of U it holds that $U^* = U = \sum_l \alpha_l \phi_l^m$, i.e. $\alpha^* = \alpha$. Thus we have shown that $\mathcal{S}(\boldsymbol{\beta}^{n_k}) \to \alpha^* = \alpha = S(\boldsymbol{\beta})$. Since the limit function U is unique we have that the whole sequence converges and we have proved that $\mathcal{S}(\cdot)$ is continuous.

Let $u_m := \sum_{j=1}^m \boldsymbol{\gamma}_j \phi_j^m$ where $\boldsymbol{\gamma}$ is a fixed point of $\mathcal{S}(\cdot)$. It follows that u_m solves $\mathbf{P_m}$.

Passage to the limit

We now wish to consider the convergence of u_m as $m \to \infty$. First observe that

$$||u_m||_V \leq c^* \quad \forall m.$$

Using the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ there exists a subsequence u_{m_k} such that

$$u_{m_k} \to u \;\text{ weakly in }\; H_0^1(\Omega),\; u_{m_k} \to u \text{ in } L^2(\Omega),\; u_{m_k} \to u \;\; a.e. \;\text{ in }\; \Omega.$$

Let $v \in V$ and

$$v_m \to v \;\text{ in }\; V.$$

Applying the dominated convergence theorem we have

$$a(\cdot, u_{m_k}) \nabla v_{m_k} \to a(\cdot, u) \nabla v \;\text{ in }\; L^2(\Omega)$$

which allows us to pass to the limit in

$$\int_\Omega a(x, u_{m_k}) \nabla u_{m_k} \cdot \nabla v_{m_k} dx = \int_\Omega f v_{m_k} dx$$

and obtain

$$\int_\Omega a(x,u) \nabla u \cdot \nabla v dx = \int_\Omega f v dx.$$

Thus we have shown existence of a solution.

□

Remark 3.3.4. Let $a(x,u) = k(u)$, i.e. there is no x dependence in the coefficient a. We suppose that

$$K_M \geq k(r) \geq K_m > 0 \;\; \forall r.$$

By considering the *Kirchoff* transformation

$$w := \int_0^u k(r) dr \tag{3.12}$$

we may show the existence and uniqueness of a solution to

$$\begin{cases} -\nabla(k(u)\nabla u) = f \;\text{ in }\; \Omega \\ u = 0 \;\text{ on }\; \partial\Omega \end{cases} \tag{3.13}$$

where $f \in L^2(\Omega)$ is given.

3.4 Monotonicity method

In this section we consider an example of a *monotone operator*. Our setting is that of a quasilinear second order elliptic equation.

Definition 3.4.1.

- A vector field $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^n$ is called *monotone* provided

$$\sum_{k=1}^{n} (A_k(\boldsymbol{p}) - A_k(\boldsymbol{q}))(p_k - q_k) \geq 0 \tag{3.14}$$

  for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$.

- A vector field $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^n$ is called *strictly monotone* provided there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} (A_k(\boldsymbol{p}) - A_k(\boldsymbol{q}))(p_k - q_k) \geq \delta |\boldsymbol{p} - \boldsymbol{q}|^2 \tag{3.15}$$

  for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$.

- We say that $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^n$ is *bounded* provided

$$|\boldsymbol{A}(\mathbf{p})| \leq C(|\mathbf{p}| + 1). \tag{3.16}$$

- We say that $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^n$ is *coercive* provided there exist $\gamma > 0, \beta \geq 0$ such that

$$\boldsymbol{A}(\mathbf{p}) \cdot \mathbf{p} \geq \gamma |\mathbf{p}|^2 - \beta. \tag{3.17}$$

PDE and variational form

Let $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping , $f \in L^2(\Omega)$ and Ω be a bounded domain in $\mathbb{R}^n$. We seek a solution to the boundary value problem

$$\begin{aligned} -\nabla \cdot \boldsymbol{A}(\nabla u) &= f && (3.18) \\ u &= 0 \quad \text{on} \quad \partial\Omega && (3.19) \end{aligned}$$

which has the variational formulation

$$u \in V : \quad \int_\Omega \boldsymbol{A}(\nabla u) \cdot \nabla v dx = \int_\Omega f v dx \quad \forall v \in V. \tag{3.20}$$

Finite dimensional approximation

Let V_m be a finite dimensional subspace of $V := H_0^1(\Omega)$ with a basis $\{\phi_j^m\}$. We assume the approximation property that

$$\forall v \;\in V \; \exists \; v_m \in V \text{ such that } v_m \to v \;\text{ in }\; V.$$

Consider the Galerkin approximation:

$$u_m = \sum_{j=1}^{m} \alpha_j \phi_j^m \in V_m : \quad \int_\Omega \boldsymbol{A}(\nabla u_m) \cdot \nabla v_m dx = \int_\Omega f v_m dx \quad \forall v_m \in V_m. \tag{3.21}$$

Lemma 3.4.2. *If $\boldsymbol{A}$ is coercive then the discrete problem has a solution. If $\boldsymbol{A}$ is strictly monotone then the discrete problem has at most one solution. If $\boldsymbol{A}$ is coercive then the discrete solution satisfies*

$$||\nabla u_m||_{L^2(\Omega)} \leq C(1 + ||f||_{L^2(\Omega)}) \tag{3.22}$$

where C depends on $\boldsymbol{A}$ and Ω.

Proof. Existence

The Galerkin approximation is equivalent to

$$\boldsymbol{F}(\boldsymbol{\alpha}) = \boldsymbol{0}$$

where

$$\boldsymbol{F}(\boldsymbol{\alpha})_i := \int_\Omega \boldsymbol{A}(\nabla \sum_{j=1}^m \alpha_j \phi_j^m) \cdot \nabla \phi_i^m dx - \int_\Omega f \phi_i^m dx.$$

Observe that

$$\boldsymbol{F}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} = \int_\Omega \boldsymbol{A}(\nabla \sum_{j=1}^m \alpha_j \phi_j^m) \cdot \sum_{i=1}^m \alpha_i \nabla \phi_i^m dx - \int_\Omega f \sum_{i=1}^m \alpha_i \phi_i^m dx.$$

Using the coercivity of $\boldsymbol{A}$ we find that

$$\boldsymbol{F}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} \geq \gamma \boldsymbol{\alpha} \cdot \mathcal{S} \boldsymbol{\alpha} - \beta |\Omega| - \boldsymbol{\alpha} \cdot \boldsymbol{f}$$

where $\mathcal{S}_{ij} = \int_\Omega \nabla \phi_i^m \nabla \phi_j^m dx$ and $\boldsymbol{f}_j = \int_\Omega f \phi_i^m dx$. The (stiffness) matrix $\mathcal{S}$ is positive definite so for all $|\boldsymbol{\alpha}|$ sufficiently large $\boldsymbol{F}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} \geq 0$ and we can apply the lemma 3.4.3 to yield the existence of a solution to $\boldsymbol{F}(\boldsymbol{\alpha}) = \boldsymbol{0}$.

Lemma 3.4.3. *Let a continuous mapping* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *satisfy*

$$\mathbf{F}(\boldsymbol{x}) \cdot \boldsymbol{x} \geq 0 \quad \textit{if} \quad |\boldsymbol{x}| = r \tag{3.23}$$

for some $r > 0$. *Then there exists* $\boldsymbol{x} \in B(\boldsymbol{0}, r)$ *such that*

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}.$$

Proof. Suppose the assertion is false. Then $\boldsymbol{F}(\boldsymbol{x}) \neq 0 \ \forall \boldsymbol{x} \in B(\boldsymbol{0}, r)$. Define the continuous mapping $S : B(\boldsymbol{0}, r) \to \partial B(\boldsymbol{0}, r)$ by

$$S(\boldsymbol{x}) := -\frac{r}{|\boldsymbol{F}(\boldsymbol{x})|} \boldsymbol{F}(\boldsymbol{x}) \ , \quad \boldsymbol{x} \in B(\boldsymbol{0}, r).$$

By the Brouwer fixed point theorem there exists a point $\boldsymbol{z} \in B(\boldsymbol{0}, r)$ such that

$$S(\boldsymbol{z}) = \boldsymbol{z}.$$

But it also holds that $\boldsymbol{z} \in \partial B(\boldsymbol{0}, r)$ so that

$$r^2 = |\boldsymbol{z}|^2 = S(\boldsymbol{z}) \cdot \boldsymbol{z} = -\frac{r}{|\boldsymbol{F}(\boldsymbol{z})|} \boldsymbol{F}(\boldsymbol{z}) \cdot \boldsymbol{z} \leq 0$$

which is a contradiction.

□

Uniqueness

Strict monotonicity of $\boldsymbol{A}$ immediately implies uniqueness since for two solutions we have

$$\int_\Omega (\boldsymbol{A}(\nabla u_m^1) - \boldsymbol{A}(\nabla u_m^2)) \cdot \nabla v_m dx = 0, \ \forall v_m \in V_m$$

and we may take $v_m = u_m^1 - u_m^2$.

Energy estimate

Take $v_m = u_m$ in the variational form to give

$$\int_\Omega \boldsymbol{A}(\nabla u_m) \cdot \nabla u_m dx = \int_\Omega f u_m dx$$

and using coercivity in the left, Young's inequality and Poincare on the right hand side gives the desired bound.

□

Passage to the limit

The uniform $H_0^1(\Omega)$ a priori bound on the sequence $\{u_m\}$ implies that there is a subsequence $\{u_{m_k}\}$ converging weakly in $H_0^1(\Omega)$ to $u \in H_0^1(\Omega)$ so that

$$u_{m_k} \to u \text{ in } L^2(\Omega),\ \nabla u_{m_k} \to \nabla u \ \text{ weakly in } \ L^2(\Omega).$$

In particular we have that $\xi_m := \boldsymbol{A}(\nabla u_m)$ is uniformly bounded in $L^2(\Omega)$ from which we deduce that

$$\boldsymbol{\xi}_{m_k} \to \boldsymbol{\xi} \ \text{ weakly in } \ L^2(\Omega)$$

and we easily deduce that

$$\int_\Omega \boldsymbol{\xi} \cdot \nabla v dx = \int_\Omega f v dx \ \ \forall v \in V.$$

However a fundamental issue in the study of nonlinear PDEs is that of passage to the limit in non-linear functions with respect to weak convergence. That is we unable to deduce simply from the stated convergence facts that

$$\boldsymbol{\xi} = \boldsymbol{A}(\nabla u).$$

In order establish this we use the method of *monotonicity*. First we note that for a monotonic vector field $\boldsymbol{A}$

$$\int_\Omega (\boldsymbol{A}(\nabla u_m) - \boldsymbol{A}(\nabla w)) \cdot (\nabla u_m - \nabla w) dx \geq 0 \ \ \forall w \in V.$$

Observe that from the discrete equation

$$\int_\Omega \boldsymbol{A}(\nabla u_m) \cdot \nabla u_m dx = \int_\Omega f u_m dx$$

which implies that

$$\int_\Omega \boldsymbol{A}(\nabla u_{m_k}) \cdot \nabla u_{m_k} dx = \int_\Omega f u_{m_k} dx \to \int_\Omega f u dx = \int_\Omega \boldsymbol{\xi} \cdot \nabla u dx.$$

Furthermore using the weak convergences of the subsequences we have

$$\int_\Omega \boldsymbol{A}(\nabla u_{m_k}) \cdot \nabla w dx \to \int_\Omega \boldsymbol{\xi} \cdot \nabla w dx, \quad \int_\Omega \boldsymbol{A}(\nabla w) \cdot \nabla u_{m_k} dx \to \int_\Omega \boldsymbol{A}(\nabla w) \cdot \nabla u dx.$$

Thus from the discrete problem, the weak convergence and the monotonicity of $\boldsymbol{A}$ we deduce

$$\int_\Omega (\boldsymbol{\xi} - \boldsymbol{A}(\nabla w)) \cdot (\nabla u - \nabla w) dx \geq 0 \ \ \forall w \in V. \tag{3.24}$$

The method of *Minty and Browder* is the observation that the above inequality yields the desired equation by consideration of

$$w = u - \lambda v, \quad v \in V, \lambda > 0.$$

Immediately we have

$$\int_\Omega (\boldsymbol{\xi} - \boldsymbol{A}(\nabla u - \lambda \nabla v)) \cdot \nabla v dx \geq 0 \ \forall v \in V.$$

Passing to the limit $\lambda \to 0$ and using the continuity of $\boldsymbol{A}$ we find

$$\int_\Omega (\boldsymbol{\xi} - \boldsymbol{A}(\nabla u)) \cdot \nabla v dx \geq 0 \ \forall v \in V$$

and since this inequality is also true with v replaced by $-v$ we find

$$\int_\Omega (\boldsymbol{\xi} - \boldsymbol{A}(\nabla u)) \cdot \nabla v dx = 0 \ \forall v \in V$$

and

$$\boldsymbol{\xi} = \boldsymbol{A}(\nabla u) \ \text{ in } \ L^2(\Omega).$$

Theorem 3.4.4. *Let $\boldsymbol{A}$ be a continuous, coercive, bounded and monotonic vector field and $f \in L^2(\Omega)$.. Then there exists a solution to*

$$u \in V : \int_\Omega \boldsymbol{A}(\nabla u) \cdot \nabla v = \int_\Omega f v dx \ \ \forall v \in V. \tag{3.25}$$

The solution is unique provided $\boldsymbol{A}$ is strictly monotone.

3.5 Applications

3.5.1 Conservation with a diffusive flux

Consider a scalar quantity (temperature, concentration of mass) $u : \Omega \to \mathbb{R}$ for which there is a flux $\boldsymbol{q}$ such that in any material domain $D \subset \Omega$ the following conservation law holds:-

flux out of D is balanced by the production of the quantity u in Ω.

This is written as

$$\int_{\partial D} \boldsymbol{q} \cdot \nu = \int_D f, \quad \forall D \subset \Omega$$

where f denotes the production rate. This equation may be rewritten using the divergence theorem to obtain

$$\int_D \nabla \cdot \boldsymbol{q} - f = 0 \quad \forall D \subset \Omega$$

and since this is true for all arbitrary D we have

$$\nabla \cdot \boldsymbol{q} - f = 0 \text{ in } \Omega. \tag{3.26}$$

This is the fundamental conservation law.

If we now assume a *constitutive relation* of the form

$$\boldsymbol{q} = -A\nabla u \tag{3.27}$$

then we obtain the following PDE

$$-\nabla \cdot A\nabla u = f \text{ in } \Omega. \tag{3.28}$$

Here A may be an $n \times n$ matrix. We say that the flux $\boldsymbol{q}$ is a *diffusive* flux and that A is a diffusivity tensor. In the case that

$$A = a(x, u)\mathcal{I}$$

we obtain a nonlinear elliptic equation. We may obtain more complicated equations by assuming that the diffusivity depends on ∇u. On the other hand the production rate f may also depend on u and ∇u.

3.5.2 Steady state problems

The *heat (diffusion)* equation

$$u_t = \nabla \cdot k\nabla u + f(x, u) \; x \in \Omega, \; t > 0$$

for $u(x, t)$ *(temperature in the heat equation and density/concentration in the diffusion equation)* is usually posed as an initial value problem for given u_0 with

$$u(x, 0) = u_0(x) \quad x \in \Omega$$

and with some suitable boundary condition. If the solution tends to a time independent function w as $t \to \infty$ then we obtain the steady state elliptic equation

$$-\nabla \cdot k\nabla w = f(x, w) \; x \in \Omega.$$

Here k (*conductivity or diffusivity*) is a given positive constant and f is a source term modelling the generation of heat or a reaction term. Note that k might also depend on the solution.

3.5.3 Advection -diffusion equation

We place ourselves now in the context of the previous subsection but assume a constitutive law of the form

$$\boldsymbol{q} = -A\nabla u + \boldsymbol{v} \cdot u \tag{3.29}$$

where we interpret $\boldsymbol{v}$ as a material velocity field which transports (advects) the scalar field u. Then we are led to the equation

$$-\nabla \cdot A\nabla u + \nabla \cdot u\boldsymbol{v} = f \text{ in } \Omega. \tag{3.30}$$

We call this an advection-diffusion equation.

3.5.4 Surfaces of prescribed curvature

Let Γ be an $n-$dimensional hypersurface in $\mathbb{R}^{n+1}$ and which is a graph $x_{n+1} = u(x), x \in \Omega$ over the $n-$dimensional bounded domain Ω where $u : \Omega \to \mathbb{R}$ so that

$$\Gamma := \{x' \in \mathbb{R}^{n+1} : x' = (x, u(x)), x \in \Omega\}.$$

The area of Γ may be written as

$$|\Gamma| = \mathcal{E}(u) :== \int_\Omega (1 + |\nabla u|^2)^{\frac{1}{2}})dx. \tag{3.31}$$

The *mean curvature* of Γ is given by (see the later chapter on surface partial differential equations)

$$-\nabla \cdot \frac{\nabla u}{(1 + |\nabla u|^2)^{\frac{1}{2}}}. \tag{3.32}$$

Given u, v we may write $G(t) = \mathcal{E}(u + tv)$ and see that

$$G'(0) = \int_\Omega \frac{\nabla u \cdot \nabla v}{(1 + |\nabla u|^2)^{\frac{1}{2}}} dx.$$

Graph like minimal surfaces

It follows that if we seek to find a graph like surface over the domain Ω which has a prescribed height at the boundary of Ω and which has minimal area we are led to the boundary value problem:

(**M**): Find u

$$-\nabla \cdot \frac{\nabla u}{(1 + |\nabla u|^2)^{\frac{1}{2}}} = 0 \quad \text{in} \quad \Omega \tag{3.33}$$

$$u = g \quad \text{on} \quad \partial\Omega \tag{3.34}$$

or equivalently in the weak form

$$\int_\Omega \frac{\nabla u \cdot \nabla v}{(1 + |\nabla u|^2)^{\frac{1}{2}}} dx = 0, \tag{3.35}$$

for v in a suitable test space.

Surfaces of prescribed curvature

Given f find a graph like surface Γ spanning Ω by solving the boundary value problem: (**P.C.**): Find u

$$-\nabla \cdot \frac{\nabla u}{(1 + |\nabla u|^2)^{\frac{1}{2}}} = f \quad \text{in} \quad \Omega \tag{3.36}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.37}$$

Remark 3.5.1. The vector field

$$\boldsymbol{A}(\boldsymbol{p}) = \frac{\boldsymbol{p}}{(1 + |\boldsymbol{p}|^2)^{1/2}}$$

is not coercive in the sense of the definition (3.4.1).

3.5.5 Flow in porous media

Flow in porous media is modelled using D'Arcy's law which states that the velocity field $\boldsymbol{q}$ in a saturated porous medium is given by

$$\boldsymbol{q} = -\frac{K}{\mu}\nabla p \tag{3.38}$$

where p is the pressure, K is the permeability tensor and μ is the fluid viscosity. In the case of an incompressible fluid

$$\nabla \cdot \boldsymbol{q} = 0. \tag{3.39}$$

This leads to an elliptic equation for p.

Chapter 4

Variational Inequalities

4.1 Projection theorem

Theorem 4.1.1. *Let K be a closed convex subset of a Hilbert space H. It follows that*

- *For all $w \in H$ there exists a unique $u \in K$ such that*
$$||u - w|| = \inf_{\eta \in K} ||\eta - w||_H.$$
We set
$$u := \mathbb{P}_K w$$
and call $\mathbb{P}_K : H \to K$ the projection operator from H onto K.
-
$$u = \mathbb{P}_K w \iff u \in K \quad \textit{and} \quad \langle u, \eta - u \rangle_H \geq \langle w, \eta - u \rangle_H \ \forall \eta \in K.$$
- *The operator $\mathbb{P}$ is non-expansive:-*
$$||\mathbb{P}_K w_1 - \mathbb{P}_K w_2||_H \leq ||w_1 - w_2||_H.$$

Proof. Set
$$\bar{J}(v) := \frac{1}{2}||v - w||^2.$$
Clearly the problem can be formulated in terms of minimzing $\bar{J}(\cdot)$ over K. Also note that
$$\bar{J}(v) = \frac{1}{2}\langle v, v \rangle - \langle w, v \rangle + \frac{1}{2}\langle w, w \rangle$$
so that we may also pose the problem as minimizing
$$J(v) = \frac{1}{2}\langle v, v \rangle - \langle w, v \rangle$$
over K. A more general case is considered in the next section. □

4.2 Elliptic variational inequality

Let K be a closed convex subset of a Hilbert space V. Let $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a bounded coercive bilinear form satisfying

1) $a(\cdot,\cdot)$ is bounded, i.e.,

$$\exists \gamma > 0 \text{ s.t. } |a(v,w)| \leq \gamma \|v\|_V \|w\|_V \ \forall v, w \in V.$$

2) $a(\cdot,\cdot)$ is coercive i.e.,

$$\exists \alpha > 0 \text{ s.t. } a(v,v) \geq \alpha \|v\|_V^2 \ \forall v \in V.$$

and $l(\cdot) : V \to \mathbb{R}$ be a bounded linear functional, i.e.,

$$\exists c_l > 0 \text{ s.t. } |l(v)| \leq c_l \|v\|_V \ \forall v \in V.$$

Theorem 4.2.1. *There exists a unique $u \in K$ such that*

$$\textit{(VI)} \quad a(u, v-u) \geq l(v-u) \quad \forall \quad v \in K. \tag{4.1}$$

Furthermore

$$||u_1 - u_2||_V \leq \frac{1}{\alpha} ||l_1 - l_2||_{V^*} \tag{4.2}$$

where $u_i, i = 1, 2$ solve (VI) for the linear forms l_1 and l_2, respectively.

Proof. Suppose $a(\cdot,\cdot)$ is symmetric.

Existence

Set $J(\cdot) : V \to \mathbb{R}$ to be the continuous quadratic functional

$$J(v) := \frac{1}{2} a(v,v) - l(v). \tag{4.3}$$

Note that

$$J(v) \geq \frac{\alpha}{2} ||v||_V^2 - c_l ||v||_V \geq \frac{\alpha}{2} ||v||_V^2 - \frac{1}{2\alpha} c_l^2 - \frac{\alpha}{2} ||v||_V^2$$

which implies that, since K is non-empty,

$$d := \inf_K J(v) \geq -\frac{1}{2\alpha} c_l^2 > -\infty$$

so that there is a minimizing sequence $u_n \in K$ such that

$$J(u_n) \to d.$$

We wish to establish that there exists $u \in K$ such that

$$u_n \to u \text{ and} J(u) = d.$$

We may choose the minimizing sequence so that

$$d \leq J(u_n) \leq d + \frac{1}{n}.$$

The bilinear form is symmetric so we have (e.g. from the parallelogram law)

$$\alpha ||u_n - u_m||^2 \leq a(u_n - u_m, u_n - u_m) = 4J(u_n) + 4J(u_m) - 8J(\frac{1}{2}(u_n + u_m)) \leq 4(\frac{1}{n} + \frac{1}{m}).$$

Hence the sequence $\{u_n\}$ is Cauchy and has a limit u which because K is closed lies in K. Furthermore from the contiuity of $J(\cdot)$ we have

$$J(u_n) \to J(u) = d.$$

We now show that u solves the variational inequality (4.1). For any $v \in K$, because K is convex $u + t(v-u) \in K, \forall\ t \in [0,1]$ and we have that $G(t) := J(u + \lambda(v-u)) \geq J(u) = G(0)$ and a calculation gives

$$ta(u, v-u) + \frac{t^2}{2} a(u-v, u-v) - tl(v-u) \geq 0 \quad t \in (0,1)$$

and dividing by t and taking the limit we obtain the variational inequality.

Uniqueness/Well posedness/Stability

From the variational inequalities for two solutions u_1 and u_2 we hve

$$a(u_i, u_j - u_i) \geq l_i(u_j - u_i) \quad i \neq j$$

and adding

$$\alpha||u_1 - u_2||_V^2 \leq a(u_1 - u_2, u_1 - u_2) \leq l_1(u_1 - u_2) - l_2(u_2 - u_1) \leq ||l_1 - l_2||_{V^*}||u_1 - u_2||_V.$$

General case

We consider the general case without assuming symmetry of the bilinear form. By the Riesz representation theorem we have the existence of $\mathcal{A} \in \mathcal{L}(V, V)$ and $L \in V$ such that

$$\langle \mathcal{A}u, v\rangle = a(u, v) \quad \forall u, v \in V \quad \text{and} \quad \langle L, v\rangle = l(v) \ \forall v \in V$$

where $\mathcal{A}^* = \mathcal{A}$ if $a(\cdot, \cdot)$ is symmetric.

Fix any ρ strictly positive. Then the variational inequality is equivalent to

$$\rho\langle \mathcal{A}u - L, v - u\rangle \geq 0 \ \forall v \in K$$

which may be rewritten as

$$\langle u, v - u\rangle \geq \langle u - \rho(\mathcal{A}u - L), v - u\rangle \ \forall v \in K$$

which is equivalent to finding $u \in K$ such that

$$u = \mathbb{P}_K(u - \rho(\mathcal{A}u - L)). \tag{4.4}$$

Here $\mathbb{P}_K$ is the projection operator from V to K in the Hilbert space V. Recall that it is non-expansive.

For convenience set

$$W_\rho(v) := \mathbb{P}_K(v - \rho(\mathcal{A}v - L)) \ , \forall v \in V.$$

We will show for suitable ρ that $W_\rho(\cdot) : V \to K$ is a strict contraction. Since $\mathbb{P}_K$ is nonexapansive we have

$$||W_\rho(v_1) - W_\rho(v_2)||^2 \leq ||v_1 - v_2||^2 + \rho^2||\mathcal{A}(v_1 - v_2)||^2 - 2\rho a(v_1 - v_2, v_1 - v_2)$$

and

$$||W_\rho(v_1) - W_\rho(v_2)||^2 \leq ||v_1 - v_2||^2 + \rho^2||\mathcal{A}||^2||v_1 - v_|||^2 - 2\rho\alpha||v_1 - v_2||^2$$

and

$$||W_\rho(v_1) - |W_\rho(v_2)|| \leq (1 - \rho||\mathcal{A}||^2(\rho^* - \rho))^{1/2}||v_1 - v_2||.$$

Thus $W_K(\cdot)$ is a strict contraction provided

$$0 < \rho < \frac{2\alpha}{||A||^2} := \rho^*.$$

By taking ρ in this range we have that there is a unique fixed point $u = W_\rho(u) = \mathbb{P}_K(u - \rho(\mathcal{A}u - l)) \in K$ and so the variational inequality has a unique solution.

$\square$

4.3 Truncation in L^2

Let $\Omega \subset \mathbb{R}^n$ be measurable and choose $\varphi \in L^2(\Omega)$. Set

$$K := \{v \in L^2(\Omega) : v \geq \varphi \text{ a.e. in } \Omega\}.$$

Clearly K is non-empty, convex and closed. Set

$$a(u, v) := (u, v)$$

where $(\cdot, \cdot)$ is the $L^2(\Omega)$ inner product. The problem: find $u \in K$ such that

$$(u - f, v - u) \geq 0 \ \forall v \in K$$

has a unique solution. A calculation reveals that

$$u = max(\varphi, f) := \begin{cases} \varphi(x) \text{ if } f(x) \leq \varphi(x) \\ f(x) \text{ if } \varphi(x) \leq f(x) \end{cases}$$

is satisfies the variational inequality and hence is the unique solution.

4.4 Obstacle problem

Let $V := H_0^1(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^d, d = 1, 2, 3$. Set

$$a(w, v) := \int_\Omega \nabla w \cdot \nabla v \quad , \quad l(v) := \int_\Omega fv$$

where $f \in L^2(\Omega)$ is given. Let $\psi \in H^1(\Omega) \cap C^0(\bar{\Omega})$ and $\psi|_{\partial\Omega} \leq 0$ and set

$$K := \{v \in H_0^1(\Omega) : v \geq \psi a.e. \in \Omega\}.$$

It follows that

1. K is non-empty.
   Set $\psi^+ := \frac{1}{2}(\psi + |\psi|) = \max(\psi, 0))$. Recall the following lemma

   Lemma 4.4.1. *Let $\theta : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, i.e.,*

   $$|\theta(t_1) - \theta(t_2)| \leq \lambda_\theta |t_1 - t_2| \ \ \forall \ \ t_1, t_2 \in \mathbb{R}.$$

   Suppose $\theta^{'}$ has a finite number of points of discontinuity. Then $\theta : H^1(\Omega) \to H^1(\Omega)$ is continuous and $\theta : H_0^1(\Omega) \to H_0^1(\Omega)$ is continuous in the case $\theta(0) = 0$.

   Hence $\psi^+ \in H_0^1(\Omega)$ and since $\psi^+ \geq \psi$ it follows that K is non-empty.

2. K is convex because for $t \in [0, 1]$, $t\eta + (1 - t)v \geq \psi \forall \eta, v \in K$.

3. K is closed.
   This follows from the fact that convergence in $H_0^1(\Omega)$ implies convergence in $L^2(\Omega)$ and hence convergence almost everywhere for a sub-sequence. From which we find that $v_n \to v$ in V implies $v_{n_i} \to v$ *a.e.* in *in* Ω and if $v_n \in K$ then $v_{n_i} \geq \psi$ *a.e. in* Ω which implies by the convergence of v_n that $v \geq \psi$ *a.e. in* Ω and so $v \in K$.

Theorem 4.4.2. *There exists a unique solution to the obstacle problem: Find $u \in K := \{v \in H_0^1(\Omega) : v \geq \psi$ a.e. in $\Omega\}$ such that*

$$\int_\Omega \nabla u \cdot (\nabla v - \nabla u) \geq \int_\Omega f(v - u) \ \ \forall v \in K.$$

Linear Complementarity Problem

Suppose we have the regularity result that the unique solution satisfies $u \in H^2(\Omega)$. Then integration by parts in the variational inequality yields

$$\int_\Omega (-\Delta u - f)(v - u) \geq 0 \quad \forall v \in K. \tag{4.5}$$

Choosing $v = u + \eta$ where $\eta \geq 0$ and $\eta \in C_0^\infty(\Omega)$ yields

$$\int_\Omega (-\Delta u - f)\eta \geq 0$$

from which we obtain

$$-\Delta u - f \geq 0 \;\; a.e. \;\; \Omega.$$

Suppose u is continuous. (This is true automatically for $u \in H^2(\Omega)$ when $d = 1, 2$.) Then the set

$$\Omega^+ := \{x \in \Omega : u(x) > \psi(x)\} \tag{4.6}$$

is open. For any $\eta \in C_0^\infty(\Omega^+)$ the function $v = u \pm \epsilon\eta \in K$ provided $|\epsilon|$ is small enough. For such an η we have

$$\int_{\Omega^+} (-\Delta u - f)\eta = 0$$

which gives

$$-\Delta u - f = 0 \quad \text{in} \quad \Omega^+.$$

Thus we have shown that if the solution $u \in H_0^1(\Omega)$ also satisfies the regularity $u \in H^2(\Omega) \cap C(\Omega)$ then it satisfies the *linear complementarity system*:

$$-\Delta u - f \geq 0, \quad u \geq \psi \;\; a.e. \;\; \Omega \tag{4.7}$$

$$(-\Delta u - f)(u - \psi) = 0 \;\; a.e. \;\; \Omega \tag{4.8}$$

$$\tag{4.9}$$

The set

$$\Omega^0 := \{x \in \Omega : u(x) = \psi(x)\} \tag{4.10}$$

is called the *coincidence set* wheres ω^+ is called the non-coincidence set. The boundary of the non-coincidence set in Ω

$$\Gamma := \partial\Omega^+ \cap \Omega \tag{4.11}$$

is called the *free boundary*.

Free boundary problem

It can be shown that (for suitable smooth f, ψ and $\partial\Omega$) that the solution of the above obstacle problem satisfies $u \in C^1(\Omega)$ or $u \in W^{2,p}(\Omega)$. It follows that

$$u - \psi = 0, \quad \nabla(u - \psi) = 0 \;\; \text{on} \;\; \Gamma.$$

Thus we may view u as being the solution of the following *free boundary problem*: Find u, Γ, Ω^+ such that

$$-\Delta u = f \;\; \text{in} \;\; \Omega^+ \tag{4.12}$$

$$u = 0 \;\; \text{on} \;\; \partial\Omega \tag{4.13}$$

$$u = \psi, \; \frac{\partial u}{\partial \nu} = g \qquad \text{on} \;\; \Gamma + \partial\Omega^+ \cap \Omega \tag{4.14}$$

where we have set $g := \frac{\partial \psi}{\partial \nu}$ and denoted by ν the normal to Γ.

Note that two conditions hold on Γ. This is because Γ is *unknown*. Such free boundary problems arise in many applications and are formulated as boundary value problems for PDEs rather than in variational form. Often they have a structure which enables them to be formulated as a variational inequality. Accounts of free boundary problems may be found in[10, 13, 19].

4.5 Obstacle problem for a membrane

Consider the situation of a membrane stretched over a rigid obstacle. Here suppose that the membrane is a hyper-surface described by the graph $x_3 = u(x), x = (x_1, x_2) \in \Omega$ where Ω is an open bounded planar domain with boundary $\partial\Omega$. Suppose that the surface of the rigid obstacle is also a graph $x_3 = \psi(x), x = (x_1, x_2) \in \Omega$. The membrane lies over the obstacle so

$$u \geq \psi \text{ in } \Omega.$$

We have that the domain is decomposed into two domains $\bar{\Omega} = \bar{\Omega}^+ \cup \bar{\Omega}_I$ where

$$u > \psi \text{ in } \Omega^+, \quad u = \psi \text{ in } \Omega_I.$$

In equilibrium away from the obstacle the membrane satisfies Laplace's equation and on the obstacle the vertical force on the membrane is non-positive so that

$$-\Delta u = 0 \text{ in } \Omega^+, \quad -\Delta u \geq 0 \text{ in } \Omega_I.$$

At the contact interface $\Gamma := \partial\Omega^+ \cap \partial\Omega_I$ the smoothness conditions

$$u = \psi, \quad \nabla u \cdot \nu = \nabla \psi \cdot \nu$$

hold where the second condition is continuity of the tension within the membrane.

Suppose $u = 0$ and $\psi \leq 0$ on the boundary $\partial\Omega$. This problem may be posed as the obstacle variational inequality described in the previous section with $f = 0$. This is achieved by using integration by parts

$$\int_\Omega \nabla u \cdot \nabla(v - u) dx \geq 0$$

for $v \in K$ and u satisfying the above conditions.

Note that the energy for this problem

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2$$

is an approximation to the area functional

$$A(u) := \int_\Omega (1 + |\nabla u|^2)^{1/2}$$

when the gradient of the displacement is small.

4.6 Hele-Shaw problem

A free boundary problems for a PDE is boundary value problems in which not only the solution of the PDE is to be determined but also the domain in which the PDE holds! The *Hele-Shaw* free boundary problem is a famous example which has been widely studied in many different contexts and with widely varying mathematical ideas. It concerns a model for the flow of incompressible viscous fluid in a narrow gap of width h between two parallel plates. The fluid then occupies the region $\{(x_1, x_2, x_3) : x =$

$(x_1, x_2) \in \Omega(t), x_3 \in (0, h)\}$. Because the gap is narrow so called *lubrication* approximation methods may be applied to the three dimensional Navier-Stokes equations to yield a problem in two space dimensions. In short only the velocity components parallel to the plates are considered and they are averaged over the thickness of the gap. The upshot is that the average velocity field $\mathbf{q} = \mathbf{q}(x, t)$ is related to the pressure $p = p(x, t)$ by the equation

$$\mathbf{q} = -\frac{h^2}{12\mu}\nabla p \tag{4.15}$$

where μ is the fluid viscosity. Since the fluid is incompressible $\nabla \cdot \mathbf{q} = \mathbf{0}$ we are led to the equation

$$-\nabla \cdot \frac{h^2}{12\mu}\nabla p = 0, \quad \text{in} \quad \Omega(t). \tag{4.16}$$

The fluid blob $\Omega(t)$ is separated from the part of the cell not occupied by fluid by an *interface* $\Gamma(t)$. Conservation of mass then yields that V_ν, the normal velocity of $\Gamma(t)$, i.e. the velocity of $\Gamma(t)$ in the outward pointing normal direction, ν, to $\Omega(t)$, is given by the fluid velocity yielding

$$V_\nu = -\frac{h^2}{12\mu}\nabla p \cdot \nu. \tag{4.17}$$

In certain physical circumstances the balance of momentum at the fluid/void interface yields that the pressure is constant on that interface. Since the pressure is undetermined up to an additive constant then we may take

$$p = 0, \quad \text{on} \quad \Gamma(t). \tag{4.18}$$

Let us suppose that the fluid initially occupies the domain $\Omega(0) = \Omega_0$.

In order to *drive* this process we need to specify some way of injecting or sucking fluid out of the cell. There are fundamental mathematical differences between injection and suction. For our purpose we consider only the simpler situation of injection. We denote by $Q(t) \geq 0$ the injection rate.

1. **Point source injection**

   Let $\Omega(t) \subset \mathbb{R}^2$ be an open bounded domain with boundary $\partial\Omega(t) = \Gamma(t)$. Let $0 \in \Omega(t)$. Consider the problem of given $\Omega(0) = \Omega_0$ and $Q : (0, T) \to \mathbb{R}_+$, finding:$\{p, \Omega(t), \Gamma(t)\}$ such that

$$-\nabla \cdot \frac{h^2}{12\mu}\nabla p = Q(t)\delta, \quad \text{in} \quad \Omega(t) \tag{4.19}$$

$$V_\nu = -\frac{h^2}{12\mu}\nabla p, \quad p = 0 \quad \text{on} \quad \Gamma(t) \tag{4.20}$$

   where δ is the Dirac delta measure.

2. **Surface injection**

   Let $\Omega(t) \subset \mathbb{R}^2$ be an open bounded annular domain with inner boundary Γ_I and outer moving boundary $\Gamma(t)$ so that $\partial\Omega(t) = \Gamma_I \cup \Gamma(t)$. Consider the *moving boundary problem* of given $\Omega(0) = \Omega_0$ and $Q : (0, T) \to \mathbb{R}_+$, finding:$\{p, \Omega(t), \Gamma(t)\}$ such that

$$-\nabla \cdot \frac{h^2}{12\mu}\nabla p = 0, \quad \text{in} \quad \Omega(t) \tag{4.21}$$

$$V_\nu = -\frac{h^2}{12\mu}\nabla p, \quad p = 0 \quad \text{on} \quad \Gamma(t) \tag{4.22}$$

$$\frac{h^2}{12\mu}\nabla p \cdot \nu = Q(t) \quad \text{on} \quad \Gamma_I. \tag{4.23}$$

Note that we may rescale so for convenience in the following we take $\frac{h^2}{12\mu} = 1$.

The above situations may be complicated further by supposing that the fluid lies in a container D bounded by container walls ∂D which are impervious to the flow. It follows that the above systems should be supplemented by the equation

$$\frac{\partial p}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_C(t) := \partial\Omega(t) \cap \partial D. \tag{4.24}$$

Here we are supposing that $\partial\Omega(t) = \Gamma(t) \cup \Gamma_I \cup \Gamma_C(t)$.

Assume that fluid is being injected. As a consequence the pressure in $\Omega(t)$ is positive and the normal derivative of p on $\Gamma(t)$ is non-positive which implies that the expanding fluid blob $\Omega(t))$ has a non-negative normal velocity. Thus we may describe the fluid interface in the following way: there exists a function $\omega(x)$ such that for each $t \geq 0$

$$\Omega(t) = \{x \in D : t > \omega(x)\}, \quad \Gamma(t) = \{x \in D : t = \omega(x)\}, \quad \omega(x) = 0 \quad x \in \Omega_0.$$

The evolution equation for $\Gamma(t)$ may now be written as

$$\nabla p \cdot \nabla \omega = -1. \tag{4.25}$$

Remark 4.6.1. The function

$$\phi(x,t) := t - \omega(x)$$

may be considered as a *level set function* whose zero level set defines the evolving hypersurface $\Gamma(t)$. The outward pointing unit normal to $\Omega(t)$ and the normal velocity of $\Gamma(t)$ are

$$\nu = -\frac{\nabla\phi}{|\nabla\phi|} = \frac{\nabla\omega}{|\nabla\omega|}, \quad V_\nu = \frac{\phi_t}{|\nabla\phi|} = 1/|\nabla\omega|$$

Remark 4.6.2. Let us observe that

$$\frac{d|\Omega(t)|}{dt} = \int_{\Gamma(t)} V_\nu = -\int_{\Gamma(t)} p_\nu = \int_{\Gamma_I} Q(t)$$

from which we deduce that

$$|\Omega(t)| = |\Omega_0| + \mathcal{Q}(t)|\Gamma_I|.$$

Since when the cell is filled with fluid we can no longer inject fluid there can only be a solution as long as

$$\mathcal{Q}(t) \leq \frac{|D| - |\Omega_0|}{|\Gamma_I|}.$$

Since the injection rate is non-negative we have that there can only be a solution for $t \in [0,T]$ where T is the smallest solution of

$$\mathcal{Q}(T) = \frac{|D| - |\Omega_0|}{|\Gamma_I|}.$$

This should be reflected in the mathematical formulation of the problem.

The Hele-Shaw problem for injection of fluid into a bounded cell D across a portion $\Gamma_I \subset \partial D$ may be formulated as an elliptic variational inequality, see [9], for

$$u(x,t) = \int_0^t p(x,\tau)d\tau, \quad x \in D \tag{4.26}$$

where we have extended the definition of p to all of D using the constant zero extension, i.e.

$$p(x,t) = 0 \quad x \in D \setminus \Omega(t).$$

Calculations reveal that, for each t, u solves the following *free boundary problem*

$$-\Delta u = f \quad \text{in} \quad \Omega(t) \tag{4.27}$$
$$u = 0 \quad \text{in} \quad D \setminus \Omega(t) \tag{4.28}$$
$$u \geq 0 \quad \text{in} \quad D \tag{4.29}$$
$$u = u_\nu = 0 \quad \text{on} \quad \Gamma(t) \tag{4.30}$$
$$u_\nu = \mathcal{Q}(t) \quad \text{on} \quad \Gamma_I \tag{4.31}$$

where

$$f = \chi_{\Omega_0} - 1.$$

By consideration of

$$\int_D (-\Delta u - f)(v - u) dx$$

for $v \in K$ we may derive a variational inequality using integration by parts and assuming regularity of $\Gamma(t)$ and u

Set

$$a(u, v) := \int_D \nabla u \cdot \nabla v dx, \quad \langle v, w \rangle = \int_{\Gamma_I} vw, \quad .$$

Then the *elliptic variational inequality* is: Find $u \in K := \{v \in H^1(D) : v \geq 0 \ \ a.e. \ \text{in} \ \ D\}$ such that

$$a(u, v - u) \geq (f, v - u) + \langle \mathcal{Q}(t), v - u \rangle \quad \forall v \in K \tag{4.32}$$

where

$$\mathcal{Q}(t) = \int_0^t Q(\tau) d\tau.$$

Note that in this setting the bilinear form is not coercive on K. However by considering for $v \in K$

$$v = \bar{v} + Pv$$

where

$$Pv := \frac{\int_D v}{|D|}$$

and the fact that for $v \in K$, (using the Poincare inequality for functions with zero mean),

$$||v||_{H^1(\Omega)} \to \infty \iff ||\nabla v||_{L^2(\Omega)} \to \infty \quad \text{or} \quad Pv \to \infty$$

we may show that the functional

$$J(v) := \frac{1}{2} a(v, v) - (f, v) - \langle \mathcal{Q}(t), v \rangle$$

is coercive on K provided

$$0 \leq t < T.$$

This shows existence. Uniqueness of u in this time interval follows by noting that the standard argument yields that the difference of two solutions is a positive constant c and that this constant satisfies

$$(f, c) + \langle \mathcal{Q}(t), c \rangle.$$

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